Intermediate Models of Prikry Type Forcings

Tom Benhamou

Department of Mathematics
Tel Aviv University

December 14, 2020

Background

Background

A general basic problem is to classify the complete subalgebras of a given complete boolean algebra B. Alternatively, one can study the intermediate models of a V-generic extensions $V[G]$ of B i.e. transitive ZFC models $V \subsetneq M \subseteq V[G]$ definable in $V[G]$.

Background

A general basic problem is to classify the complete subalgebras of a given complete boolean algebra B. Alternatively, one can study the intermediate models of a V-generic extensions $V[G]$ of B i.e. transitive ZFC models $V \subsetneq M \subseteq V[G]$ definable in $V[G]$. Any intermediate model M of a forcing extension by B is of the form $M = V[H]$, where H is V-generic for some complete subalgebra of B.

Background

A general basic problem is to classify the complete subalgebras of a given complete boolean algebra B. Alternatively, one can study the intermediate models of a V-generic extensions $V[G]$ of B i.e. transitive ZFC models $V \subsetneq M \subseteq V[G]$ definable in $V[G]$. Any intermediate model M of a forcing extension by B is of the form $M = V[H]$, where H is V-generic for some complete subalgebra of B.

For the Cohen and Random forcings we have the following classification:

Background

A general basic problem is to classify the complete subalgebras of a given complete boolean algebra B. Alternatively, one can study the intermediate models of a V-generic extensions $V[G]$ of B i.e. transitive ZFC models $V \subsetneq M \subseteq V[G]$ definable in $V[G]$. Any intermediate model M of a forcing extension by B is of the form $M = V[H]$, where H is V-generic for some complete subalgebra of B.

For the Cohen and Random forcings we have the following classification:

Theorem

Background

A general basic problem is to classify the complete subalgebras of a given complete boolean algebra B. Alternatively, one can study the intermediate models of a V-generic extensions $V[G]$ of B i.e. transitive ZFC models $V \subsetneq M \subseteq V[G]$ definable in $V[G]$. Any intermediate model M of a forcing extension by B is of the form $M = V[H]$, where H is V-generic for some complete subalgebra of B.

For the Cohen and Random forcings we have the following classification:

Theorem

1. *(Folklore [7]) Any intermediate model of a Cohen generic extension is a Cohen generic extension.*

Background

A general basic problem is to classify the complete subalgebras of a given complete boolean algebra B. Alternatively, one can study the intermediate models of a V-generic extensions $V[G]$ of B i.e. transitive ZFC models $V \subsetneq M \subseteq V[G]$ definable in $V[G]$. Any intermediate model M of a forcing extension by B is of the form $M = V[H]$, where H is V-generic for some complete subalgebra of B.

For the Cohen and Random forcings we have the following classification:

Theorem

1. *(Folklore [7]) Any intermediate model of a Cohen generic extension is a Cohen generic extension.*
2. *(D.Maharam) Any intermediate model of a Random real generic extension is a Random real generic extension.*

Background

Background

Already here there is a simple open question:

Already here there is a simple open question:

Problem

Background

Already here there is a simple open question:

Problem

What are the complete subforcings of Random $\times$ Random?

Background

Already here there is a simple open question:

Problem

What are the complete subforcings of Random $\times$ Random?

Clearly, the Random forcing is an example, and by Steinhouse, Random $\times$ Random adds a Cohen real. But is that all?

Background

Already here there is a simple open question:

Problem

What are the complete subforcings of Random $\times$ Random?

Clearly, the Random forcing is an example, and by Steinhouse, Random $\times$ Random adds a Cohen real. But is that all?
The same phenomena of the structure of intermediate models also hold for the standard Prikry forcing:

Background

Already here there is a simple open question:

Problem

What are the complete subforcings of Random $\times$ Random?

Clearly, the Random forcing is an example, and by Steinhouse, Random $\times$ Random adds a Cohen real. But is that all?
The same phenomena of the structure of intermediate models also hold for the standard Prikry forcing:

Theorem (Gitik, Kanovei, Koepke, 2010 [6])

Background

Already here there is a simple open question:

Problem

What are the complete subforcings of Random $\times$ Random?

Clearly, the Random forcing is an example, and by Steinhouse, Random $\times$ Random adds a Cohen real. But is that all?
The same phenomena of the structure of intermediate models also hold for the standard Prikry forcing:

Theorem (Gitik, Kanovei, Koepke, 2010 [6])

Let U be a normal measure over κ and $G \subseteq \mathbb{P}(U)$ be a V-generic set producing the Prikry sequence $C_G := \{C_G(n) \mid n < \omega\}$. Then for every $A \in V[G]$ there is $C \subseteq C_G$, such that $V[A] = V[C]$.

Background

Background

Corollary

Background

Corollary

In the settings of the last theorem, let $V \subsetneq M \subseteq V[G]$ *be an intermediate model, then* $M = V[G']$ *where* $G' \subseteq \mathbb{P}(U)$ *is another* V*-generic filter.*

Background

Corollary

In the settings of the last theorem, let $V \subsetneq M \subseteq V[G]$ *be an intermediate model, then* $M = V[G']$ *where* $G' \subseteq \mathbb{P}(U)$ *is another* V*-generic filter.*

Proof.

Background

Corollary

In the settings of the last theorem, let $V \subsetneq M \subseteq V[G]$ be an intermediate model, then $M = V[G']$ where $G' \subseteq \mathbb{P}(U)$ is another V-generic filter.

Proof.

Every such model is of the form $M = V[A]$ for some set $A \in V[G]$. By the theorem, $M = V[C]$ for some subsequence C of the Prikry sequence. By the Mathias criteria[10], C is itself a Prikry sequence for U.

Background

Corollary

In the settings of the last theorem, let $V \subsetneq M \subseteq V[G]$ *be an intermediate model, then* $M = V[G']$ *where* $G' \subseteq \mathbb{P}(U)$ *is another* V*-generic filter.*

Proof.

Every such model is of the form $M = V[A]$ for some set $A \in V[G]$. By the theorem, $M = V[C]$ for some subsequence C of the Prikry sequence. By the Mathias criteria[10], C is itself a Prikry sequence for U. □

In this talk we will examine the intermediate models of **the tree Prikry** and the **Magidor-Radin** forcings.

The tree-Prikry forcing

The tree-Prikry forcing

Let $\langle S, \leq_S \rangle$ be any poset, denote by $[S]^{<\omega}$ the tree of finite $\leq_S$-increasing sequences ordered by end-extension. Let $\vec{U} = \langle U_a \mid a \in [S]^{<\omega}\rangle$ be a tree of $|S|$-complete uniform ultrafilters over S.

The tree-Prikry forcing

Let $\langle S, \leq_S \rangle$ be any poset, denote by $[S]^{<\omega}$ the tree of finite $\leq_S$-increasing sequences ordered by end-extension. Let $\vec{U} = \langle U_a \mid a \in [S]^{<\omega} \rangle$ be a tree of $|S|$-complete uniform ultrafilters over S.

Definition (Tree Prikry Focring-$P_T(\vec{U})$)

The tree-Prikry forcing

Let $\langle S, \leq_S \rangle$ be any poset, denote by $[S]^{<\omega}$ the tree of finite $\leq_S$-increasing sequences ordered by end-extension. Let $\vec{U} = \langle U_a \mid a \in [S]^{<\omega} \rangle$ be a tree of $|S|$-complete uniform ultrafilters over S.

Definition (Tree Prikry Focring-$P_T(\vec{U})$)

Conditions of $P_T(\vec{U})$ are pairs $\langle t, T \rangle$, where T is a subtree of $[S]^{<\omega}$ with stem t, which is $\vec{U}$-splitting:

$$\forall s \in T. s \geq t \rightarrow \mathrm{Succ}_T(s) := \{x \in S \mid s^\frown x \in T\} \in U_s$$

The tree-Prikry forcing

Let $\langle S, \leq_S \rangle$ be any poset, denote by $[S]^{<\omega}$ the tree of finite $\leq_S$-increasing sequences ordered by end-extension. Let $\vec{U} = \langle U_a \mid a \in [S]^{<\omega} \rangle$ be a tree of $|S|$-complete uniform ultrafilters over S.

Definition (Tree Prikry Focring-$P_T(\vec{U})$)

Conditions of $P_T(\vec{U})$ are pairs $\langle t, T \rangle$, where T is a subtree of $[S]^{<\omega}$ with stem t, which is $\vec{U}$-splitting:

$$\forall s \in T. s \geq t \rightarrow \mathrm{Succ}_T(s) := \{x \in S \mid s^\frown x \in T\} \in U_s$$

The order is defined (Israel convention: $q \leq p$ then $p \Vdash q \in \dot{G}$)
$\langle t, T \rangle \leq \langle s, S \rangle$ iff $S \subseteq T$ (hence $s \in T$)

The tree-Prikry forcing

Let $\langle S, \leq_S \rangle$ be any poset, denote by $[S]^{<\omega}$ the tree of finite $\leq_S$-increasing sequences ordered by end-extension. Let $\vec{U} = \langle U_a \mid a \in [S]^{<\omega} \rangle$ be a tree of $|S|$-complete uniform ultrafilters over S.

Definition (Tree Prikry Focring-$P_T(\vec{U})$)

Conditions of $P_T(\vec{U})$ are pairs $\langle t, T \rangle$, where T is a subtree of $[S]^{<\omega}$ with stem t, which is $\vec{U}$-splitting:

$$\forall s \in T. s \geq t \rightarrow \mathrm{Succ}_T(s) := \{x \in S \mid s^\frown x \in T\} \in U_s$$

The order is defined (Israel convention: $q \leq p$ then $p \Vdash q \in \dot{G}$)
$\langle t, T \rangle \leq \langle s, S \rangle$ iff $S \subseteq T$ (hence $s \in T$)

We will mostly be interested in ultrafilters over κ itself.

The tree-Prikry forcing

Let $\langle S, \leq_S \rangle$ be any poset, denote by $[S]^{<\omega}$ the tree of finite $\leq_S$-increasing sequences ordered by end-extension. Let $\vec{U} = \langle U_a \mid a \in [S]^{<\omega} \rangle$ be a tree of $|S|$-complete uniform ultrafilters over S.

Definition (Tree Prikry Focring-$P_T(\vec{U})$)

Conditions of $P_T(\vec{U})$ are pairs $\langle t, T \rangle$, where T is a subtree of $[S]^{<\omega}$ with stem t, which is $\vec{U}$-splitting:

$$\forall s \in T. s \geq t \to \mathrm{Succ}_T(s) := \{x \in S \mid s^\frown x \in T\} \in U_s$$

The order is defined (Israel convention: $q \leq p$ then $p \Vdash q \in \dot{G}$)
$\langle t, T \rangle \leq \langle s, S \rangle$ iff $S \subseteq T$ (hence $s \in T$)

We will mostly be interested in ultrafilters over κ itself. It turns out (not surprisingly) that the structure of the intermediate models of the tree Prikry forcing depends on the combinatorical properties of the measures in $\vec{U}$.

Intermediate Models of Tree Prikry forcing

Intermediate Models of Tree Prikry forcing

Theorem (Koepke, Räsch, Schlicht (2013)[8])

Intermediate Models of Tree Prikry forcing

Theorem (Koepke, Räsch, Schlicht (2013)[8])

Assume that $\vec{U} = \langle U_\alpha \mid \alpha < \kappa \rangle$ is a sequence of distinct normal measures. Then for every V-generic filter $G \subseteq P_T(\vec{U})$[a], there is no proper intermediate model $V \subsetneq M \subsetneq V[G]$.

[a] We view $\vec{U}$ as a tree by defining for every $a \in [\kappa]^{<\omega}$, $U_a = U_{\max(a)}$.

Intermediate Models of Tree Prikry forcing

Theorem (Koepke, Räsch, Schlicht (2013)[8])

Assume that $\vec{U} = \langle U_\alpha \mid \alpha < \kappa\rangle$ is a sequence of distinct normal measures. Then for every V-generic filter $G \subseteq P_T(\vec{U})$[a], there is no proper intermediate model $V \subsetneq M \subsetneq V[G]$.

[a] We view $\vec{U}$ as a tree by defining for every $a \in [\kappa]^{<\omega}$, $U_a = U_{\max(a)}$.

On the other hand:

Theorem (Gitik, B. (2021)[3])

Intermediate Models of Tree Prikry forcing

Theorem (Koepke, Räsch, Schlicht (2013)[8])

Assume that $\vec{U} = \langle U_\alpha \mid \alpha < \kappa \rangle$ is a sequence of distinct normal measures. Then for every V-generic filter $G \subseteq P_T(\vec{U})^a$, there is no proper intermediate model $V \subsetneq M \subsetneq V[G]$.

[a] We view $\vec{U}$ as a tree by defining for every $a \in [\kappa]^{<\omega}$, $U_a = U_{\max(a)}$.

On the other hand:

Theorem (Gitik, B. (2021)[3])

Assume GCH and let κ be a measurable cardinal. There is a cofinality preserving forcing extension $V \subseteq N$ and an ultrafilter $W \in N$ such that forcing with $P_T(W)^a$ over N adds a κ-Cohen real.

[a] We view $\vec{U}$ as a tree by defining for every $a \in [\kappa]^{<\omega}$, $U_a = W$.

Intermediate Models of Tree Prikry forcing

Theorem (Koepke, Räsch, Schlicht (2013)[8])

Assume that $\vec{U} = \langle U_\alpha \mid \alpha < \kappa \rangle$ is a sequence of distinct normal measures. Then for every V-generic filter $G \subseteq P_T(\vec{U})$[a], there is no proper intermediate model $V \subsetneq M \subsetneq V[G]$.

[a] We view $\vec{U}$ as a tree by defining for every $a \in [\kappa]^{<\omega}$, $U_a = U_{\max(a)}$.

On the other hand:

Theorem (Gitik, B. (2021)[3])

Assume GCH and let κ be a measurable cardinal. There is a cofinality preserving forcing extension $V \subseteq N$ and an ultrafilter $W \in N$ such that forcing with $P_T(W)$[a] over N adds a κ-Cohen real.

[a] We view $\vec{U}$ as a tree by defining for every $a \in [\kappa]^{<\omega}$, $U_a = W$.

Let us sketch the main ideas of the proof:

Prikry introduce Cohen- Proof

Proof.

The model N is obtained by forcing the Easton support iteration $\langle P_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa, \beta < \kappa \rangle$:

Prikry introduce Cohen- Proof

Proof.

The model N is obtained by forcing the Easton support iteration $\langle P_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa, \beta < \kappa \rangle$: Each $\underset{\sim}{Q}_\beta$ is trivial, unless β is inaccessible. For inaccessible β, $\underset{\sim}{Q}_\beta$ is the lottery sum of the trivial forcing $\{0\}$ and the β-Cohen real forcing $Add(\beta, 1)$.

Prikry introduce Cohen- Proof

Proof.

The model N is obtained by forcing the Easton support iteration $\langle P_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa, \beta < \kappa \rangle$: Each $\underset{\sim}{Q}_\beta$ is trivial, unless β is inaccessible. For inaccessible β, $\underset{\sim}{Q}_\beta$ is the lottery sum of the trivial forcing $\{0\}$ and the β-Cohen real forcing $Add(\beta, 1)$. Let $G_\kappa \subseteq P_\kappa$ be V-generic and $N := V[G_\kappa]$. Let $U \in V$ be a normal measure over κ and $j_1 := j_U : V \to M_U$ be the corresponding elementary embedding, denote $j_1(\kappa) = \kappa_1$.

Prikry introduce Cohen- Proof

Proof.

The model N is obtained by forcing the Easton support iteration $\langle P_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa, \beta < \kappa\rangle$: Each $\underset{\sim}{Q}_\beta$ is trivial, unless β is inaccessible. For inaccessible β, $\underset{\sim}{Q}_\beta$ is the lottery sum of the trivial forcing $\{0\}$ and the β-Cohen real forcing $Add(\beta, 1)$. Let $G_\kappa \subseteq P_\kappa$ be V-generic and $N := V[G_\kappa]$. Let $U \in V$ be a normal measure over κ and $j_1 := j_U : V \to M_U$ be the corresponding elementary embedding, denote $j_1(\kappa) = \kappa_1$. Let us extend j_U to $j_1^* : V[G_\kappa] \to M_U[H]$, note that $j_1(P_\kappa) = P_{\kappa_1}$ is an iteration defined similar to P_κ inside M_U, so we only need to find a generic for the part $P_{[\kappa,\kappa_1)}$.

Prikry introduce Cohen- Proof

Proof.

The model N is obtained by forcing the Easton support iteration $\langle P_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa, \beta < \kappa \rangle$: Each $\underset{\sim}{Q}_\beta$ is trivial, unless β is inaccessible. For inaccessible β, $\underset{\sim}{Q}_\beta$ is the lottery sum of the trivial forcing $\{0\}$ and the β-Cohen real forcing $Add(\beta, 1)$. Let $G_\kappa \subseteq P_\kappa$ be V-generic and $N := V[G_\kappa]$. Let $U \in V$ be a normal measure over κ and $j_1 := j_U : V \to M_U$ be the corresponding elementary embedding, denote $j_1(\kappa) = \kappa_1$. Let us extend j_U to $j_1^* : V[G_\kappa] \to M_U[H]$, note that $j_1(P_\kappa) = P_{\kappa_1}$ is an iteration defined similar to P_κ inside M_U, so we only need to find a generic for the part $P_{[\kappa,\kappa_1)}$. Since κ is inaccessible in M_U, the forcing $\underset{\sim}{Q}_\kappa$ is a lottery sum, which is not κ^+-closed. By choosing $\{0\}$, we gain sufficient closure to construct a M_U-generic $G_{\kappa_1} \in V[G_\kappa]$.

Prikry introduce Cohen- Proof

Proof.

The model N is obtained by forcing the Easton support iteration $\langle P_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa, \beta < \kappa\rangle$: Each $\underset{\sim}{Q}_\beta$ is trivial, unless β is inaccessible. For inaccessible β, $\underset{\sim}{Q}_\beta$ is the lottery sum of the trivial forcing $\{0\}$ and the β-Cohen real forcing $Add(\beta, 1)$. Let $G_\kappa \subseteq P_\kappa$ be V-generic and $N := V[G_\kappa]$. Let $U \in V$ be a normal measure over κ and $j_1 := j_U : V \to M_U$ be the corresponding elementary embedding, denote $j_1(\kappa) = \kappa_1$. Let us extend j_U to $j_1^* : V[G_\kappa] \to M_U[H]$, note that $j_1(P_\kappa) = P_{\kappa_1}$ is an iteration defined similar to P_κ inside M_U, so we only need to find a generic for the part $P_{[\kappa,\kappa_1)}$. Since κ is inaccessible in M_U, the forcing $\underset{\sim}{Q}_\kappa$ is a lottery sum, which is not κ^+-closed. By choosing $\{0\}$, we gain sufficient closure to construct a M_U-generic $G_{\kappa_1} \in V[G_\kappa]$. Note that, the ultrafilter $\{X \in P^N(\kappa) \mid \kappa \in j_1^*(X)\}$ concentrates on $\{\alpha < \kappa \mid \{0\} \text{ was forced at } \alpha\}$.

Prikry introduce Cohen- Proof

Prikry introduce Cohen- Proof

Back in V, consider the second iteration by U, $j_{j_1(U)} : M_U \to M_{j_1(U)}$, and denote by $j_2 = j_{j_1(U)} \circ j_1$, $\kappa_2 = j_2(\kappa)$.

Prikry introduce Cohen- Proof

Back in V, consider the second iteration by U, $j_{j_1(U)} : M_U \to M_{j_1(U)}$, and denote by $j_2 = j_{j_1(U)} \circ j_1$, $\kappa_2 = j_2(\kappa)$. Note that from the point of view of V, the forcing $P_{[\kappa_1,\kappa_2)}$ is κ^+ closed.

Prikry introduce Cohen- Proof

Back in V, consider the second iteration by U, $j_{j_1(U)} : M_U \to M_{j_1(U)}$, and denote by $j_2 = j_{j_1(U)} \circ j_1$, $\kappa_2 = j_2(\kappa)$. Note that from the point of view of V, the forcing $P_{[\kappa_1,\kappa_2)}$ is κ^+ closed. Hence a generic filter can be constructed to extend j_2 to j_2^*, only this time we can choose the Cohen part in the lottery of $\underset{\sim}{Q}_{\kappa_1}$.

Prikry introduce Cohen- Proof

Back in V, consider the second iteration by U, $j_{j_1(U)} : M_U \to M_{j_1(U)}$, and denote by $j_2 = j_{j_1(U)} \circ j_1$, $\kappa_2 = j_2(\kappa)$. Note that from the point of view of V, the forcing $P_{[\kappa_1,\kappa_2)}$ is κ^+ closed. Hence a generic filter can be constructed to extend j_2 to j_2^*, only this time we can choose the Cohen part in the lottery of $\underset{\sim}{Q}_{\kappa_1}$. Let

$$W = \{X \in P^N(\kappa) \mid \kappa_1 \in j_2^*(X)\}$$

Clearly W concentrates on the set Y of point on which the Cohen part was forced in G_κ.

Prikry introduce Cohen- Proof

Back in V, consider the second iteration by U, $j_{j_1(U)} : M_U \to M_{j_1(U)}$, and denote by $j_2 = j_{j_1(U)} \circ j_1$, $\kappa_2 = j_2(\kappa)$. Note that from the point of view of V, the forcing $P_{[\kappa_1,\kappa_2)}$ is κ^+ closed. Hence a generic filter can be constructed to extend j_2 to j_2^*, only this time we can choose the Cohen part in the lottery of $\underset{\sim}{Q}_{\kappa_1}$. Let

$$W = \{X \in P^N(\kappa) \mid \kappa_1 \in j_2^*(X)\}$$

Clearly W concentrates on the set Y of point on which the Cohen part was forced in G_κ. For each $\alpha \in Y$, let f_α be the Cohen function added by G_κ.

Prikry introduce Cohen- Proof

Back in V, consider the second iteration by U, $j_{j_1(U)} : M_U \to M_{j_1(U)}$, and denote by $j_2 = j_{j_1(U)} \circ j_1$, $\kappa_2 = j_2(\kappa)$. Note that from the point of view of V, the forcing $P_{[\kappa_1,\kappa_2)}$ is κ^+ closed. Hence a generic filter can be constructed to extend j_2 to j_2^*, only this time we can choose the Cohen part in the lottery of $\underset{\sim}{Q}_{\kappa_1}$. Let

$$W = \{X \in P^N(\kappa) \mid \kappa_1 \in j_2^*(X)\}$$

Clearly W concentrates on the set Y of point on which the Cohen part was forced in G_κ. For each $\alpha \in Y$, let f_α be the Cohen function added by G_κ. Force $P_T(W)$ over N, and denote by $C_G := \{\kappa_n \mid n < \omega\}$ the Prikry sequence.

Prikry introduce Cohen- Proof

Back in V, consider the second iteration by U, $j_{j_1(U)} : M_U \to M_{j_1(U)}$, and denote by $j_2 = j_{j_1(U)} \circ j_1$, $\kappa_2 = j_2(\kappa)$. Note that from the point of view of V, the forcing $P_{[\kappa_1, \kappa_2)}$ is κ^+ closed. Hence a generic filter can be constructed to extend j_2 to j_2^*, only this time we can choose the Cohen part in the lottery of $\underset{\sim}{Q}_{\kappa_1}$. Let

$$W = \{X \in P^N(\kappa) \mid \kappa_1 \in j_2^*(X)\}$$

Clearly W concentrates on the set Y of point on which the Cohen part was forced in G_κ. For each $\alpha \in Y$, let f_α be the Cohen function added by G_κ. Force $P_T(W)$ over N, and denote by $C_G := \{\kappa_n \mid n < \omega\}$ the Prikry sequence. There is $n_0 < \omega$ such that for every $n \geq n_0$, $\kappa_n \in Y$ and therefore f_{κ_n} is defined.

Prikry introduce Cohen- Proof

Back in V, consider the second iteration by U, $j_{j_1(U)} : M_U \to M_{j_1(U)}$, and denote by $j_2 = j_{j_1(U)} \circ j_1$, $\kappa_2 = j_2(\kappa)$. Note that from the point of view of V, the forcing $P_{[\kappa_1, \kappa_2)}$ is κ^+ closed. Hence a generic filter can be constructed to extend j_2 to j_2^*, only this time we can choose the Cohen part in the lottery of $\underset{\sim}{Q}_{\kappa_1}$. Let

$$W = \{X \in P^N(\kappa) \mid \kappa_1 \in j_2^*(X)\}$$

Clearly W concentrates on the set Y of point on which the Cohen part was forced in G_κ. For each $\alpha \in Y$, let f_α be the Cohen function added by G_κ. Force $P_T(W)$ over N, and denote by $C_G := \{\kappa_n \mid n < \omega\}$ the Prikry sequence. There is $n_0 < \omega$ such that for every $n \geq n_0$, $\kappa_n \in Y$ and therefore f_{κ_n} is defined. It remains to see that

$$f = \cup_{n_0 \leq n < \omega} f_{\kappa_n} \restriction [\kappa_{n-1}, \kappa_n) \in N[C_G]$$

is N-generic for $Add(\kappa, 1)$. □

the class of projection of $P_T(U)$

the class of projection of $P_T(U)$

Question

the class of projection of $P_T(U)$

Question

What is the class of forcing extensions which can be intermediate to a tree Prikry generic extension?

the class of projection of $P_T(U)$

Question

What is the class of forcing extensions which can be intermediate to a tree Prikry generic extension?

Obviously, such a class cannot add bounded subsets to κ as $P_T(U)$ does not add bounded subsets of κ. Also there are forcing of cardinality κ^+ which cannot be projections of $P_T(\vec{U})$, for example $Add(\kappa^+, 1)$.

the class of projection of $P_T(U)$

Question

What is the class of forcing extensions which can be intermediate to a tree Prikry generic extension?

Obviously, such a class cannot add bounded subsets to κ as $P_T(U)$ does not add bounded subsets of κ. Also there are forcing of cardinality κ^+ which cannot be projections of $P_T(\vec{U})$, for example $Add(\kappa^+, 1)$. A natural subclass to consider is the class of κ-distributive forcing of cardinality κ.

the class of projection of $P_T(U)$

Question

What is the class of forcing extensions which can be intermediate to a tree Prikry generic extension?

Obviously, such a class cannot add bounded subsets to κ as $P_T(U)$ does not add bounded subsets of κ. Also there are forcing of cardinality κ^+ which cannot be projections of $P_T(\vec{U})$, for example $Add(\kappa^+, 1)$. A natural subclass to consider is the class of κ-distributive forcing of cardinality κ.

Theorem

the class of projection of $P_T(U)$

Question

What is the class of forcing extensions which can be intermediate to a tree Prikry generic extension?

Obviously, such a class cannot add bounded subsets to κ as $P_T(U)$ does not add bounded subsets of κ. Also there are forcing of cardinality κ^+ which cannot be projections of $P_T(\vec{U})$, for example $Add(\kappa^+, 1)$. A natural subclass to consider is the class of κ-distributive forcing of cardinality κ.

Theorem

Let $\mathbb{P}$ be a σ-distributive forcing of size κ. The following are equivalent:

the class of projection of $P_T(U)$

Question

What is the class of forcing extensions which can be intermediate to a tree Prikry generic extension?

Obviously, such a class cannot add bounded subsets to κ as $P_T(U)$ does not add bounded subsets of κ. Also there are forcing of cardinality κ^+ which cannot be projections of $P_T(\vec{U})$, for example $Add(\kappa^+, 1)$. A natural subclass to consider is the class of κ-distributive forcing of cardinality κ.

Theorem

Let $\mathbb{P}$ be a σ-distributive forcing of size κ. The following are equivalent:

1. *There is a projection $\pi \colon \mathbb{P}_T(\vec{\mathcal{U}}) \to \mathbb{P}$, where $\vec{\mathcal{U}} = \langle U_\eta \mid \eta \in [S]^{<\omega} \rangle$ is a sequence of ultrafilters on S and $|S| = \kappa$.*

the class of projection of $P_T(U)$

Question

What is the class of forcing extensions which can be intermediate to a tree Prikry generic extension?

Obviously, such a class cannot add bounded subsets to κ as $P_T(U)$ does not add bounded subsets of κ. Also there are forcing of cardinality κ^+ which cannot be projections of $P_T(\vec{U})$, for example $Add(\kappa^+, 1)$. A natural subclass to consider is the class of κ-distributive forcing of cardinality κ.

Theorem

Let $\mathbb{P}$ be a σ-distributive forcing of size κ. The following are equivalent:

1. *There is a projection $\pi: \mathbb{P}_T(\vec{\mathcal{U}}) \to \mathbb{P}$, where $\vec{\mathcal{U}} = \langle U_\eta \mid \eta \in [S]^{<\omega} \rangle$ is a sequence of ultrafilters on S and $|S| = \kappa$.*
2. *For every $p \in \mathbb{P}$ there is a κ-complete ultrafilter $U_p \supseteq \mathcal{D}_p(\mathbb{P})$. Where $\mathcal{D}_p(\mathbb{P})$ is the filter of dense open subsets of $\mathbb{P}$ above p.*

Assuming κ is κ-compact

Assuming κ is κ-compact

The last theorem indicates that we are only interested in extending κ-complete filters of sets of cardinality κ.

Definition (κ-compact Cardinal)

Assuming κ is κ-compact

The last theorem indicates that we are only interested in extending κ-complete filters of sets of cardinality κ.

Definition (κ-compact Cardinal)

κ is called a κ-compact cardinal if every κ-complete filter over κ can be extended to a κ-complete ultrafilter over κ

Assuming κ is κ-compact

The last theorem indicates that we are only interested in extending κ-complete filters of sets of cardinality κ.

Definition (κ-compact Cardinal)

κ is called a κ-compact cardinal if every κ-complete filter over κ can be extended to a κ-complete ultrafilter over κ

Corollary

Assuming κ is κ-compact

The last theorem indicates that we are only interested in extending κ-complete filters of sets of cardinality κ.

Definition (κ-compact Cardinal)

κ is called a κ-compact cardinal if every κ-complete filter over κ can be extended to a κ-complete ultrafilter over κ

Corollary

Assume that κ is κ-compact then every κ distributive forcing of cardinality κ is a projection of the tree-Prikry forcing.

Assuming κ is κ-compact

The last theorem indicates that we are only interested in extending κ-complete filters of sets of cardinality κ.

Definition (κ-compact Cardinal)

κ is called a κ-compact cardinal if every κ-complete filter over κ can be extended to a κ-complete ultrafilter over κ

Corollary

Assume that κ is κ-compact then every κ distributive forcing of cardinality κ is a projection of the tree-Prikry forcing.

The assumption that κ is κ-compact is quit strong:

The last theorem indicates that we are only interested in extending κ-complete filters of sets of cardinality κ.

Definition (κ-compact Cardinal)

κ is called a κ-compact cardinal if every κ-complete filter over κ can be extended to a κ-complete ultrafilter over κ

Corollary

Assume that κ is κ-compact then every κ distributive forcing of cardinality κ is a projection of the tree-Prikry forcing.

The assumption that κ is κ-compact is quit strong:

Theorem (Gitik [5])

Assuming κ is κ-compact

The last theorem indicates that we are only interested in extending κ-complete filters of sets of cardinality κ.

Definition (κ-compact Cardinal)

κ is called a κ-compact cardinal if every κ-complete filter over κ can be extended to a κ-complete ultrafilter over κ

Corollary

Assume that κ is κ-compact then every κ distributive forcing of cardinality κ is a projection of the tree-Prikry forcing.

The assumption that κ is κ-compact is quit strong:

Theorem (Gitik [5])

If κ is κ-compact then there is an inner model with a Woodin cardinal.

Lower bound for all the κ-distributive

Lower bound for all the κ-distributive

Question

Lower bound for all the κ-distributive

Question

Can the assumption that κ is κ-compact be relaxed?

Lower bound for all the κ-distributive

Question

Can the assumption that κ is κ-compact be relaxed?

In a recent joint work with Gitik and Hayut, we have found that there is a non trivial lower bound:

Theorem (Gitik, Hayut, B.)

Lower bound for all the κ-distributive

Question

Can the assumption that κ is κ-compact be relaxed?

In a recent joint work with Gitik and Hayut, we have found that there is a non trivial lower bound:

Theorem (Gitik, Hayut, B.)

Let Q be the forcing shooting a club through the singulars below κ[a]*. Assume that there is a κ-complete ultrafilter extending the filter $D(Q)$ of dense open subset of Q. Then either there is an inner model for $\exists\lambda, o(\lambda) = \lambda^{++}$, or $o^{\mathcal{K}}(\kappa) \geq \kappa + \kappa$.*

[a] Thus Making κ not Mahlo. It is $< \kappa$-strategically closed.

Lower bound for all the κ-distributive

Question

Can the assumption that κ is κ-compact be relaxed?

In a recent joint work with Gitik and Hayut, we have found that there is a non trivial lower bound:

Theorem (Gitik, Hayut, B.)

Let Q be the forcing shooting a club through the singulars below κ[a]. Assume that there is a κ-complete ultrafilter extending the filter $D(Q)$ of dense open subset of Q. Then either there is an inner model for $\exists\lambda, o(\lambda) = \lambda^{++}$, or $o^{\mathcal{K}}(\kappa) \geq \kappa + \kappa$.

[a] Thus Making κ not Mahlo. It is $< \kappa$-strategically closed.

Minimal assumption- Masterable forcing

Another direction taken in our work is the following:

Question

Minimal assumption- Masterable forcing

Another direction taken in our work is the following:

Question

Under the minimal assumption that κ is measurable. What is the class of forcing $\mathcal{P}$ which can be intermediate to a tree Prikry extension?

Minimal assumption- Masterable forcing

Another direction taken in our work is the following:

Question

Under the minimal assumption that κ is measurable. What is the class of forcing $\mathcal{P}$ which can be intermediate to a tree Prikry extension?

We have constructed a Class of forcing called **Masterable forcing** (Denoted by $\mathcal{N}_\kappa$) which is obtained by isolating the properties of a forcing which are sufficient for the argument given for Cohen forcing to work.

Minimal assumption- Masterable forcing

Another direction taken in our work is the following:

Question

Under the minimal assumption that κ is measurable. What is the class of forcing $\mathcal{P}$ which can be intermediate to a tree Prikry extension?

We have constructed a Class of forcing called **Masterable forcing** (Denoted by $\mathcal{N}_\kappa$) which is obtained by isolating the properties of a forcing which are sufficient for the argument given for Cohen forcing to work. $\mathcal{N}_\kappa$ is closed under complete subforcings, and proven to consistently include many forcing notions $\mathcal{N}_\kappa$.

Minimal assumption- Masterable forcing

Another direction taken in our work is the following:

Question

Under the minimal assumption that κ is measurable. What is the class of forcing $\mathcal{P}$ which can be intermediate to a tree Prikry extension?

We have constructed a Class of forcing called **Masterable forcing** (Denoted by $\mathcal{N}_\kappa$) which is obtained by isolating the properties of a forcing which are sufficient for the argument given for Cohen forcing to work. $\mathcal{N}_\kappa$ is closed under complete subforcings, and proven to consistently include many forcing notions $\mathcal{N}_\kappa$.

Theorem

Minimal assumption- Masterable forcing

Another direction taken in our work is the following:

Question

Under the minimal assumption that κ is measurable. What is the class of forcing $\mathcal{P}$ which can be intermediate to a tree Prikry extension?

We have constructed a Class of forcing called **Masterable forcing** (Denoted by $\mathcal{N}_\kappa$) which is obtained by isolating the properties of a forcing which are sufficient for the argument given for Cohen forcing to work. $\mathcal{N}_\kappa$ is closed under complete subforcings, and proven to consistently include many forcing notions $\mathcal{N}_\kappa$.

Theorem

Assume GCH and let κ be a measurable cardinal.
Then there is a cofinality preserving forcing extension in which for any $\mathbb{Q} \in \mathcal{N}_\kappa$, there is a κ-complete ultrafilter $\mathcal{U}$ extending $\mathcal{D}_p(\mathbb{Q})$ for every $p \in \mathbb{Q}$.

Magidor Forcing

Magidor Forcing

We follow the description from [4]:

Magidor Forcing

We follow the description from [4]:

Definition ($\mathbb{M}[\vec{U}]$-Magidor Forcing [9])

We follow the description from [4]:

Definition ($\mathbb{M}[\vec{U}]$-Magidor Forcing [9])

Let $\vec{U} = \langle U(\alpha, \beta) \mid \alpha \leq \kappa, \beta < o^{\vec{U}}(\alpha)\rangle$ be a coherent sequence. The conditions of $\mathbb{M}[\vec{U}]$ are of the form $\langle\langle\alpha_1, A_1\rangle, ..., \langle\alpha_n, A_n\rangle, \langle\kappa, A\rangle\rangle$ where:

Magidor Forcing

We follow the description from [4]:

Definition ($\mathbb{M}[\vec{U}]$-Magidor Forcing [9])

Let $\vec{U} = \langle U(\alpha, \beta) \mid \alpha \leq \kappa, \beta < o^{\vec{U}}(\alpha)\rangle$ be a coherent sequence. The conditions of $\mathbb{M}[\vec{U}]$ are of the form $\langle\langle\alpha_1, A_1\rangle, ..., \langle\alpha_n, A_n\rangle, \langle\kappa, A\rangle\rangle$ where:

1. $\alpha_1 < ... < \alpha_n$ is an increasing sequence below κ.

We follow the description from [4]:

Definition ($\mathbb{M}[\vec{U}]$-Magidor Forcing [9])

Let $\vec{U} = \langle U(\alpha, \beta) \mid \alpha \leq \kappa, \beta < o^{\vec{U}}(\alpha) \rangle$ be a coherent sequence. The conditions of $\mathbb{M}[\vec{U}]$ are of the form $\langle \langle \alpha_1, A_1 \rangle, ..., \langle \alpha_n, A_n \rangle, \langle \kappa, A \rangle \rangle$ where:

1. $\alpha_1 < ... < \alpha_n$ is an increasing sequence below κ.
2. $A_i = \emptyset$ unless $o^{\vec{U}}(\alpha_i) > 0$ in which case, $A_i \in \bigcap_{\beta < o^{\vec{U}}(\alpha_i)} U(\alpha_i, \beta)$ is a measure one set with respect to **all** the measures given on α_i.

Magidor Forcing

We follow the description from [4]:

Definition ($\mathbb{M}[\vec{U}]$-Magidor Forcing [9])

Let $\vec{U} = \langle U(\alpha, \beta) \mid \alpha \leq \kappa, \beta < o^{\vec{U}}(\alpha) \rangle$ be a coherent sequence. The conditions of $\mathbb{M}[\vec{U}]$ are of the form $\langle \langle \alpha_1, A_1 \rangle, ..., \langle \alpha_n, A_n \rangle, \langle \kappa, A \rangle \rangle$ where:

1. $\alpha_1 < ... < \alpha_n$ is an increasing sequence below κ.
2. $A_i = \emptyset$ unless $o^{\vec{U}}(\alpha_i) > 0$ in which case, $A_i \in \bigcap_{\beta < o^{\vec{U}}(\alpha_i)} U(\alpha_i, \beta)$ is a measure one set with respect to **all** the measures given on α_i.

   The order is define as follows,

   $\langle \langle \alpha_1, A_1 \rangle, ..., \langle \alpha_n, A_n \rangle, \langle \kappa, A \rangle \rangle \leq \langle \langle \beta_1, B_1 \rangle, ..., \langle \beta_m, B_m \rangle, \langle \kappa, B \rangle \rangle$ iff:

Magidor Forcing

We follow the description from [4]:

Definition ($\mathbb{M}[\vec{U}]$-Magidor Forcing [9])

Let $\vec{U} = \langle U(\alpha, \beta) \mid \alpha \leq \kappa, \beta < o^{\vec{U}}(\alpha) \rangle$ be a coherent sequence. The conditions of $\mathbb{M}[\vec{U}]$ are of the form $\langle \langle \alpha_1, A_1 \rangle, ..., \langle \alpha_n, A_n \rangle, \langle \kappa, A \rangle \rangle$ where:

1. $\alpha_1 < ... < \alpha_n$ is an increasing sequence below κ.
2. $A_i = \emptyset$ unless $o^{\vec{U}}(\alpha_i) > 0$ in which case, $A_i \in \bigcap_{\beta < o^{\vec{U}}(\alpha_i)} U(\alpha_i, \beta)$ is a measure one set with respect to **all** the measures given on α_i.

   The order is define as follows,

   $\langle \langle \alpha_1, A_1 \rangle, ..., \langle \alpha_n, A_n \rangle, \langle \kappa, A \rangle \rangle \leq \langle \langle \beta_1, B_1 \rangle, ..., \langle \beta_m, B_m \rangle, \langle \kappa, B \rangle \rangle$ iff:

   $\exists 1 \leq i_1 < ... < i_n \leq m$ such that for every $1 \leq j \leq m$:

Magidor Forcing

We follow the description from [4]:

Definition ($\mathbb{M}[\vec{U}]$-Magidor Forcing [9])

Let $\vec{U} = \langle U(\alpha,\beta) \mid \alpha \leq \kappa, \beta < o^{\vec{U}}(\alpha)\rangle$ be a coherent sequence. The conditions of $\mathbb{M}[\vec{U}]$ are of the form $\langle\langle\alpha_1, A_1\rangle, ..., \langle\alpha_n, A_n\rangle, \langle\kappa, A\rangle\rangle$ where:

1. $\alpha_1 < ... < \alpha_n$ is an increasing sequence below κ.
2. $A_i = \emptyset$ unless $o^{\vec{U}}(\alpha_i) > 0$ in which case, $A_i \in \bigcap_{\beta < o^{\vec{U}}(\alpha_i)} U(\alpha_i, \beta)$ is a measure one set with respect to **all** the measures given on α_i.
   The order is define as follows,
   $\langle\langle\alpha_1, A_1\rangle, ..., \langle\alpha_n, A_n\rangle, \langle\kappa, A\rangle\rangle \leq \langle\langle\beta_1, B_1\rangle, ..., \langle\beta_m, B_m\rangle, \langle\kappa, B\rangle\rangle$ iff:
   $\exists 1 \leq i_1 < ... < i_n \leq m$ such that for every $1 \leq j \leq m$:
   1. If $\exists 1 \leq r \leq n$ such that $i_r = j$ then $\beta_{i_r} = \alpha_r$ and $B_{i_r} \subseteq A_r$.

Magidor Forcing

We follow the description from [4]:

Definition ($\mathbb{M}[\vec{U}]$-Magidor Forcing [9])

Let $\vec{U} = \langle U(\alpha,\beta) \mid \alpha \leq \kappa, \beta < o^{\vec{U}}(\alpha)\rangle$ be a coherent sequence. The conditions of $\mathbb{M}[\vec{U}]$ are of the form $\langle\langle\alpha_1, A_1\rangle, ..., \langle\alpha_n, A_n\rangle, \langle\kappa, A\rangle\rangle$ where:

1. $\alpha_1 < ... < \alpha_n$ is an increasing sequence below κ.
2. $A_i = \emptyset$ unless $o^{\vec{U}}(\alpha_i) > 0$ in which case, $A_i \in \bigcap_{\beta < o^{\vec{U}}(\alpha_i)} U(\alpha_i, \beta)$ is a measure one set with respect to **all** the measures given on α_i.

   The order is define as follows,

   $\langle\langle\alpha_1, A_1\rangle, ..., \langle\alpha_n, A_n\rangle, \langle\kappa, A\rangle\rangle \leq \langle\langle\beta_1, B_1\rangle, ..., \langle\beta_m, B_m\rangle, \langle\kappa, B\rangle\rangle$ iff:

   $\exists 1 \leq i_1 < ... < i_n \leq m$ such that for every $1 \leq j \leq m$:

   1. If $\exists 1 \leq r \leq n$ such that $i_r = j$ then $\beta_{i_r} = \alpha_r$ and $B_{i_r} \subseteq A_r$.
   2. Otherwise let $1 \leq r \leq n+1$ such that $i_{r-1} < j < i_r$ then:
      $\beta_j \in A_r,\ B_j \subseteq A_r \cap \alpha_r,\ o^{\vec{U}}(\beta_j) < o^{\vec{U}}(\alpha_r)$

Magidor Forcing- Examples of Intermediate Models

Intermediate Models of a generic extension by $\mathbb{M}[\vec{U}]$ are not necessarily generic extensions of $\mathbb{M}[\vec{U}]$:

Magidor Forcing- Examples of Intermediate Models

Intermediate Models of a generic extension by $\mathbb{M}[\vec{U}]$ are not necessarily generic extensions of $\mathbb{M}[\vec{U}]$:

Example

Magidor Forcing- Examples of Intermediate Models

Intermediate Models of a generic extension by $\mathbb{M}[\vec{U}]$ are not necessarily generic extensions of $\mathbb{M}[\vec{U}]$:

Example

Assume that $o^{\vec{U}}(\kappa) = 2$. Then κ carries two measures: $U(\kappa, 0), U(\kappa, 1)$. This means that typically the generic club generated is of order type ω^2, denote it by $C_G = \{C_G(i) \mid i < \omega^2\}$. If we take for example the intermediate model $V[\{C_G(n) \mid n < \omega\}]$, it is a Prikry $P(U(C_G(\omega), 0))$ generic extension (By the Mathias criteria), which is not a generic extension for $\mathbb{M}[\vec{U}]$.

Example

Magidor Forcing- Examples of Intermediate Models

Intermediate Models of a generic extension by $\mathbb{M}[\vec{U}]$ are not necessarily generic extensions of $\mathbb{M}[\vec{U}]$:

Example

Assume that $o^{\vec{U}}(\kappa) = 2$. Then κ carries two measures: $U(\kappa, 0), U(\kappa, 1)$. This means that typically the generic club generated is of order type ω^2, denote it by $C_G = \{C_G(i) \mid i < \omega^2\}$. If we take for example the intermediate model $V[\{C_G(n) \mid n < \omega\}]$, it is a Prikry $P(U(C_G(\omega), 0))$ generic extension (By the Mathias criteria), which is not a generic extension for $\mathbb{M}[\vec{U}]$.

Example

Assume that $o^{\vec{U}}(\kappa) = \omega$, thus $\mathrm{otp}(C_G) = \omega^\omega$. Consider the intermediate extension $V[\{C_G(\omega^n) \mid n < \omega\}]$ it is a diagonal Prikry generic extension for the sequence of measures $\langle U(\kappa, n) \mid n < \omega\rangle$.

Intermediate models of Magidor Extensions

Intermediate models of Magidor Extensions

Theorem (Gitik, B.[3])

Theorem (Gitik, B.[3])

Let κ be a cardinal such that $o^{\vec{U}}(\kappa) < \delta_0 := \min(\alpha \mid o^{\vec{U}}(\alpha) = 1)$. Let $G \subseteq \mathbb{M}[\vec{U}]$ be a V-generic set producing the Magidor club C_G. Then for every $A \in V[G]$ there is $C \subseteq C_G$, such that $V[A] = V[C]$.

Intermediate models of Magidor Extensions

Theorem (Gitik, B.[3])

Let κ be a cardinal such that $o^{\vec{U}}(\kappa) < \delta_0 := \min(\alpha \mid o^{\vec{U}}(\alpha) = 1)$. Let $G \subseteq \mathbb{M}[\vec{U}]$ be a V-generic set producing the Magidor club C_G. Then for every $A \in V[G]$ there is $C \subseteq C_G$, such that $V[A] = V[C]$.

As we have seen in the examples, it is not clear which are the forcings such that the models $V[C]$ are generic for. In our paper, we defined in the ground model a class of "Magidor-Type" forcing notions, denoted by $\mathbb{M}_I[\vec{U}]$, which is basically a Magidor forcing adding elements prescribed by the set I, where I is the set of indices of C inside C_G.

Intermediate models of Magidor Extensions

Theorem (Gitik, B.[3])

Let κ be a cardinal such that $o^{\vec{U}}(\kappa) < \delta_0 := \min(\alpha \mid o^{\vec{U}}(\alpha) = 1)$. Let $G \subseteq \mathbb{M}[\vec{U}]$ be a V-generic set producing the Magidor club C_G. Then for every $A \in V[G]$ there is $C \subseteq C_G$, such that $V[A] = V[C]$.

As we have seen in the examples, it is not clear which are the forcings such that the models $V[C]$ are generic for. In our paper, we defined in the ground model a class of "Magidor-Type" forcing notions, denoted by $\mathbb{M}_I[\vec{U}]$, which is basically a Magidor forcing adding elements prescribed by the set I, where I is the set of indices of C inside C_G.

Theorem

Intermediate models of Magidor Extensions

Theorem (Gitik, B.[3])

Let κ be a cardinal such that $o^{\vec{U}}(\kappa) < \delta_0 := \min(\alpha \mid o^{\vec{U}}(\alpha) = 1)$. Let $G \subseteq \mathbb{M}[\vec{U}]$ be a V-generic set producing the Magidor club C_G. Then for every $A \in V[G]$ there is $C \subseteq C_G$, such that $V[A] = V[C]$.

As we have seen in the examples, it is not clear which are the forcings such that the models $V[C]$ are generic for. In our paper, we defined in the ground model a class of "Magidor-Type" forcing notions, denoted by $\mathbb{M}_I[\vec{U}]$, which is basically a Magidor forcing adding elements prescribed by the set I, where I is the set of indices of C inside C_G.

Theorem

In the settings of the last theorem, let $V \subseteq M \subseteq V[G]$ be an intermediate ZFC model definable $V[G]$, $M = V[G']$ where $G' \subseteq \mathbb{M}_I[\vec{U}]$ is a generic filter for some $I \in V$.

The case $\delta_0 \leq o^{\vec{U}}(\kappa) \leq \kappa$

The case $\delta_0 \leq o^{\vec{U}}(\kappa) \leq \kappa$

Example

The case $\delta_0 \leq o^{\vec{U}}(\kappa) \leq \kappa$

Example

Let $o^{\vec{U}}(\kappa) = \delta_0$. There is $G \subseteq \mathbb{M}[\vec{U}]$ which produces a Magidor sequence $\{C_G(\alpha) \mid \alpha < \delta_0\}$ such that $C_G(\omega) = \delta_0$. The first Prikry sequence $\{C_G(n) \mid n < \omega\} \in V[G]$ is a cofinal sequence in $C_G(\omega) = \delta_0$. Consider the sequence $C = \{C_G(C_G(n)) \mid n < \omega\}$. It is unbounded in κ and witnesses that κ changes cofinality to ω. This example does not fall under the classification of the last theorem since the indices of C inside C_G are $I := \{C_G(n) \mid n < \omega\} \notin V$.

The case $\delta_0 \leq o^{\vec{U}}(\kappa) \leq \kappa$

Example

Let $o^{\vec{U}}(\kappa) = \delta_0$. There is $G \subseteq \mathbb{M}[\vec{U}]$ which produces a Magidor sequence $\{C_G(\alpha) \mid \alpha < \delta_0\}$ such that $C_G(\omega) = \delta_0$. The first Prikry sequence $\{C_G(n) \mid n < \omega\} \in V[G]$ is a cofinal sequence in $C_G(\omega) = \delta_0$. Consider the sequence $C = \{C_G(C_G(n)) \mid n < \omega\}$. It is unbounded in κ and witnesses that κ changes cofinality to ω. This example does not fall under the classification of the last theorem since the indices of C inside C_G are $I := \{C_G(n) \mid n < \omega\} \notin V$.

Example

The case $\delta_0 \leq o^{\vec{U}}(\kappa) \leq \kappa$

Example

Let $o^{\vec{U}}(\kappa) = \delta_0$. There is $G \subseteq \mathbb{M}[\vec{U}]$ which produces a Magidor sequence $\{C_G(\alpha) \mid \alpha < \delta_0\}$ such that $C_G(\omega) = \delta_0$. The first Prikry sequence $\{C_G(n) \mid n < \omega\} \in V[G]$ is a cofinal sequence in $C_G(\omega) = \delta_0$. Consider the sequence $C = \{C_G(C_G(n)) \mid n < \omega\}$. It is unbounded in κ and witnesses that κ changes cofinality to ω. This example does not fall under the classification of the last theorem since the indices of C inside C_G are $I := \{C_G(n) \mid n < \omega\} \notin V$.

Example

Assume that $o^{\vec{U}}(\kappa) = \kappa$, and let $C_G = \{C_G(\alpha) \mid \alpha < \kappa\}$, and let $\kappa^* \in C_G$ is such that for any $\beta \in C_G \setminus \kappa^*$, $o^{\vec{U}}(\beta) < \beta$. In $V[G]$, define $\alpha_0 = \kappa^*$, and $\alpha_{n+1} = C_G(\alpha_n)$. Then $\{\alpha_n \mid n < \omega\}$ is a cofinal ω-sequence in κ. Also, it satisfy the Mathias criteria [1] for the tree Prikry forcing with respect to the measures on κ, $\langle U(\kappa, \alpha) \mid \alpha < \kappa\rangle$.

Lately we have proven the following result:

Lately we have proven the following result:

Theorem (Gitik, B.)

Lately we have proven the following result:

Theorem (Gitik, B.)

Let κ be a cardinal such that $o^{\vec{U}}(\kappa) \leq \kappa$. Let $G \subseteq \mathbb{M}[\vec{U}]$ be a V-generic set producing the Magidor sequence C_G. Then for every $A \in V[G]$ there is $C \subseteq C_G$, such that $V[A] = V[C]$.

Lately we have proven the following result:

Theorem (Gitik, B.)

Let κ be a cardinal such that $o^{\vec{U}}(\kappa) \leq \kappa$. Let $G \subseteq \mathbb{M}[\vec{U}]$ be a V-generic set producing the Magidor sequence C_G. Then for every $A \in V[G]$ there is $C \subseteq C_G$, such that $V[A] = V[C]$.

Let us sketch some of the ideas from the proof:

Lately we have proven the following result:

Theorem (Gitik, B.)

Let κ be a cardinal such that $o^{\vec{U}}(\kappa) \leq \kappa$. Let $G \subseteq \mathbb{M}[\vec{U}]$ be a V-generic set producing the Magidor sequence C_G. Then for every $A \in V[G]$ there is $C \subseteq C_G$, such that $V[A] = V[C]$.

Let us sketch some of the ideas from the proof: The theorem is by induction on κ. Note that it suffices to prove the theorem for sets of ordinals.

Lately we have proven the following result:

Theorem (Gitik, B.)

Let κ be a cardinal such that $o^{\vec{U}}(\kappa) \leq \kappa$. Let $G \subseteq \mathbb{M}[\vec{U}]$ be a V-generic set producing the Magidor sequence C_G. Then for every $A \in V[G]$ there is $C \subseteq C_G$, such that $V[A] = V[C]$.

Let us sketch some of the ideas from the proof: The theorem is by induction on κ. Note that it suffices to prove the theorem for sets of ordinals.

Lemma

Lately we have proven the following result:

Theorem (Gitik, B.)

Let κ be a cardinal such that $o^{\vec{U}}(\kappa) \leq \kappa$. Let $G \subseteq \mathbb{M}[\vec{U}]$ be a V-generic set producing the Magidor sequence C_G. Then for every $A \in V[G]$ there is $C \subseteq C_G$, such that $V[A] = V[C]$.

Let us sketch some of the ideas from the proof: The theorem is by induction on κ. Note that it suffices to prove the theorem for sets of ordinals.

Lemma

If $A \subseteq V, A \in V[G]$, $|A| < \kappa$, then there is $C \subseteq C_G$ such that $V[A] = V[C]$.

Lately we have proven the following result:

Theorem (Gitik, B.)

Let κ be a cardinal such that $o^{\vec{U}}(\kappa) \leq \kappa$. Let $G \subseteq \mathbb{M}[\vec{U}]$ be a V-generic set producing the Magidor sequence C_G. Then for every $A \in V[G]$ there is $C \subseteq C_G$, such that $V[A] = V[C]$.

Let us sketch some of the ideas from the proof: The theorem is by induction on κ. Note that it suffices to prove the theorem for sets of ordinals.

Lemma

If $A \subseteq V, A \in V[G]$, $|A| < \kappa$, then there is $C \subseteq C_G$ such that $V[A] = V[C]$.

The proof of the lemma used the strong Prikry property.

The strong Prikry property for $\mathbb{M}[\vec{U}]$

The strong Prikry property for $\mathbb{M}[\vec{U}]$

Definition

The strong Prikry property for $\mathbb{M}[\vec{U}]$

Definition

A tree $T \subseteq [\kappa]^{<\omega}$ is called a $\vec{U}$-fat tree, if $ht(T) < \omega$ and for every $t \in T$, either t is a maximal element of the tree, or
$succ_T(t) := \{\alpha < \kappa \mid t^\frown\alpha \in T\} \in U(\beta, i)$ for some $\beta \leq \kappa$ and $i < o^{\vec{U}}(\beta)$.

The strong Prikry property for $\mathbb{M}[\vec{U}]$

Definition

A tree $T \subseteq [\kappa]^{<\omega}$ is called a $\vec{U}$-fat tree, if $ht(T) < \omega$ and for every $t \in T$, either t is a maximal element of the tree, or
$succ_T(t) := \{\alpha < \kappa \mid t^\frown\alpha \in T\} \in U(\beta, i)$ for some $\beta \leq \kappa$ and $i < o^{\vec{U}}(\beta)$.

Proposition (The strong Prikry Property)

The strong Prikry property for $\mathbb{M}[\vec{U}]$

Definition

A tree $T \subseteq [\kappa]^{<\omega}$ is called a $\vec{U}$-fat tree, if $ht(T) < \omega$ and for every $t \in T$, either t is a maximal element of the tree, or
$succ_T(t) := \{\alpha < \kappa \mid t^\frown \alpha \in T\} \in U(\beta, i)$ for some $\beta \leq \kappa$ and $i < o^{\vec{U}}(\beta)$.

Proposition (The strong Prikry Property)

Suppose that $p \in \mathbb{M}[\vec{U}]$ and $D \subseteq \mathbb{M}[\vec{U}]$ is a dense open subset. Then there is $p^* \geq^* p$ and a $\vec{U}$-fat tree T, such that for every maximal branch $\vec{b} \in T$, $p^{*\frown}\vec{b} \in D$.

The strong Prikry property for $\mathbb{M}[\vec{U}]$

Definition

A tree $T \subseteq [\kappa]^{<\omega}$ is called a $\vec{U}$-fat tree, if $ht(T) < \omega$ and for every $t \in T$, either t is a maximal element of the tree, or
$succ_T(t) := \{\alpha < \kappa \mid t^\frown\alpha \in T\} \in U(\beta, i)$ for some $\beta \leq \kappa$ and $i < o^{\vec{U}}(\beta)$.

Proposition (The strong Prikry Property)

Suppose that $p \in \mathbb{M}[\vec{U}]$ and $D \subseteq \mathbb{M}[\vec{U}]$ is a dense open subset. Then there is $p^* \geq^* p$ and a $\vec{U}$-fat tree T, such that for every maximal branch $\vec{b} \in T$, $p^{*\frown}\vec{b} \in D$.

Proof of lemma.

The strong Prikry property for $\mathbb{M}[\vec{U}]$

Definition

A tree $T \subseteq [\kappa]^{<\omega}$ is called a $\vec{U}$-fat tree, if $ht(T) < \omega$ and for every $t \in T$, either t is a maximal element of the tree, or
$succ_T(t) := \{\alpha < \kappa \mid t^\frown\alpha \in T\} \in U(\beta, i)$ for some $\beta \leq \kappa$ and $i < o^{\vec{U}}(\beta)$.

Proposition (The strong Prikry Property)

Suppose that $p \in \mathbb{M}[\vec{U}]$ and $D \subseteq \mathbb{M}[\vec{U}]$ is a dense open subset. Then there is $p^* \geq^* p$ and a $\vec{U}$-fat tree T, such that for every maximal branch $\vec{b} \in T$, $p^{*\frown}\vec{b} \in D$.

Proof of lemma.

Assume for example that $A = \{a_n \mid n < \omega\}$ and let $\langle \underset{\sim}{a}_n \mid n < \omega\rangle$ be a sequence of $\mathbb{M}[\vec{U}]$-names for A.

The strong Prikry property for $\mathbb{M}[\vec{U}]$

Definition

A tree $T \subseteq [\kappa]^{<\omega}$ is called a $\vec{U}$-fat tree, if $ht(T) < \omega$ and for every $t \in T$, either t is a maximal element of the tree, or $succ_T(t) := \{\alpha < \kappa \mid t^\frown\alpha \in T\} \in U(\beta, i)$ for some $\beta \leq \kappa$ and $i < o^{\vec{U}}(\beta)$.

Proposition (The strong Prikry Property)

Suppose that $p \in \mathbb{M}[\vec{U}]$ and $D \subseteq \mathbb{M}[\vec{U}]$ is a dense open subset. Then there is $p^* \geq^* p$ and a $\vec{U}$-fat tree T, such that for every maximal branch $\vec{b} \in T$, $p^{*\frown}\vec{b} \in D$.

Proof of lemma.

Assume for example that $A = \{a_n \mid n < \omega\}$ and let $\langle \underset{\sim}{a}_n \mid n < \omega\rangle$ be a sequence of $\mathbb{M}[\vec{U}]$-names for A. Let $p \in \mathbb{M}[\vec{U}]$, for each n apply the Strong Prikry property to find $p \leq^* p_n$ and a $\vec{U}$-fat tree T_n such that for every $\vec{\beta} \in mb(T_n)$, there is γ $p_n^\frown\vec{\beta} \Vdash \underset{\sim}{a}_n = \gamma$. Denote by $f_n(\vec{\beta}) = \gamma$.

Sketch of the Proof

Using combinatorical properties of $\vec{U}$-fat trees, we can extend p_n to some p_n^* and collapse some of the levels of T_n to T_n^* such that the restriction of f_n to T_n^*, will be $1-1$.

Sketch of the Proof

Using combinatorical properties of $\vec{U}$-fat trees, we can extend p_n to some p_n^* and collapse some of the levels of T_n to T_n^* such that the restriction of f_n to T_n^*, will be $1-1$. By $\leq^*$-closure, find a single p_ω such that $p_n \leq^* p_\omega$ for every $n < \omega$ and by density find such $p^* \in G$.

Sketch of the Proof

Using combinatorical properties of $\vec{U}$-fat trees, we can extend p_n to some p_n^* and collapse some of the levels of T_n to T_n^* such that the restriction of f_n to T_n^*, will be $1-1$. By $\leq^*$-closure, find a single p_ω such that $p_n \leq^* p_\omega$ for every $n < \omega$ and by density find such $p^* \in G$. There will always be a branch D_n such that $p^{*\frown} D_n \in G$.

Sketch of the Proof

Using combinatorical properties of $\vec{U}$-fat trees, we can extend p_n to some p_n^* and collapse some of the levels of T_n to T_n^* such that the restriction of f_n to T_n^*, will be $1-1$. By $\leq^*$-closure, find a single p_ω such that $p_n \leq^* p_\omega$ for every $n<\omega$ and by density find such $p^* \in G$. There will always be a branch D_n such that $p^{*\frown} D_n \in G$. Since $(\underset{\sim}{a}_n)_G = a_n$ it follows that $f_n(D_n) = a_n$, define $C = \cup_{n<\omega} D_n$.

Sketch of the Proof

Using combinatorical properties of $\vec{U}$-fat trees, we can extend p_n to some p_n^* and collapse some of the levels of T_n to T_n^* such that the restriction of f_n to T_n^*, will be $1-1$. By $\leq^*$-closure, find a single p_ω such that $p_n \leq^* p_\omega$ for every $n < \omega$ and by density find such $p^* \in G$. There will always be a branch D_n such that $p^{*\frown} D_n \in G$. Since $(\underset{\sim}{a}_n)_G = a_n$ it follows that $f_n(D_n) = a_n$, define $C = \cup_{n<\omega} D_n$. In $V[C]$ we can construct $A = \{f_n(D_n) \mid n < \omega\}$ and in $V[A]$ we can calculate each D_n, $D_n = f_n^{-1}(a_n)$, then $C = \cup_{n<\omega} f_n^{-1}(a_n)$. Thus $V[A] = V[C]$. □

Sketch of the Proof

Using combinatorical properties of $\vec{U}$-fat trees, we can extend p_n to some p_n^* and collapse some of the levels of T_n to T_n^* such that the restriction of f_n to T_n^*, will be $1-1$. By $\leq^*$-closure, find a single p_ω such that $p_n \leq^* p_\omega$ for every $n < \omega$ and by density find such $p^* \in G$. There will always be a branch D_n such that $p^{*\frown} D_n \in G$. Since $(\underset{\sim}{a}_n)_G = a_n$ it follows that $f_n(D_n) = a_n$, define $C = \cup_{n<\omega} D_n$. In $V[C]$ we can construct $A = \{f_n(D_n) \mid n < \omega\}$ and in $V[A]$ we can calculate each D_n, $D_n = f_n^{-1}(a_n)$, then $C = \cup_{n<\omega} f_n^{-1}(a_n)$. Thus $V[A] = V[C]$. $\square$

Next we let $A \subseteq \kappa$. We split into two cases according to the following definition:

Sketch of the Proof

Using combinatorical properties of $\vec{U}$-fat trees, we can extend p_n to some p_n^* and collapse some of the levels of T_n to T_n^* such that the restriction of f_n to T_n^*, will be $1-1$. By $\leq^*$-closure, find a single p_ω such that $p_n \leq^* p_\omega$ for every $n<\omega$ and by density find such $p^* \in G$. There will always be a branch D_n such that $p^{*\frown} D_n \in G$. Since $(\underset{\sim}{a}_n)_G = a_n$ it follows that $f_n(D_n) = a_n$, define $C = \cup_{n<\omega} D_n$. In $V[C]$ we can construct $A = \{f_n(D_n) \mid n<\omega\}$ and in $V[A]$ we can calculate each D_n, $D_n = f_n^{-1}(a_n)$, then $C = \cup_{n<\omega} f_n^{-1}(a_n)$. Thus $V[A] = V[C]$. □

Next we let $A \subseteq \kappa$. We split into two cases according to the following definition:

Definition

Sketch of the Proof

Using combinatorical properties of $\vec{U}$-fat trees, we can extend p_n to some p_n^* and collapse some of the levels of T_n to T_n^* such that the restriction of f_n to T_n^*, will be $1-1$. By $\leq^*$-closure, find a single p_ω such that $p_n \leq^* p_\omega$ for every $n < \omega$ and by density find such $p^* \in G$. There will always be a branch D_n such that $p^{*\frown} D_n \in G$. Since $(\underset{\sim}{a}_n)_G = a_n$ it follows that $f_n(D_n) = a_n$, define $C = \cup_{n<\omega} D_n$. In $V[C]$ we can construct $A = \{f_n(D_n) \mid n < \omega\}$ and in $V[A]$ we can calculate each D_n, $D_n = f_n^{-1}(a_n)$, then $C = \cup_{n<\omega} f_n^{-1}(a_n)$. Thus $V[A] = V[C]$. □

Next we let $A \subseteq \kappa$. We split into two cases according to the following definition:

Definition

Say that $A \subseteq On$ stabilizes, if there is $\beta < \kappa$ such that $\forall \alpha < \sup(A). A \cap \alpha \in V[C_G \cap \beta]$.

Sketch of the Proof

Using combinatorical properties of $\vec{U}$-fat trees, we can extend p_n to some p_n^* and collapse some of the levels of T_n to T_n^* such that the restriction of f_n to T_n^*, will be $1-1$. By $\leq^*$-closure, find a single p_ω such that $p_n \leq^* p_\omega$ for every $n<\omega$ and by density find such $p^* \in G$. There will always be a branch D_n such that $p^{*\frown}D_n \in G$. Since $(\underset{\sim}{a}_n)_G = a_n$ it follows that $f_n(D_n) = a_n$, define $C = \cup_{n<\omega} D_n$. In $V[C]$ we can construct $A = \{f_n(D_n) \mid n<\omega\}$ and in $V[A]$ we can calculate each D_n, $D_n = f_n^{-1}(a_n)$, then $C = \cup_{n<\omega} f_n^{-1}(a_n)$. Thus $V[A] = V[C]$. □

Next we let $A \subseteq \kappa$. We split into two cases according to the following definition:

Definition

Say that $A \subseteq On$ stabilizes, if there is $\beta < \kappa$ such that $\forall \alpha < \sup(A). A \cap \alpha \in V[C_G \cap \beta]$.

Sketch of the Proof

Example

Sketch of the Proof

Example

An example for A that stabilizes if for example the canonical ω-sequence $\langle \alpha_n \mid n < \omega \rangle$ we defined in previous examples. Any bounded initial segment of it is final and therefore belongs to the ground model. So we can take $\beta = 0$ for example and $\forall \delta < \kappa . \{\alpha_n \mid n < \omega\} \cap \delta \in V = V[C_G \cap 0]$.

Sketch of the Proof

Example

An example for A that stabilizes if for example the canonical ω-sequence $\langle \alpha_n \mid n < \omega \rangle$ we defined in previous examples. Any bounded initial segment of it is final and therefore belongs to the ground model. So we can take $\beta = 0$ for example and $\forall \delta < \kappa . \{\alpha_n \mid n < \omega\} \cap \delta \in V = V[C_G \cap 0]$.

For A which stabilizes there is a nice argument which we wont prove here.

Sketch of the Proof

Example

An example for A that stabilizes if for example the canonical ω-sequence $\langle\alpha_n \mid n<\omega\rangle$ we defined in previous examples. Any bounded initial segment of it is final and therefore belongs to the ground model. So we can take $\beta=0$ for example and $\forall\delta<\kappa.\{\alpha_n \mid n<\omega\}\cap\delta\in V=V[C_G\cap 0]$.

For A which stabilizes there is a nice argument which we wont prove here.

Example

Sketch of the Proof

Example

An example for A that stabilizes if for example the canonical ω-sequence $\langle \alpha_n \mid n < \omega \rangle$ we defined in previous examples. Any bounded initial segment of it is final and therefore belongs to the ground model. So we can take $\beta = 0$ for example and $\forall \delta < \kappa . \{\alpha_n \mid n < \omega\} \cap \delta \in V = V[C_G \cap 0]$.

For A which stabilizes there is a nice argument which we wont prove here.

Example

An example of A which does not stabilize, we can take the $A = C_G$ or any $C \subseteq C_G$ such that $Lim(C)$ is unbounded in κ.

Sketch of the Proof

Example

An example for A that stabilizes if for example the canonical ω-sequence $\langle \alpha_n \mid n < \omega \rangle$ we defined in previous examples. Any bounded initial segment of it is final and therefore belongs to the ground model. So we can take $\beta = 0$ for example and $\forall \delta < \kappa. \{\alpha_n \mid n < \omega\} \cap \delta \in V = V[C_G \cap 0]$.

For A which stabilizes there is a nice argument which we wont prove here.

Example

An example of A which does not stabilize, we can take the $A = C_G$ or any $C \subseteq C_G$ such that $Lim(C)$ is unbounded in κ.

Let us show some of the main ideas in the proof for a non stabilizing A. First we note that:

Sketch of the Proof

Example

An example for A that stabilizes if for example the canonical ω-sequence $\langle \alpha_n \mid n < \omega \rangle$ we defined in previous examples. Any bounded initial segment of it is final and therefore belongs to the ground model. So we can take $\beta = 0$ for example and $\forall \delta < \kappa . \{\alpha_n \mid n < \omega\} \cap \delta \in V = V[C_G \cap 0]$.

For A which stabilizes there is a nice argument which we wont prove here.

Example

An example of A which does not stabilize, we can take the $A = C_G$ or any $C \subseteq C_G$ such that $Lim(C)$ is unbounded in κ.

Let us show some of the main ideas in the proof for a non stabilizing A. First we note that:

Lemma

Sketch of the Proof

Example

An example for A that stabilizes if for example the canonical ω-sequence $\langle \alpha_n \mid n < \omega \rangle$ we defined in previous examples. Any bounded initial segment of it is final and therefore belongs to the ground model. So we can take $\beta = 0$ for example and $\forall \delta < \kappa. \{\alpha_n \mid n < \omega\} \cap \delta \in V = V[C_G \cap 0]$.

For A which stabilizes there is a nice argument which we wont prove here.

Example

An example of A which does not stabilize, we can take the $A = C_G$ or any $C \subseteq C_G$ such that $Lim(C)$ is unbounded in κ.

Let us show some of the main ideas in the proof for a non stabilizing A. First we note that:

Lemma

If $A \subseteq \kappa$ does not stabilize, then $\theta_A := cf^{V[A]}(\kappa) < \kappa$.

Sketch of the Proof- A which does not stabilize

Sketch of the Proof- A which does not stabilize

proof

Sketch of the Proof- A which does not stabilize

proof

Consider the set $X_A = \{\nu < \kappa \mid cf^{V[A]}(\nu) < cf^{V}(\nu) = \nu\}$. Note that $Cl(X_A) \subseteq Lim(C_G)$, since only the points in $Lim(C_G)$ change cofinality in $V[G]$ (thus in $V[A]$).

Sketch of the Proof- A which does not stabilize

proof

Consider the set $X_A = \{\nu < \kappa \mid cf^{V[A]}(\nu) < cf^V(\nu) = \nu\}$. Note that $Cl(X_A) \subseteq Lim(C_G)$, since only the points in $Lim(C_G)$ change cofinality in $V[G]$ (thus in $V[A]$). Let us argue that X_A is unbounded, if $\delta < \kappa$, since A does not stabilize, there is $\beta < \kappa$ such that $A \cap \beta \notin V[C_G \cap \delta]$.

Sketch of the Proof- A which does not stabilize

proof

Consider the set $X_A = \{\nu < \kappa \mid cf^{V[A]}(\nu) < cf^V(\nu) = \nu\}$. Note that $Cl(X_A) \subseteq Lim(C_G)$, since only the points in $Lim(C_G)$ change cofinality in $V[G]$ (thus in $V[A]$). Let us argue that X_A is unbounded, if $\delta < \kappa$, since A does not stabilize, there is $\beta < \kappa$ such that $A \cap \beta \notin V[C_G \cap \delta]$. By the previous part, there is $C \subseteq C_G$ such that $V[C] = V[A \cap \beta] \subseteq V[A]$.

Sketch of the Proof- A which does not stabilize

proof

Consider the set $X_A = \{\nu < \kappa \mid cf^{V[A]}(\nu) < cf^V(\nu) = \nu\}$. Note that $Cl(X_A) \subseteq Lim(C_G)$, since only the points in $Lim(C_G)$ change cofinality in $V[G]$ (thus in $V[A]$). Let us argue that X_A is unbounded, if $\delta < \kappa$, since A does not stabilize, there is $\beta < \kappa$ such that $A \cap \beta \notin V[C_G \cap \delta]$. By the previous part, there is $C \subseteq C_G$ such that $V[C] = V[A \cap \beta] \subseteq V[A]$. It is impossible that $C \setminus \delta$ is finite, otherwise, $C \in V[C_G \cap \delta]$ and also $A \cap \beta \in V[C_G \cap \delta]$.

Sketch of the Proof- A which does not stabilize

proof

Consider the set $X_A = \{\nu < \kappa \mid cf^{V[A]}(\nu) < cf^V(\nu) = \nu\}$. Note that $Cl(X_A) \subseteq Lim(C_G)$, since only the points in $Lim(C_G)$ change cofinality in $V[G]$ (thus in $V[A]$). Let us argue that X_A is unbounded, if $\delta < \kappa$, since A does not stabilize, there is $\beta < \kappa$ such that $A \cap \beta \notin V[C_G \cap \delta]$. By the previous part, there is $C \subseteq C_G$ such that $V[C] = V[A \cap \beta] \subseteq V[A]$. It is impossible that $C \setminus \delta$ is finite, otherwise, $C \in V[C_G \cap \delta]$ and also $A \cap \beta \in V[C_G \cap \delta]$. Let $\rho = \min(Lim(C) \setminus \delta)$. By minimality, $\mathrm{otp}(C \cap (\delta, \rho)) = \omega$, hence $cf^{V[A]}(\rho) = \omega$. If $\rho = \kappa$, the we are done.

Sketch of the Proof- A which does not stabilize

proof

Consider the set $X_A = \{\nu < \kappa \mid cf^{V[A]}(\nu) < cf^V(\nu) = \nu\}$. Note that $Cl(X_A) \subseteq Lim(C_G)$, since only the points in $Lim(C_G)$ change cofinality in $V[G]$ (thus in $V[A]$). Let us argue that X_A is unbounded, if $\delta < \kappa$, since A does not stabilize, there is $\beta < \kappa$ such that $A \cap \beta \notin V[C_G \cap \delta]$. By the previous part, there is $C \subseteq C_G$ such that $V[C] = V[A \cap \beta] \subseteq V[A]$. It is impossible that $C \setminus \delta$ is finite, otherwise, $C \in V[C_G \cap \delta]$ and also $A \cap \beta \in V[C_G \cap \delta]$. Let $\rho = \min(Lim(C) \setminus \delta)$. By minimality, $\mathrm{otp}(C \cap (\delta, \rho)) = \omega$, hence $cf^{V[A]}(\rho) = \omega$. If $\rho = \kappa$, the we are done. Otherwise, $\rho < \kappa$, $\rho \in Lim(C_G)$, therefore, ρ was measurable in V which means that $\rho \in X_A \setminus \delta$.

Sketch of the Proof- A which does not stabilize

proof

Consider the set $X_A = \{\nu < \kappa \mid cf^{V[A]}(\nu) < cf^{V}(\nu) = \nu\}$. Note that $Cl(X_A) \subseteq Lim(C_G)$, since only the points in $Lim(C_G)$ change cofinality in $V[G]$ (thus in $V[A]$). Let us argue that X_A is unbounded, if $\delta < \kappa$, since A does not stabilize, there is $\beta < \kappa$ such that $A \cap \beta \notin V[C_G \cap \delta]$. By the previous part, there is $C \subseteq C_G$ such that $V[C] = V[A \cap \beta] \subseteq V[A]$. It is impossible that $C \setminus \delta$ is finite, otherwise, $C \in V[C_G \cap \delta]$ and also $A \cap \beta \in V[C_G \cap \delta]$. Let $\rho = \min(Lim(C) \setminus \delta)$. By minimality, $\mathrm{otp}(C \cap (\delta, \rho)) = \omega$, hence $cf^{V[A]}(\rho) = \omega$. If $\rho = \kappa$, the we are done. Otherwise, $\rho < \kappa$, $\rho \in Lim(C_G)$, therefore, ρ was measurable in V which means that $\rho \in X_A \setminus \delta$. If $|X_A| < \kappa$, then $\theta_A < \kappa$. Otherwise, enumerate $X_A = \{x_i \mid i < \kappa\}$, define in $V[A]$: $\beta_0 = \kappa^*$ and $\beta_{n+1} = x_{\beta_n}$.

Sketch of the Proof- A which does not stabilize

proof

Consider the set $X_A = \{\nu < \kappa \mid cf^{V[A]}(\nu) < cf^V(\nu) = \nu\}$. Note that $Cl(X_A) \subseteq Lim(C_G)$, since only the points in $Lim(C_G)$ change cofinality in $V[G]$ (thus in $V[A]$). Let us argue that X_A is unbounded, if $\delta < \kappa$, since A does not stabilize, there is $\beta < \kappa$ such that $A \cap \beta \notin V[C_G \cap \delta]$. By the previous part, there is $C \subseteq C_G$ such that $V[C] = V[A \cap \beta] \subseteq V[A]$. It is impossible that $C \setminus \delta$ is finite, otherwise, $C \in V[C_G \cap \delta]$ and also $A \cap \beta \in V[C_G \cap \delta]$. Let $\rho = \min(Lim(C) \setminus \delta)$. By minimality, $\mathrm{otp}(C \cap (\delta, \rho)) = \omega$, hence $cf^{V[A]}(\rho) = \omega$. If $\rho = \kappa$, the we are done. Otherwise, $\rho < \kappa$, $\rho \in Lim(C_G)$, therefore, ρ was measurable in V which means that $\rho \in X_A \setminus \delta$. If $|X_A| < \kappa$, then $\theta_A < \kappa$. Otherwise, enumerate $X_A = \{x_i \mid i < \kappa\}$, define in $V[A]$: $\beta_0 = \kappa^*$ and $\beta_{n+1} = x_{\beta_n}$. Since $X_A \subseteq C_G$, if follows that for every n, $\beta_n \geq \alpha_n$, and therefore $\sup_{n<\omega}\beta_n = \kappa$, hence $\theta_A = \omega$. □

Sketch of the Proof- A which does not stabilize

Sketch of the Proof- A which does not stabilize

how do we construct C?

Sketch of the Proof- A which does not stabilize

how do we construct C? Fix a cofinal sequence $\langle \alpha_i \mid i < \theta_A \rangle \in V[A]$. For $i < \theta_A$, apply induction to find $C_i \subseteq C_G$ such that $V[C_i] = V[A \cap \alpha_i]$.

Sketch of the Proof- A which does not stabilize

how do we construct C? Fix a cofinal sequence $\langle \alpha_i \mid i < \theta_A \rangle \in V[A]$. For $i < \theta_A$, apply induction to find $C_i \subseteq C_G$ such that $V[C_i] = V[A \cap \alpha_i]$. The sequence $\langle C_i \mid i < \theta_A \rangle$ entails the information needed to construct A. It is tempting to define $C = \cup_{i<\theta_A} C_i$. However there are two problems here:

Sketch of the Proof- A which does not stabilize

how do we construct C? Fix a cofinal sequence $\langle \alpha_i \mid i < \theta_A \rangle \in V[A]$. For $i < \theta_A$, apply induction to find $C_i \subseteq C_G$ such that $V[C_i] = V[A \cap \alpha_i]$. The sequence $\langle C_i \mid i < \theta_A \rangle$ entails the information needed to construct A. It is tempting to define $C = \cup_{i<\theta_A} C_i$. However there are two problems here:

1. Although each $C_i \in V[A \cap \alpha_i] \subseteq V[A]$, the entire sequence $\langle C_i \mid i < \theta_A \rangle$ is not necessarily in $V[A]$.

Sketch of the Proof- A which does not stabilize

how do we construct C? Fix a cofinal sequence $\langle \alpha_i \mid i < \theta_A \rangle \in V[A]$. For $i < \theta_A$, apply induction to find $C_i \subseteq C_G$ such that $V[C_i] = V[A \cap \alpha_i]$. The sequence $\langle C_i \mid i < \theta_A \rangle$ entails the information needed to construct A. It is tempting to define $C = \cup_{i<\theta_A} C_i$. However there are two problems here:

1. Although each $C_i \in V[A \cap \alpha_i] \subseteq V[A]$, the entire sequence $\langle C_i \mid i < \theta_A \rangle$ is not necessarily in $V[A]$.
2. When taking the union we might loss information i.e it is possible that some $C_i \notin V[C]$.

Sketch of the Proof- A which does not stabilize

how do we construct C? Fix a cofinal sequence $\langle \alpha_i \mid i < \theta_A \rangle \in V[A]$. For $i < \theta_A$, apply induction to find $C_i \subseteq C_G$ such that $V[C_i] = V[A \cap \alpha_i]$. The sequence $\langle C_i \mid i < \theta_A \rangle$ entails the information needed to construct A. It is tempting to define $C = \cup_{i<\theta_A} C_i$. However there are two problems here:

1. Although each $C_i \in V[A \cap \alpha_i] \subseteq V[A]$, the entire sequence $\langle C_i \mid i < \theta_A \rangle$ is not necessarily in $V[A]$.
2. When taking the union we might loss information i.e it is possible that some $C_i \notin V[C]$.

Definition (Mathias set)

Sketch of the Proof- A which does not stabilize

how do we construct C? Fix a cofinal sequence $\langle \alpha_i \mid i < \theta_A \rangle \in V[A]$. For $i < \theta_A$, apply induction to find $C_i \subseteq C_G$ such that $V[C_i] = V[A \cap \alpha_i]$. The sequence $\langle C_i \mid i < \theta_A \rangle$ entails the information needed to construct A. It is tempting to define $C = \cup_{i<\theta_A} C_i$. However there are two problems here:

1. Although each $C_i \in V[A \cap \alpha_i] \subseteq V[A]$, the entire sequence $\langle C_i \mid i < \theta_A \rangle$ is not necessarily in $V[A]$.
2. When taking the union we might loss information i.e it is possible that some $C_i \notin V[C]$.

Definition (Mathias set)

A set $D \in V[A]$ is called a *Mathias set*, if

Sketch of the Proof- A which does not stabilize

how do we construct C? Fix a cofinal sequence $\langle \alpha_i \mid i < \theta_A \rangle \in V[A]$. For $i < \theta_A$, apply induction to find $C_i \subseteq C_G$ such that $V[C_i] = V[A \cap \alpha_i]$. The sequence $\langle C_i \mid i < \theta_A \rangle$ entails the information needed to construct A. It is tempting to define $C = \cup_{i<\theta_A} C_i$. However there are two problems here:

1. Although each $C_i \in V[A \cap \alpha_i] \subseteq V[A]$, the entire sequence $\langle C_i \mid i < \theta_A \rangle$ is not necessarily in $V[A]$.
2. When taking the union we might loss information i.e it is possible that some $C_i \notin V[C]$.

Definition (Mathias set)

A set $D \in V[A]$ is called a *Mathias set*, if

1. $Lim(D) \subseteq Cl(X_A)$.

Sketch of the Proof- A which does not stabilize

how do we construct C? Fix a cofinal sequence $\langle \alpha_i \mid i < \theta_A \rangle \in V[A]$. For $i < \theta_A$, apply induction to find $C_i \subseteq C_G$ such that $V[C_i] = V[A \cap \alpha_i]$. The sequence $\langle C_i \mid i < \theta_A \rangle$ entails the information needed to construct A. It is tempting to define $C = \cup_{i<\theta_A} C_i$. However there are two problems here:

1. Although each $C_i \in V[A \cap \alpha_i] \subseteq V[A]$, the entire sequence $\langle C_i \mid i < \theta_A \rangle$ is not necessarily in $V[A]$.
2. When taking the union we might loss information i.e it is possible that some $C_i \notin V[C]$.

Definition (Mathias set)

A set $D \in V[A]$ is called a *Mathias set*, if

1. $Lim(D) \subseteq Cl(X_A)$.
2. For all $\delta \in Lim(D)$ and $Y \in \cap_{i<o^{\vec{U}}(\delta)} U(\delta, i)$, there is $\xi < \delta$ such that $D \cap (\xi, \delta) \subseteq Y$.

Sketch of the Proof- A which does not stabilize

how do we construct C? Fix a cofinal sequence $\langle \alpha_i \mid i < \theta_A \rangle \in V[A]$. For $i < \theta_A$, apply induction to find $C_i \subseteq C_G$ such that $V[C_i] = V[A \cap \alpha_i]$. The sequence $\langle C_i \mid i < \theta_A \rangle$ entails the information needed to construct A. It is tempting to define $C = \cup_{i<\theta_A} C_i$. However there are two problems here:

1. Although each $C_i \in V[A \cap \alpha_i] \subseteq V[A]$, the entire sequence $\langle C_i \mid i < \theta_A \rangle$ is not necessarily in $V[A]$.
2. When taking the union we might loss information i.e it is possible that some $C_i \notin V[C]$.

Definition (Mathias set)

A set $D \in V[A]$ is called a *Mathias set*, if

1. $Lim(D) \subseteq Cl(X_A)$.
2. For all $\delta \in Lim(D)$ and $Y \in \cap_{i<o^{\vec{U}}(\delta)} U(\delta, i)$, there is $\xi < \delta$ such that $D \cap (\xi, \delta) \subseteq Y$.

The main property of a Mathias set is:

Sketch of the Proof- A which does not stabilize

Sketch of the Proof- A which does not stabilize

Proposition

Sketch of the Proof- A which does not stabilize

Proposition

Let $D \in V[A]$. A is a Mathias set iff $D \subseteq^* C_G$ i.e. $D \setminus C_G$ is finite.

Sketch of the Proof- A which does not stabilize

Proposition

Let $D \in V[A]$. A is a Mathias set iff $D \subseteq^* C_G$ i.e. $D \setminus C_G$ is finite.

The direction $D \subseteq^* C_G$ implies that D is a Mathias set, is a standard density argument in C_G.

Sketch of the Proof- A which does not stabilize

Proposition

Let $D \in V[A]$. A is a Mathias set iff $D \subseteq^* C_G$ i.e. $D \setminus C_G$ is finite.

The direction $D \subseteq^* C_G$ implies that D is a Mathias set, is a standard density argument in C_G. For the other direction, we have use the following property which is known also for other Prikry-type forcing [1],[2]:

Sketch of the Proof- A which does not stabilize

Proposition

Let $D \in V[A]$. A is a Mathias set iff $D \subseteq^* C_G$ i.e. $D \setminus C_G$ is finite.

The direction $D \subseteq^* C_G$ implies that D is a Mathias set, is a standard density argument in C_G. For the other direction, we have use the following property which is known also for other Prikry-type forcing [1],[2]:

Lemma (Hausdorff Property)

Sketch of the Proof- A which does not stabilize

Proposition

Let $D \in V[A]$. A is a Mathias set iff $D \subseteq^* C_G$ i.e. $D \setminus C_G$ is finite.

The direction $D \subseteq^* C_G$ implies that D is a Mathias set, is a standard density argument in C_G. For the other direction, we have use the following property which is known also for other Prikry-type forcing [1],[2]:

Lemma (Hausdorff Property)

Let $Y \in V[C_G]$ be a set of ordinals, $|Y| < \kappa$, such that $C_G \cap Y = \emptyset$. Then there is $X \in \cap_{i<o(\kappa)} U(\kappa, i)$ such that $X \cap Y = \emptyset$.

Sketch of the Proof- A which does not stabilize

Proposition

Let $D \in V[A]$. A is a Mathias set iff $D \subseteq^* C_G$ i.e. $D \setminus C_G$ is finite.

The direction $D \subseteq^* C_G$ implies that D is a Mathias set, is a standard density argument in C_G. For the other direction, we have use the following property which is known also for other Prikry-type forcing [1],[2]:

Lemma (Hausdorff Property)

Let $Y \in V[C_G]$ be a set of ordinals, $|Y| < \kappa$, such that $C_G \cap Y = \emptyset$. Then there is $X \in \cap_{i<o(\kappa)} U(\kappa, i)$ such that $X \cap Y = \emptyset$.

To overcome the first problem we pick in $V[A]$ a sequence of Mathias sets $\langle D_i \mid i < \theta_A \rangle$ such that $V[A \cap \alpha_i] = V[D_i]$.

Sketch of the Proof- A which does not stabilize

Proposition

Let $D \in V[A]$. A is a Mathias set iff $D \subseteq^* C_G$ i.e. $D \setminus C_G$ is finite.

The direction $D \subseteq^* C_G$ implies that D is a Mathias set, is a standard density argument in C_G. For the other direction, we have use the following property which is known also for other Prikry-type forcing [1],[2]:

Lemma (Hausdorff Property)

Let $Y \in V[C_G]$ be a set of ordinals, $|Y| < \kappa$, such that $C_G \cap Y = \emptyset$. Then there is $X \in \cap_{i<o(\kappa)} U(\kappa, i)$ such that $X \cap Y = \emptyset$.

To overcome the first problem we pick in $V[A]$ a sequence of Mathias sets $\langle D_i \mid i < \theta_A \rangle$ such that $V[A \cap \alpha_i] = V[D_i]$. Note that a new problem was created, $D := \cup_{i<\theta_A} D_i$ can accumulate infinite noise and seize to be Mathias.

Sketch of the Proof- A which does not stabilize

Proposition

Let $D \in V[A]$. A is a Mathias set iff $D \subseteq^* C_G$ i.e. $D \setminus C_G$ is finite.

The direction $D \subseteq^* C_G$ implies that D is a Mathias set, is a standard density argument in C_G. For the other direction, we have use the following property which is known also for other Prikry-type forcing [1],[2]:

Lemma (Hausdorff Property)

Let $Y \in V[C_G]$ be a set of ordinals, $|Y| < \kappa$, such that $C_G \cap Y = \emptyset$. Then there is $X \in \cap_{i<o(\kappa)} U(\kappa, i)$ such that $X \cap Y = \emptyset$.

To overcome the first problem we pick in $V[A]$ a sequence of Mathias sets $\langle D_i \mid i < \theta_A \rangle$ such that $V[A \cap \alpha_i] = V[D_i]$. Note that a new problem was created, $D := \cup_{i<\theta_A} D_i$ can accumulate infinite noise and seize to be Mathias. Fortunately, $|D \setminus C_G| \leq \theta_A < \kappa$, and by the Hausdorff property, there is Y which filters this noise.

Sketch of the Proof- A which does not stabilize

Proposition

Let $D \in V[A]$. A is a Mathias set iff $D \subseteq^* C_G$ i.e. $D \setminus C_G$ is finite.

The direction $D \subseteq^* C_G$ implies that D is a Mathias set, is a standard density argument in C_G. For the other direction, we have use the following property which is known also for other Prikry-type forcing [1],[2]:

Lemma (Hausdorff Property)

Let $Y \in V[C_G]$ be a set of ordinals, $|Y| < \kappa$, such that $C_G \cap Y = \emptyset$. Then there is $X \in \cap_{i<o(\kappa)} U(\kappa, i)$ such that $X \cap Y = \emptyset$.

To overcome the first problem we pick in $V[A]$ a sequence of Mathias sets $\langle D_i \mid i < \theta_A \rangle$ such that $V[A \cap \alpha_i] = V[D_i]$. Note that a new problem was created, $D := \cup_{i<\theta_A} D_i$ can accumulate infinite noise and seize to be Mathias. Fortunately, $|D \setminus C_G| \leq \theta_A < \kappa$, and by the Hausdorff property, there is Y which filters this noise. The second problem is a bit more technical to fix.

Sketch of the Proof- $\sup(A) \geq \kappa^+$

Sketch of the Proof- $\sup(A) \geq \kappa^+$

For sets A such that $\sup(A) \geq \kappa^+$, we continue by induction on $sup(A)$. The idea at the induction step it to first find a single $C \subseteq C_G$, such that for every $\alpha < \sup(A)$, $A \cap \alpha \in V[C]$.

Sketch of the Proof- $\sup(A) \geq \kappa^+$

For sets A such that $\sup(A) \geq \kappa^+$, we continue by induction on $sup(A)$. The idea at the induction step it to first find a single $C \subseteq C_G$, such that for every $\alpha < \sup(A)$, $A \cap \alpha \in V[C]$. Then we claim that A must also be in $V[C]$. Otherwise, A is fresh with respect to $V[C] \subseteq V[G]$, which is a forcing extension by the quotient $\mathbb{M}[\vec{U}]/C$. the following two results says that such a situation is impossible:

Sketch of the Proof- $\sup(A) \geq \kappa^+$

For sets A such that $\sup(A) \geq \kappa^+$, we continue by induction on $sup(A)$. The idea at the induction step it to first find a single $C \subseteq C_G$, such that for every $\alpha < \sup(A)$, $A \cap \alpha \in V[C]$. Then we claim that A must also be in $V[C]$. Otherwise, A is fresh with respect to $V[C] \subseteq V[G]$, which is a forcing extension by the quotient $\mathbb{M}[\vec{U}]/C$. the following two results says that such a situation is impossible:

Theorem (No Fresh Subsets of λ)

Sketch of the Proof- $\sup(A) \geq \kappa^+$

For sets A such that $\sup(A) \geq \kappa^+$, we continue by induction on $sup(A)$. The idea at the induction step it to first find a single $C \subseteq C_G$, such that for every $\alpha < \sup(A)$, $A \cap \alpha \in V[C]$. Then we claim that A must also be in $V[C]$. Otherwise, A is fresh with respect to $V[C] \subseteq V[G]$, which is a forcing extension by the quotient $\mathbb{M}[\vec{U}]/C$. the following two results says that such a situation is impossible:

Theorem (No Fresh Subsets of λ)

Let $W \models ZFC$ and $\mathbb{P} \in W$ a forcing notion. Let $T \subseteq \mathbb{P}$ be any W-generic filter and λ a regular cardinal in $W[T]$. Assume $\mathbb{P}$ is λ-c.c. in $W[T]$. Then in $W[T]$ there are no fresh subsets of λ with respect to W.

Sketch of the Proof- $\sup(A) \geq \kappa^+$

For sets A such that $\sup(A) \geq \kappa^+$, we continue by induction on $sup(A)$. The idea at the induction step it to first find a single $C \subseteq C_G$, such that for every $\alpha < \sup(A)$, $A \cap \alpha \in V[C]$. Then we claim that A must also be in $V[C]$. Otherwise, A is fresh with respect to $V[C] \subseteq V[G]$, which is a forcing extension by the quotient $\mathbb{M}[\vec{U}]/C$. the following two results says that such a situation is impossible:

Theorem (No Fresh Subsets of λ)

Let $W \models ZFC$ and $\mathbb{P} \in W$ a forcing notion. Let $T \subseteq \mathbb{P}$ be any W-generic filter and λ a regular cardinal in $W[T]$. Assume $\mathbb{P}$ is λ-c.c. in $W[T]$. Then in $W[T]$ there are no fresh subsets of λ with respect to W.

Theorem

Sketch of the Proof- $\sup(A) \geq \kappa^+$

For sets A such that $\sup(A) \geq \kappa^+$, we continue by induction on $sup(A)$. The idea at the induction step it to first find a single $C \subseteq C_G$, such that for every $\alpha < \sup(A)$, $A \cap \alpha \in V[C]$. Then we claim that A must also be in $V[C]$. Otherwise, A is fresh with respect to $V[C] \subseteq V[G]$, which is a forcing extension by the quotient $\mathbb{M}[\vec{U}]/C$. the following two results says that such a situation is impossible:

Theorem (No Fresh Subsets of λ)

Let $W \models ZFC$ and $\mathbb{P} \in W$ a forcing notion. Let $T \subseteq \mathbb{P}$ be any W-generic filter and λ a regular cardinal in $W[T]$. Assume $\mathbb{P}$ is λ-c.c. in $W[T]$. Then in $W[T]$ there are no fresh subsets of λ with respect to W.

Theorem

$\mathbb{M}[\vec{U}]/C$ is $\kappa^+ -$ c.c. in $V[G]$.

Sketch of the Proof- $\sup(A) \geq \kappa^+$

Note that the standard argument for κ^+-c.c. does not work:

Sketch of the Proof- $\sup(A) \geq \kappa^+$

Note that the standard argument for κ^+-c.c. does not work: Assume otherwise, and let $\langle p_i \mid i < \kappa^+ \rangle \in V[G]$ be an antichain in $\mathbb{M}[\vec{U}]/C$. Each p_i is of the form $p_{i,\downarrow}^\frown \langle \kappa, A_i \rangle$. Since κ^+ is still regular in $V[G]$ and there are κ many possibilities for $p_{i,\downarrow}$, there are $i \neq j$ such that $p_\downarrow := p_{i,\downarrow} = p_{j,\downarrow}$.

Sketch of the Proof- $\sup(A) \geq \kappa^+$

Note that the standard argument for κ^+-c.c. does not work: Assume otherwise, and let $\langle p_i \mid i < \kappa^+ \rangle \in V[G]$ be an antichain in $\mathbb{M}[\vec{U}]/C$. Each p_i is of the form $p_{i,\downarrow}^{\frown}\langle \kappa, A_i \rangle$. Since κ^+ is still regular in $V[G]$ and there are κ many possibilities for $p_{i,\downarrow}$, there are $i \neq j$ such that $p_\downarrow := p_{i,\downarrow} = p_{j,\downarrow}$. It follows that $p_\downarrow^{\frown}\langle \kappa, A_i \cap A_j \rangle \geq p_i, p_j$. However, the condition $p_\downarrow^{\frown}\langle \kappa, A_i \cap A_j \rangle$ might not be in $\mathbb{M}[\vec{U}]/C$:

Sketch of the Proof- $\sup(A) \geq \kappa^+$

Note that the standard argument for κ^+-c.c. does not work: Assume otherwise, and let $\langle p_i \mid i < \kappa^+ \rangle \in V[G]$ be an antichain in $\mathbb{M}[\vec{U}]/C$. Each p_i is of the form $p_{i,\downarrow}^\frown\langle\kappa, A_i\rangle$. Since κ^+ is still regular in $V[G]$ and there are κ many possibilities for $p_{i,\downarrow}$, there are $i \neq j$ such that $p_\downarrow := p_{i,\downarrow} = p_{j,\downarrow}$. It follows that $p_\downarrow^\frown\langle\kappa, A_i \cap A_j\rangle \geq p_i, p_j$. However, the condition $p_\downarrow^\frown\langle\kappa, A_i \cap A_j\rangle$ might not be in $\mathbb{M}[\vec{U}]/C$:

Example

Sketch of the Proof- $\sup(A) \geq \kappa^+$

Note that the standard argument for κ^+-c.c. does not work: Assume otherwise, and let $\langle p_i \mid i < \kappa^+ \rangle \in V[G]$ be an antichain in $\mathbb{M}[\vec{U}]/C$. Each p_i is of the form $p_{i,\downarrow}^\frown \langle \kappa, A_i \rangle$. Since κ^+ is still regular in $V[G]$ and there are κ many possibilities for $p_{i,\downarrow}$, there are $i \neq j$ such that $p_\downarrow := p_{i,\downarrow} = p_{j,\downarrow}$. It follows that $p_\downarrow^\frown \langle \kappa, A_i \cap A_j \rangle \geq p_i, p_j$. However, the condition $p_\downarrow^\frown \langle \kappa, A_i \cap A_j \rangle$ might not be in $\mathbb{M}[\vec{U}]/C$:

Example

Consider the standard Prikry forcing, and assume that $C = \{C_G(2n) \mid n < \omega\}$. What is $P(U)/C$?

Sketch of the Proof- $\sup(A) \geq \kappa^+$

Note that the standard argument for κ^+-c.c. does not work: Assume otherwise, and let $\langle p_i \mid i < \kappa^+ \rangle \in V[G]$ be an antichain in $\mathbb{M}[\vec{U}]/C$. Each p_i is of the form $p_{i,\downarrow}^\frown \langle \kappa, A_i \rangle$. Since κ^+ is still regular in $V[G]$ and there are κ many possibilities for $p_{i,\downarrow}$, there are $i \neq j$ such that $p_\downarrow := p_{i,\downarrow} = p_{j,\downarrow}$. It follows that $p_\downarrow^\frown \langle \kappa, A_i \cap A_j \rangle \geq p_i, p_j$. However, the condition $p_\downarrow^\frown \langle \kappa, A_i \cap A_j \rangle$ might not be in $\mathbb{M}[\vec{U}]/C$:

Example

Consider the standard Prikry forcing, and assume that $C = \{C_G(2n) \mid n < \omega\}$. What is $P(U)/C$? it consist of all the conditions $\langle \alpha_0, ..., \alpha_n, A \rangle$ such that:

Sketch of the Proof- $\sup(A) \geq \kappa^+$

Note that the standard argument for κ^+-c.c. does not work: Assume otherwise, and let $\langle p_i \mid i < \kappa^+ \rangle \in V[G]$ be an antichain in $\mathbb{M}[\vec{U}]/C$. Each p_i is of the form $p_{i,\downarrow}^\frown \langle \kappa, A_i \rangle$. Since κ^+ is still regular in $V[G]$ and there are κ many possibilities for $p_{i,\downarrow}$, there are $i \neq j$ such that $p_\downarrow := p_{i,\downarrow} = p_{j,\downarrow}$. It follows that $p_\downarrow^\frown \langle \kappa, A_i \cap A_j \rangle \geq p_i, p_j$. However, the condition $p_\downarrow^\frown \langle \kappa, A_i \cap A_j \rangle$ might not be in $\mathbb{M}[\vec{U}]/C$:

Example

Consider the standard Prikry forcing, and assume that $C = \{C_G(2n) \mid n < \omega\}$. What is $P(U)/C$? it consist of all the conditions $\langle \alpha_0, ..., \alpha_n, A \rangle$ such that:

1. $\alpha_{2i} = C_G(2i)$.

Sketch of the Proof- $\sup(A) \geq \kappa^+$

Note that the standard argument for κ^+-c.c. does not work: Assume otherwise, and let $\langle p_i \mid i < \kappa^+ \rangle \in V[G]$ be an antichain in $\mathbb{M}[\vec{U}]/C$. Each p_i is of the form $p_{i,\downarrow}^\frown \langle \kappa, A_i \rangle$. Since κ^+ is still regular in $V[G]$ and there are κ many possibilities for $p_{i,\downarrow}$, there are $i \neq j$ such that $p_\downarrow := p_{i,\downarrow} = p_{j,\downarrow}$. It follows that $p_\downarrow^\frown \langle \kappa, A_i \cap A_j \rangle \geq p_i, p_j$. However, the condition $p_\downarrow^\frown \langle \kappa, A_i \cap A_j \rangle$ might not be in $\mathbb{M}[\vec{U}]/C$:

Example

Consider the standard Prikry forcing, and assume that $C = \{C_G(2n) \mid n < \omega\}$. What is $P(U)/C$? it consist of all the conditions $\langle \alpha_0, ..., \alpha_n, A \rangle$ such that:

1. $\alpha_{2i} = C_G(2i)$.
2. For $m > n/2$, $C_G(2m) \in A$.

Sketch of the Proof- $\sup(A) \geq \kappa^+$

Note that the standard argument for κ^+-c.c. does not work: Assume otherwise, and let $\langle p_i \mid i < \kappa^+ \rangle \in V[G]$ be an antichain in $\mathbb{M}[\vec{U}]/C$. Each p_i is of the form $p_{i,\downarrow}^\frown \langle \kappa, A_i \rangle$. Since κ^+ is still regular in $V[G]$ and there are κ many possibilities for $p_{i,\downarrow}$, there are $i \neq j$ such that $p_\downarrow := p_{i,\downarrow} = p_{j,\downarrow}$. It follows that $p_\downarrow^\frown \langle \kappa, A_i \cap A_j \rangle \geq p_i, p_j$. However, the condition $p_\downarrow^\frown \langle \kappa, A_i \cap A_j \rangle$ might not be in $\mathbb{M}[\vec{U}]/C$:

Example

Consider the standard Prikry forcing, and assume that $C = \{C_G(2n) \mid n < \omega\}$. What is $P(U)/C$? it consist of all the conditions $\langle \alpha_0, ..., \alpha_n, A \rangle$ such that:

1. $\alpha_{2i} = C_G(2i)$.
2. For $m > n/2$, $C_G(2m) \in A$.
3. For $m > n/2$, $(C_G(2m-2), C_G(2m)) \cap A \neq \emptyset$.

Sketch of the Proof- $\sup(A) \geq \kappa^+$

Note that the standard argument for κ^+-c.c. does not work: Assume otherwise, and let $\langle p_i \mid i < \kappa^+ \rangle \in V[G]$ be an antichain in $\mathbb{M}[\vec{U}]/C$. Each p_i is of the form $p_{i,\downarrow}^\frown\langle\kappa, A_i\rangle$. Since κ^+ is still regular in $V[G]$ and there are κ many possibilities for $p_{i,\downarrow}$, there are $i \neq j$ such that $p_\downarrow := p_{i,\downarrow} = p_{j,\downarrow}$. It follows that $p_\downarrow^\frown\langle\kappa, A_i \cap A_j\rangle \geq p_i, p_j$. However, the condition $p_\downarrow^\frown\langle\kappa, A_i \cap A_j\rangle$ might not be in $\mathbb{M}[\vec{U}]/C$:

Example

Consider the standard Prikry forcing, and assume that $C = \{C_G(2n) \mid n < \omega\}$. What is $P(U)/C$? it consist of all the conditions $\langle\alpha_0, ..., \alpha_n, A\rangle$ such that:

1. $\alpha_{2i} = C_G(2i)$.
2. For $m > n/2$, $C_G(2m) \in A$.
3. For $m > n/2$, $(C_G(2m-2), C_G(2m)) \cap A \neq \emptyset$.

The third condition might fail when intersecting large sets.

Sketch of the Proof- $\sup(A) \geq \kappa^+$

Sketch of the Proof- $\sup(A) \geq \kappa^+$

We pick names for $\underset{\sim}{p}_i$ and find

$$r \vdash \forall i < \kappa^+ . \underset{\sim}{p}_i \in \mathbb{M}[\vec{U}]/\underset{\sim}{C} \text{ is an antichian}$$

also we extend to r_i so that $r_i \vdash \underset{\sim}{p}_i = q_i$.

Sketch of the Proof- $\sup(A) \geq \kappa^+$

We pick names for $\underset{\sim}{p}_i$ and find

$$r \Vdash \forall i < \kappa^+ . \underset{\sim}{p}_i \in \mathbb{M}[\vec{U}]/\underset{\sim}{C} \text{ is an antichian}$$

also we extend to r_i so that $r_i \Vdash \underset{\sim}{p}_i = q_i$. Now it is possible to stabilize the lower part of both r_i's and q_i's, $r_i = \langle \alpha_0, ..., \alpha_n, A_i \rangle$, $q_i = \langle \beta_0, ...\beta_m, B_i \rangle$
The fact that $r_i \Vdash q_i \in \mathbb{M}[\vec{U}]/\underset{\sim}{C}$ is equivalent to:

Sketch of the Proof- $\sup(A) \geq \kappa^+$

We pick names for $\underset{\sim}{p}_i$ and find

$$r \vdash \forall i < \kappa^+ . \underset{\sim}{p}_i \in \mathbb{M}[\vec{U}]/\underset{\sim}{C} \text{ is an antichian}$$

also we extend to r_i so that $r_i \vdash \underset{\sim}{p}_i = q_i$. Now it is possible to stabilize the lower part of both r_i's and q_i's, $r_i = \langle \alpha_0, ..., \alpha_n, A_i \rangle$, $q_i = \langle \beta_0, ...\beta_m, B_i \rangle$
The fact that $r_i \vdash q_i \in \mathbb{M}[\vec{U}]/\underset{\sim}{C}$ is equivalent to:

1. $n \geq m$, and $\beta_{2k} = \alpha_{2k}$.
2. for $m/2 < k \leq n/2$, $\alpha_{2k} \in B_i$ and $(\alpha_{2k-2}, \alpha_{2k}) \cap B_i \neq \emptyset$.
3. $A_i \subseteq B_i$ and for every $x, y \in A_i$, $x < y$ implies that $(x, y) \cap B_i \neq \emptyset$.

Sketch of the Proof- $\sup(A) \geq \kappa^+$

We pick names for $\underset{\sim}{p}_i$ and find

$$r \Vdash \forall i < \kappa^+ . \underset{\sim}{p}_i \in \mathbb{M}[\vec{U}]/\underset{\sim}{C} \text{ is an antichian}$$

also we extend to r_i so that $r_i \Vdash \underset{\sim}{p}_i = q_i$. Now it is possible to stabilize the lower part of both r_i's and q_i's, $r_i = \langle \alpha_0, ..., \alpha_n, A_i \rangle$, $q_i = \langle \beta_0, ...\beta_m, B_i \rangle$ The fact that $r_i \Vdash q_i \in \mathbb{M}[\vec{U}]/\underset{\sim}{C}$ is equivalent to:

1. $n \geq m$, and $\beta_{2k} = \alpha_{2k}$.
2. for $m/2 < k \leq n/2$, $\alpha_{2k} \in B_i$ and $(\alpha_{2k-2}, \alpha_{2k}) \cap B_i \neq \emptyset$.
3. $A_i \subseteq B_i$ and for every $x, y \in A_i$, $x < y$ implies that $(x, y) \cap B_i \neq \emptyset$.

Note that we can also stabilize the part of the large set of q_i below α_n which guarantees that the intersection of r_i and r_j satisfy $(1), (2)$, as for (3), we can shrink even more $A_i \cap A_j$ to a set X so that (3) holds with respect to B_i, B_j.

Sketch of the Proof- $\sup(A) \geq \kappa^+$

We pick names for $\underset{\sim}{p}_i$ and find

$$r \vdash \forall i < \kappa^+. \underset{\sim}{p}_i \in \mathbb{M}[\vec{U}]/\underset{\sim}{C} \text{ is an antichian}$$

also we extend to r_i so that $r_i \vdash \underset{\sim}{p}_i = q_i$. Now it is possible to stabilize the lower part of both r_i's and q_i's, $r_i = \langle \alpha_0, ..., \alpha_n, A_i \rangle$, $q_i = \langle \beta_0, ...\beta_m, B_i \rangle$ The fact that $r_i \vdash q_i \in \mathbb{M}[\vec{U}]/\underset{\sim}{C}$ is equivalent to:

1. $n \geq m$, and $\beta_{2k} = \alpha_{2k}$.
2. for $m/2 < k \leq n/2$, $\alpha_{2k} \in B_i$ and $(\alpha_{2k-2}, \alpha_{2k}) \cap B_i \neq \emptyset$.
3. $A_i \subseteq B_i$ and for every $x, y \in A_i$, $x < y$ implies that $(x, y) \cap B_i \neq \emptyset$.

Note that we can also stabilize the part of the large set of q_i below α_n which guarantees that the intersection of r_i and r_j satisfy (1), (2), as for (3), we can shrink even more $A_i \cap A_j$ to a set X so that (3) holds with respect to B_i, B_j. The condition $r^* = \langle \alpha_0, ..., \alpha_n, X \rangle$ forces that $\underset{\sim}{p}_i, \underset{\sim}{p}_j$ are compatible by $\langle \beta_0, ...\beta_m, B_i \cap B_j \rangle \in \mathbb{M}[\vec{U}]/C$.

Intermediate models in the case $\delta_0 < o^{\vec{U}}(\kappa) < \kappa$

Intermediate models in the case $\delta_0 < o^{\vec{U}}(\kappa) < \kappa$

If $o^{\vec{U}}(\kappa) < \kappa$, then we have the following characterization of the intermediate models:

Intermediate models in the case $\delta_0 < o^{\vec{U}}(\kappa) < \kappa$

If $o^{\vec{U}}(\kappa) < \kappa$, then we have the following characterization of the intermediate models:

Theorem ([2])

Intermediate models in the case $\delta_0 < o^{\vec{U}}(\kappa) < \kappa$

If $o^{\vec{U}}(\kappa) < \kappa$, then we have the following characterization of the intermediate models:

Theorem ([2])

Assume that for every $\alpha <\leq \kappa$, $o^{\vec{U}}(\alpha) < \alpha$. Then for every V-generic filter $G \subseteq \mathbb{M}[\vec{U}]$ and every transitive ZFC intermediate model $V \subseteq M \subseteq V[G]$, there is a closed subset $C_{fin} \subseteq C_G$ such that:

Intermediate models in the case $\delta_0 < o^{\vec{U}}(\kappa) < \kappa$

If $o^{\vec{U}}(\kappa) < \kappa$, then we have the following characterization of the intermediate models:

Theorem ([2])

Assume that for every $\alpha <\leq \kappa$, $o^{\vec{U}}(\alpha) < \alpha$. Then for every V-generic filter $G \subseteq \mathbb{M}[\vec{U}]$ and every transitive ZFC intermediate model $V \subseteq M \subseteq V[G]$, there is a closed subset $C_{fin} \subseteq C_G$ such that:

1. $M = V[C_{fin}]$.

Intermediate models in the case $\delta_0 < o^{\vec{U}}(\kappa) < \kappa$

If $o^{\vec{U}}(\kappa) < \kappa$, then we have the following characterization of the intermediate models:

Theorem ([2])

Assume that for every $\alpha <\leq \kappa$, $o^{\vec{U}}(\alpha) < \alpha$. Then for every V-generic filter $G \subseteq \mathbb{M}[\vec{U}]$ and every transitive ZFC intermediate model $V \subseteq M \subseteq V[G]$, there is a closed subset $C_{fin} \subseteq C_G$ such that:

1. $M = V[C_{fin}]$.
2. *There is a finite iteration $\mathbb{Q} := \mathbb{M}_{f_1}[\vec{U}] * \mathbb{M}_{\mathcal{L}_2}[\vec{U}] ... * \mathbb{M}_{\mathcal{L}_n}[\vec{U}]$, and $H^* \subseteq \mathbb{Q}$, V-generic H^* filter such that $V[H^*] = V[C_{fin}] = M$.*

Intermediate models in the case $\delta_0 < o^{\vec{U}}(\kappa) < \kappa$

If $o^{\vec{U}}(\kappa) < \kappa$, then we have the following characterization of the intermediate models:

Theorem ([2])

Assume that for every $\alpha <\leq \kappa$, $o^{\vec{U}}(\alpha) < \alpha$. Then for every V-generic filter $G \subseteq \mathbb{M}[\vec{U}]$ and every transitive ZFC intermediate model $V \subseteq M \subseteq V[G]$, there is a closed subset $C_{fin} \subseteq C_G$ such that:

1. $M = V[C_{fin}]$.
2. *There is a finite iteration $\mathbb{Q} := \mathbb{M}_{f_1}[\vec{U}] * \mathbb{M}_{\underset{\sim}{f}_2}[\vec{U}] ... * \mathbb{M}_{\underset{\sim}{f}_n}[\vec{U}]$, and $H^* \subseteq \mathbb{Q}$, V-generic H^* filter such that $V[H^*] = V[C_{fin}] = M$.*

In the first example, the model is in fact a two steps iteration, the first parts adds a Prikry sequence to $C_G(\omega)$, so $f_1 : \omega \to \kappa$, $f_1(n) = 0$. The second part is of the form $\mathbb{M}_{\underset{\sim}{f}}[\vec{U}]$, where $\underset{\sim}{f}$ is a name for the function $f : \omega \to \delta_0$ defined by $f(n) = C_G(n)$.

References I

- Tom Benhamou, *Prikry Forcing and Tree Prikry Forcing of Various Filters*, Arch. Math. Logic **58** (2019), 787–817.
- Tom Benhamou and Moti Gitik, *Intermediate Models of Magidor-Radin Foring-Part I*, arXiv:2009.12775 (2020), submitted to Israely Journal of Mathematics.
- ______, *Sets in Prikry and Magidor Generic Extensions*, Annals of Pure and Applied Logic **172** (2021), no. 4, 102926.
- Moti Gitik, *Prikry-Type Forcings*, pp. 1351–1447, Springer Netherlands, Dordrecht, 2010.
- Moti Gitik, *On κ-Compact Cardinals*, preprint (to appear).
- Moti Gitik, Vladimir Kanovei, and Peter Koepke, *Intermediate Models of Prikry Generic Extensions*, Pre Print (2010).

References II

Akihiro Kanamori, *The Higher Infinite*, Springer, 1994.

Peter Koepke, Karen Rasch, and Philipp Schlicht, *Minimal Prikry-Type Forcing for Singularizing a Measurable Cardinal*, J. Symb. Logic **78** (2013), 85–100.

Menachem Magidor, *Changing the cofinality of cardinals*, Fundamenta Mathematicae **99** (1978), 61–71.

A. R. D. Mathias, *On Sequences Generic in the Sense of Prikry*, Journal of Australian Mathematical Society **15** (1973), 409–414.

Finish line

Thank you for your attention!