Changing tail types

Monroe Eskew
(from joint work with Yair Hayut)

Kurt Gödel Research Center
University of Vienna

December 14, 2020

Main Question

Can we have a forcing extension W of V such that for all countable limit ordinals $\alpha \in W$, $\aleph_{\alpha+1}^W = (\kappa^{+\omega+1})^V$ for some κ?

The motivation is to get a model of the "Singular Global Chang's Conjecture," which says that successors of singulars of the same cofinality are indistinguishable by an augmentation of FOL with a certain cardinality quantifier. The construction should work in 2 phases:

1. Construct a model in which for all $\alpha < \beta$, $(\beta^{+\omega+1}, \beta^{+\omega}) \twoheadrightarrow (\alpha^{+\omega+1}, \alpha^{+\omega})$, and there are supercomapcts. (No problem!)
2. Do a Prikry forcing with collapses as in the Main Question. Show that the instances of Chang's Conjecure are preserved. (Can do only up to $\aleph_{\omega^\omega}$. ☹)

But let's ignore Chang's Conjecture and just focus on getting the cardinal collapsing pattern.

Suppose λ is an ordinal. The **tail** of λ, $\tau(\lambda) = \lim_{\alpha \to \lambda} \text{ot}(\lambda \setminus \alpha)$.

For a cardinal λ, the **cardinal tail** of λ, $\mathcal{T}(\lambda)$ is $\tau(\alpha)$, where $\lambda = \aleph_\alpha$.

Suppose $\kappa = \lambda^+$, where $\text{cf}(\lambda) = \omega$. Can we force to preserve κ and make $\kappa = \xi^+$, where $\mathcal{T}(\xi)^{V[G]} \neq \mathcal{T}(\lambda)^V$?

Making $\mathcal{T}$ *smaller* is easy. Assume GCH and let $\lambda = \sup_{i<\omega} \lambda_i$, where each λ_i is regular. Let $\delta < \mathcal{T}(\lambda)$ be the desired $\mathcal{T}(\xi)$, and let $\delta = \sum_i \delta_i$. Force with

$$\prod_i \text{Col}(\lambda_i^{+\delta_i+1}, \lambda_{i+1}).$$

Note we actually have $\xi = \lambda$, but $\mathcal{T}(\lambda)^{V[G]} < \mathcal{T}(\lambda)^V$.

Making the tail *larger* is only possible if $\xi < \lambda$, which violates weak covering and thus requires fairly large cardinals. It is not known whether we can get $V[G] \models \text{cf}(\xi) \neq \text{cf}(\lambda)$. Cummings proved that this cannot be done by κ-c.c. forcing.

But increasing the cardinal tail to some other ordinal of countable cofinality can be achieved with the **Gitik-Sharon forcing**.

Suppose $\gamma < \delta$ are limit ordinals of countable cofinality, and $\vec{\gamma} = \langle \gamma_i : i < \omega \rangle$, $\vec{\delta} = \langle \delta_i : i < \omega \rangle$ are sequences such that:

- $\vec{\gamma}$ is strictly increasing with $\sup_i \gamma_i = \gamma$.
- $\vec{\delta}$ is nondecreasing with $\sum_i \delta_i = \delta$.
- **domination**: $\gamma_i \leq \delta_i$.

Assume GCH. Suppose $\kappa > \delta$ is $\kappa^{+\gamma_i}$-supercompact for each i, and $\mu < \kappa$ is regular. For $i < \omega$, let U_i be a κ-complete normal measure on $\mathcal{P}_\kappa(\kappa^{+\gamma_i})$, and let $j_i : V \to M_i \cong \mathrm{Ult}(V, U_i)$ be the ultrapower embedding. We choose guiding generics $K_i \subseteq \mathrm{Col}(\kappa^{+\delta_i+2}, j_i(\kappa))^{M_i}$.

With these choices made, we define the forcing $\mathbb{P}(\mu, \vec{\gamma}, \vec{\delta}, \vec{U}, \vec{K})$, which will have the following properties:

- The Prikry property!
- The forcing generates an ω-sequence $\langle x_i : i < \omega \rangle$, where $x_i \in \mathcal{P}_\kappa(\kappa^{+\gamma_i})$, and for each $\alpha \leq \kappa^{+\gamma}$, $\alpha = \bigcup_i (x_i \cap \alpha)$.
- Each $x_i \cap \kappa$ is a V-cardinal κ_i. We get a sequence of generics $G_i \subseteq \mathrm{Col}(\kappa_i^{+\delta_i+2}, \kappa_{i+1})$ and a generic $H \subseteq \mathrm{Col}(\mu, \kappa_0)$.
- The forcing is $\kappa^{+\gamma+1}$-c.c.

Hence: κ is forced to become $\mu^{+\delta}$, and $(\kappa^{+\gamma+1})^V$ becomes $(\kappa^+)^{V[G]}$.

The **domination** condition ($\gamma_i \leq \delta_i$) is important for the construction of guiding generics and the proof of the Prikry property.

To solve the first step of the Main Quesiton, we let $\vec{\delta} = \langle \omega, \omega, \omega \ldots \rangle$ and $\vec{\gamma} = \langle 0, 1, 2, \ldots, \rangle$.

To take this further, we should Radinize. A false start would be to try to force a sequence $\langle x_i : i < \lambda \rangle$, where $x_i \in \mathcal{P}_\kappa(\kappa^{+i})$. The reason this doesn't work that once we get to $i > \omega^2$, **domination** says the interleaved collapse must be of the form $\mathrm{Col}(\kappa_i^{+\xi}, \kappa_{i+1})$, where $\xi > \omega^2$. Thus we leave some singular untouched that has tail greater than ω.

So we instead force a sequence of x_i's, where the space x_i comes from is realted to the smallest component of the **Cantor normal form** of i.

We define inductively classes GS_n of higher Gitik-Sharon forcings, where GS_1 is like above.

The partial orders in GS_n are defined using sequences $\langle U_\alpha, K_\alpha : \alpha < \omega \cdot n \rangle$ such that:

1. There is a $\kappa > \omega$ such that each U_α is an ultrafilter with completeness κ.
2. For $n < \omega$, U_n is a normal ultrafilter on $\mathcal{P}_\kappa(\kappa^{+n})$ and for $\omega \leq \alpha < \omega \cdot n$, U_α is a normal ultrafilter on $\mathcal{P}_\kappa(H_{\kappa^{+\alpha}})$.
3. For $\alpha < \omega \cdot n$, if $j_\alpha : V \to M_\alpha$ is the ultrapower embedding from U_α, then K_α is $\mathrm{Col}(\kappa^{+\alpha+\omega+2}, j_\alpha(\kappa))^{M_\alpha}$-generic over M_α.

Essentially, the forcing yields an ω-sequence $\vec{z}$, where z_i is typical for $U_{\omega\cdot(n-1)+i}$, with some interleaved posets.

The point z_i is an elementary submodel, and its transitive collapse determines a smaller forcing from GS_{n-1}. Once this is determined, conditions in these are interleaved. Also interleaved just after is a $\kappa_i^{+\omega\cdot n+2}$-closed collapse.

Finally, we can diagonally stack these to define GS_ω. This turns a $\kappa^{+\omega^2}$-supercompact κ into $\aleph_{\omega^\omega}$, while making every $\aleph_{\alpha+1}$, for limit $\alpha < \omega^\omega$, formerly a cardinal of the form $\lambda^{+\omega\cdot n+1}$, where λ was very large.

We obey the **domination** condition throughout. We could go further to $\mathsf{GS}_{\omega+1}$ etc., but domination would make some of the collapses skip intervals of cardinals of length more than ω^2. ☹

So, can we get around the domination condition? Or is there a completely different forcing strategy?

Thanks for your attention!