A theory of particular sets

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- $\forall$ Every group has a least subgroup.
- $\forall$ Addition of ordinals is associative.
- $\exists$ An empty set exists.
- $\forall\exists\forall$ Every set generates a free group.
- $\neg\forall\exists\forall$ Not every set generates a free complete boolean algebra.
- $\forall$ The generalized continuum hypothesis.
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Boolos, Cartwright: there's no problem

Quantifying over all ordinals doesn't imply that they form a "totality".

Safe mathematics

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Goal

To isolate the "safe" part of mathematics

that doesn't invoke proper classes, even as the range of a quantifier.

TOPS = Theory of Particular Sets

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Two differences from ZFC:

1. Every quantifier ranges over a set.
2. Urelements are allowed.

The latter is not essential, but increases the theory's generality without affecting sentences.

Syntax

There are many equivalent formulations, so this is largely a matter of taste.

Terms

$$r, s, t, A, B \quad ::= \quad x \mid \emptyset \mid A \cup B \mid \bigcup_{a \in A} B(a) \mid \{r \mid\mid \phi\} \mid \mathsf{UE}(A)$$

$$\mid \mathsf{IR}(s/x.r) \mid \mathsf{WR}_{\in A}^{x,y.\phi/z,Y.r}(s) \mid \mathcal{P}A$$

Propositional formulas

$$\phi, \psi \quad ::= \quad \mathsf{False} \mid \mathsf{True} \mid \phi \vee \psi \mid \phi \wedge \psi \mid \phi \Rightarrow \psi \mid \neg\phi$$

$$\mid \exists x \in A.\, \phi \mid \forall x \in A.\, \phi \mid s = t$$

$$\mid \mathsf{IsSet}(A) \mid s \in A \mid \mathsf{W}_{\in A}^{x,y.\phi}(s)$$

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On the other hand:

- TOPS is weaker than ZFC.
- Mahlo TOPS is weaker than Mahlo ZFC.

Thesis

- Mahlo TOPS achieves our goal.
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- Mahlo TOPS achieves our goal.
- It captures precisely the "safe" part of mathematics that doesn't invoke proper classes.
- Caveat: it doesn't capture reflection on itself.

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The principal difference

- TOPS distinguishes between sentences and open formulas.
- This is important for our goal.

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Mahlo TOPS gives all the hyper*inaccessible cardinals without invoking an absolute totality of sets.