Actions of Polish groups

Jan Grebík

Mathematics Institute, University of Warwick

11.2.2020

Setting

Setting

- A topological space X is a *Polish space* if it is separable and completely metrizable.

Setting

- A topological space X is a *Polish space* if it is separable and completely metrizable.
- A topological group G is a *Polish group* if the underlying topology is Polish.

Setting

- A topological space X is a *Polish space* if it is separable and completely metrizable.
- A topological group G is a *Polish group* if the underlying topology is Polish.
- We say that X is a *Polish G-space* if there is a continuous action $G \curvearrowright X$.

Setting

- A topological space X is a *Polish space* if it is separable and completely metrizable.
- A topological group G is a *Polish group* if the underlying topology is Polish.
- We say that X is a *Polish G-space* if there is a continuous action $G \curvearrowright X$.

Define the corresponding orbit equivalence relation E_G^X on X as

$$(x, y) \in E_G^X \iff \exists g \in G \ (g \cdot x = y).$$

Setting

- A topological space X is a *Polish space* if it is separable and completely metrizable.
- A topological group G is a *Polish group* if the underlying topology is Polish.
- We say that X is a *Polish G-space* if there is a continuous action $G \curvearrowright X$.

Define the corresponding orbit equivalence relation E_G^X on X as

$$(x, y) \in E_G^X \iff \exists g \in G \ (g \cdot x = y) .$$

Assumption

E_G^X is Borel as a subset of $X \times X$.

Task

We want to understand E_G^X in the realm of Borel reducibility, i.e., compare it with standard concepts or examples.

Task

We want to understand E_G^X in the realm of Borel reducibility, i.e., compare it with standard concepts or examples.

Definition

Let E and F be equivalence relations on Polish spaces Y and Z. We say that E is Borel reducible to F if there is a Borel map $\varphi : Y \to Z$ such that

$$(u, v) \in E \iff (\varphi(u), \varphi(v)) \in F$$

for every $u, v \in Y$.

Task

We want to understand E_G^X in the realm of Borel reducibility, i.e., compare it with standard concepts or examples.

Definition

Let E and F be equivalence relations on Polish spaces Y and Z. We say that E is Borel reducible to F if there is a Borel map $\varphi : Y \to Z$ such that

$$(u, v) \in E \;\Leftrightarrow\; (\varphi(u), \varphi(v)) \in F$$

for every $u, v \in Y$.

There are many 'general' big results and concepts, e.g., Silver, Glimm–Effros, Harrington–Kechris–Louveau, turbulence, classification by countable strucutres, CBER, etc.

Strategy

Theorem ($\mathbb{G}_0$-dichotomy, Kechris–Solecki–Todorcevic)

Let X be a Polish space and $H \subseteq X \times X$ be an analytic graph. Then either $\chi_B(H) = \aleph_0$ or there is a continuous map $\varphi : 2^{\mathbb{N}} \to X$ that is a homomorphism from $\mathbb{G}_0$ to H.

Strategy

Theorem ($\mathbb{G}_0$-dichotomy, Kechris–Solecki–Todorcevic)

Let X be a Polish space and $H \subseteq X \times X$ be an analytic graph. Then either $\chi_B(H) = \aleph_0$ or there is a continuous map $\varphi : 2^{\mathbb{N}} \to X$ that is a homomorphism from $\mathbb{G}_0$ to H.

Use variants of the 'elementary' proof of $\mathbb{G}_0$-dichotomy (B. Miller) to prove 'finer versions' of the known dichotomies.

Strategy

Theorem ($\mathbb{G}_0$-dichotomy, Kechris–Solecki–Todorcevic)

Let X be a Polish space and $H \subseteq X \times X$ be an analytic graph. Then either $\chi_B(H) = \aleph_0$ or there is a continuous map $\varphi : 2^{\mathbb{N}} \to X$ that is a homomorphism from $\mathbb{G}_0$ to H.

Use variants of the 'elementary' proof of $\mathbb{G}_0$-dichotomy (B. Miller) to prove 'finer versions' of the known dichotomies.

Find new dichotomies and attack old problems.

Strategy

Theorem ($\mathbb{G}_0$-dichotomy, Kechris–Solecki–Todorcevic)

Let X be a Polish space and $H \subseteq X \times X$ be an analytic graph. Then either $\chi_B(H) = \aleph_0$ or there is a continuous map $\varphi : 2^{\mathbb{N}} \to X$ that is a homomorphism from $\mathbb{G}_0$ to H.

Use variants of the 'elementary' proof of $\mathbb{G}_0$-dichotomy (B. Miller) to prove 'finer versions' of the known dichotomies.

Find new dichotomies and attack old problems.

Often we need to assume that G admits two-sided invariant metric ($\mathbb{G}_0$-dichotomy $\Leftrightarrow$ $\mathbb{H}_0$-dichotomy).

Examples/Conjectures/Questions

Examples/Conjectures/Questions

1. essentially countable $= \sigma$-lacunary,

Examples/Conjectures/Questions

1. essentially countable $=\sigma$-lacunary,
2. $\mathbb{E}_3$-dichotomy,

Examples/Conjectures/Questions

1. essentially countable = σ-lacunary,
2. $\mathbb{E}_3$-dichotomy,
3. basis for turbulence,

Examples/Conjectures/Questions

1. essentially countable $=$ σ-lacunary,
2. $\mathbb{E}_3$-dichotomy,
3. basis for turbulence,
4. characterisation of classification by countable structures,

Examples/Conjectures/Questions

1. essentially countable $=$ σ-lacunary,
2. $\mathbb{E}_3$-dichotomy,
3. basis for turbulence,
4. characterisation of classification by countable structures,
5. if E_G^X is essentially countable, where does it sit in the class CBER (dependeing on G).

Thank you!