How category theorists think about sets

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Related paper: 'Rethinking set theory'

1. What do category theorists do?

What do category theorists do?

- Internal work

  Like any other subject, category theory has a constantly-developing body of definitions, theorems and proofs that are mainly of interest to category theorists.

  I won't talk about this.

- External work

  We also put a lot of effort into trying to illuminate and connect together other subjects:

  - in mathematics: algebra, logic, topology, geometry, analysis, . . .
  - outside mathematics: computer science, physics, biology, . . .

How do category theorists approach other subjects?

I'll discuss two aspects of how we approach other subjects:

- Mathematical anthropology (Section 2 of this talk)
- Organizing into categories (Section 3 of this talk)

The rest of this talk is an explanation what I mean by these two things, and how they apply to sets.

2. Mathematical anthropology

What is 'mathematical anthropology'?

The mathematical anthropologist looks at a branch of mathematics and asks:

- What do the practitioners of the subject find important?
  What do they talk about a lot?
- Why do they focus on *that particular* object, not something slightly different?
- Is there any tension between what they *say* they do and what they *actually* do? Between the grand narrative and the unglamorous detail?

What does a (categorical) mathematical anthropologist hope to achieve?

- Make precise what's so special about the central objects of a subject.
- Streamline: e.g. show that a tricky construction that appears to be subject-specific is actually an instance of a general categorical construction.
- Spot new analogies and formalize existing ones.
  Categorical language is very good for this.
- Resolve tensions between concepts/narrative and execution/detail.

(I'm not claiming category theory can always achieve these things!)

When category theorists look at set theory (done by set theorists). . .

. . . we see a deep body of work, with some connections to category theory: e.g. topos-theoretic (sheaf-theoretic) approaches to forcing.

But I won't talk about this.

The status of sets in 'ordinary' mathematics

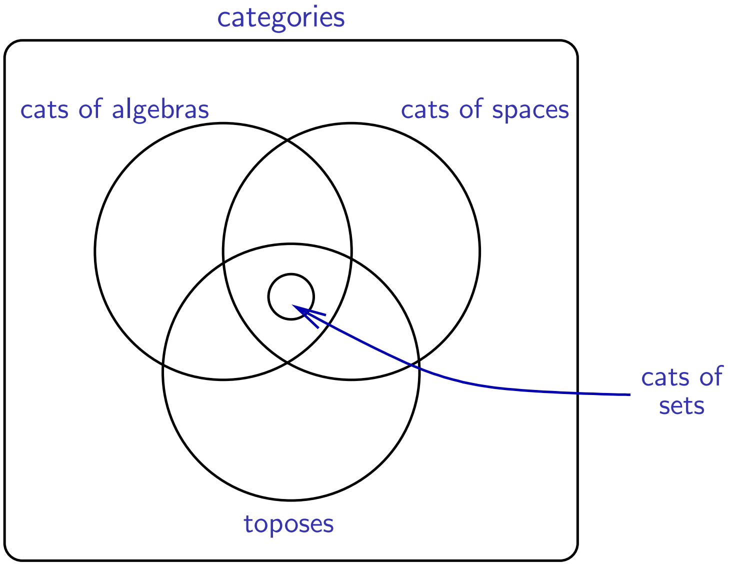
- Mathemiticans use sets all the time.
- Mathematicians rarely make mistakes in what they do with sets.
- Almost no one can state 'the' (or any) axioms for set theory.
- So apparently: there is a reliable body of (perhaps subconscious) principles that mathematicians use when manipulating sets.

More anthropology: sets in theory and practice

Axiomatic set theory à la ZFC	Sets as used by ordinary mathematicians
There are some things called sets	Some typical sets: $\mathbb{R}$, A_5, the set of measures on $[0, 1]$, solution-set of some PDE, ...
Elements of sets are always sets	Elements of sets need not be sets ($\pi \in \mathbb{R}$ etc.)
Given sets X and Y, it always makes sense to ask 'is $X \in Y$?'	Debatable
Tree structure of sets heavily used	Almost never used
Axiom of foundation	'There is a real number none of whose elements are real numbers': both meaning and relevance debatable

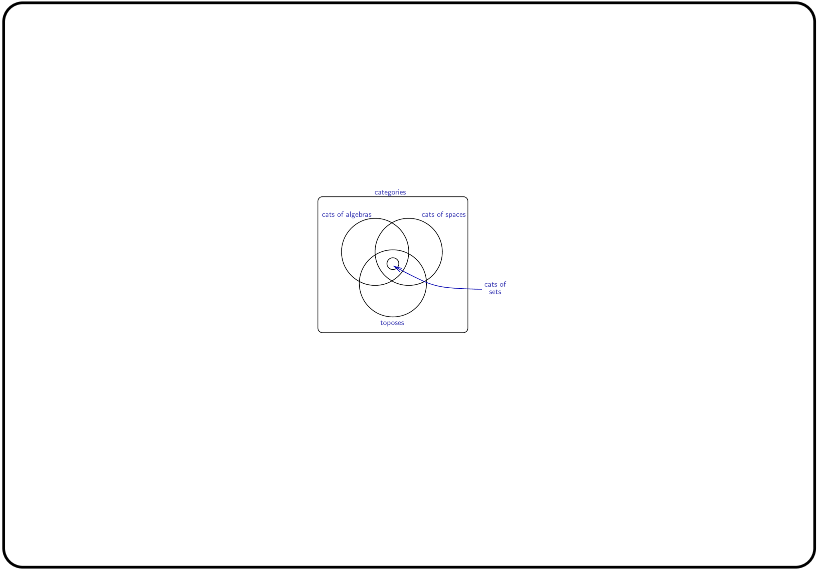
3. Organizing into categories

The world of categories



The world of categories

higher categories

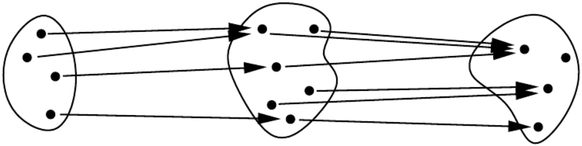


Categories of sets

A categorical axiomatization of sets takes as the primitives:

- sets
- functions from one given set to another
- composition of functions.

There will be axioms (starting with the category axioms), expressing what's special about categories *of sets*.



Deriving the concept of element

A set T is terminal if for all sets X, there is exactly one function $X \longrightarrow T$.

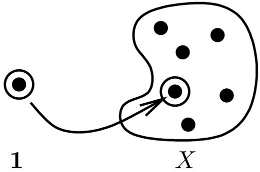
Lemma Any two terminal sets are uniquely isomorphic.

Axiom There exists a terminal set.

Fix a terminal set, 1.

Write $x \in X$ to mean $x \colon 1 \longrightarrow X$.

(Compare: ‘a sequence in X is a function $\mathbb{N} \longrightarrow X$’.)



Evaluation is composition:

$$1 \xrightarrow{x} X \xrightarrow{f} Y, \qquad 1 \xrightarrow{f(x)} Y$$

Lawvere's elementary theory of the category of sets (1964)

Informally stated, the axioms are:

1. Composition of functions is associative and has identities
2. There is a set with exactly one element
3. There is a set with no elements
4. A function is determined by its effect on elements
5. Given sets X and Y, one can form their cartesian product $X \times Y$
6. Given sets X and Y, one can form the set of functions from X to Y
7. Given $f\colon X \longrightarrow Y$ and $y \in Y$, one can form the inverse image $f^{-1}(y)$
8. The subsets of a set X correspond to the functions from X to $\{0, 1\}$
9. The natural numbers form a set
10. Every surjection has a right inverse.

Example: the product axiom

Let X and Y be sets. A product of X and Y consists of a set $X \times Y$ and functions

$$X \xleftarrow{\mathrm{pr}_1} X \times Y \xrightarrow{\mathrm{pr}_2} Y$$

with the following property: for all sets A and functions

$$X \xleftarrow{f_1} A \xrightarrow{f_2} Y,$$

there is exactly one function $(f_1, f_2)\colon A \longrightarrow X \times Y$ such that

$$\mathrm{pr}_1\big((f_1, f_2)(a)\big) = f_1(a), \qquad \mathrm{pr}_2\big((f_1, f_2)(a)\big) = f_2(a)$$

for all $a \in A$.

Axiom Every pair of sets has a product.

Beyond these axioms

Lawvere's ten axioms (ETCS) have the same consistency strength as bounded Zermelo with choice.

Myth Category theorists are particularly attached to weak set theories.

E.g. Lawvere: 'it is important to investigate first-order strengthenings of our axiom system.'

- Replacement has a natural categorical formulation in terms of coproducts.
- ETCS+R is bi-interpretable with ZFC.