Bolzano-Weierstraß properties in generalised analysis

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M. Carl, L. Galeotti, B. Löwe, The Bolzano-Weierstraß theorem in generalised analysis, *Houston Journal of Mathematics* 44:4 (2018), 1081-1109.

L. Galeotti, A. Hanafi, B. Löwe, Bolzano-Weierstraß properties in generalised analysis, *in preparation*.

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Let $\mathrm{bn}(K) := \min\{\kappa\,;$ there is a descending sequence of length κ converging to $0\}$. A field K is Archimedean if and only if $\mathrm{bn}(K) = \aleph_0$.

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Goal of generalised analysis: Fix an uncountable κ and find a field K with $\mathrm{bn}(K) = \kappa$ that is similar to the real numbers $\mathbb{R}$.

Generalised real numbers (1).

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1. Start with κ;
2. form κ-$\mathbb{Z}$;
3. take the quotient field κ-$\mathbb{Q}$.

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Unfortunately, κ-$\mathbb{Q}$ does not satisfy the intermediate value theorem.

Generalised real numbers (2).

Definition. An ordered field K is called *saturated* if for any sets L, R with $|L|, |R| < \mathrm{bn}(K)$ such that $L < R$, there is an $x \in K$ such that $L < x < R$.

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1. Start with $\mathrm{No}_{<\kappa}$, the field of surreal numbers with sequence representations of length $< \kappa$;
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Theorem (Galeotti). The field $\mathbb{R}_\kappa$ is saturated and satisfies the intermediate value theorem for κ-continuous functions.

Saturation & Bolzano-Weierstraß.

Theorem (CGHL). If K has base number κ and is saturated, then it does not satisfy Bolzano-Weierstraß.

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Definition. An ordered field K with base number κ is called *spherically complete* if for any $\lambda < \kappa$ and any family $(I_\alpha\,;\,\alpha < \lambda)$ of nested intervals, $\bigcap_{\alpha<\lambda} I_\alpha \neq \varnothing$.

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Theorem (GHL). If a spherically complete K has base number κ and satisfies Bolzano-Weierstraß, then κ is weakly compact.

Weakening Bolzano-Weierstraß (1).

A sequence s is *totally bounded* if for all $\varepsilon \in K^+$ there is $\beta_\varepsilon < \kappa$ such that for all $\beta < \kappa$ there is $\gamma < \beta_\varepsilon$ and $|s(\beta) - s(\gamma)| < \varepsilon$.

The sequence s is called *interval witnessed* if for every $\varepsilon \in K^+$ there is a family of size $< \kappa$ of intervals of length $< \varepsilon$ that covers almost all elements of s.

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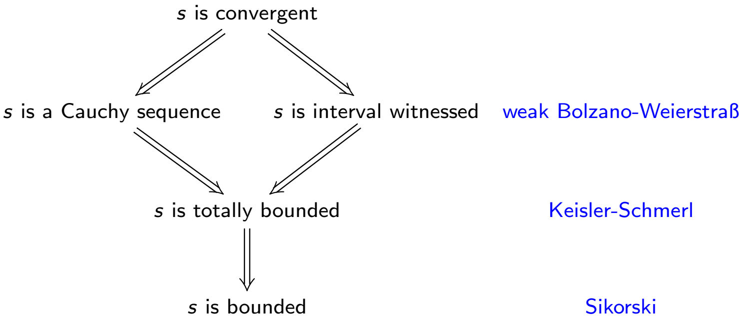

Weakening Bolzano-Weierstraß (2).

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Lemma (GHL). The weak Bolzano-Weierstraß property is preserved by thinning out the field: if K lies cofinal in L and L has the weak Bolzano-Weierstraß property, then so does K.

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Lemma (GHL). The weak Bolzano-Weierstraß property is preserved by thinning out the field: if K lies cofinal in L and L has the weak Bolzano-Weierstraß property, then so does K.

Theorem (GHL). For an ordered field K, the following are equivalent:

1. K is Cauchy complete and has the weak Bolzano-Weierstraß property and
2. K has the Keisler-Schmerl property.

Weak Bolzano-Weierstraß & the tree property.

Theorem (CGHL). Let κ be uncountable and K be a spherically complete ordered field of base number κ. Then the following are equivalent:

1. K has the weak Bolzano-Weierstraß property and
2. κ has the tree property.