Embeddings of ZFC^-

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Background

Definition

$\mathrm{ZFC}-$ is the theory

- ZF without the Power Set Axiom,
- the Replacement scheme,
- the Well-Ordering principle in place of the Axiom of Choice.

Definition

ZFC^- is the theory $\mathrm{ZFC}-$ plus the collection scheme.

Remarks

- For μ regular, $H_\mu \models \mathrm{ZFC}^-$.
- Models of $\mathrm{ZFC}-$ can behave very counter-intuitively.[1]

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[1] See *What is the Theory* ZFC *without Power Set* by Gitman, Hamkins and Johnstone for examples.

Background

Definition

Let M be a class and $j : M \to M$ an (Σ_n -)elementary embedding. Then $M \models \mathrm{ZF(C)}_j^-$ if:

- $M \models \mathrm{ZF(C)}^-$
- M satisfies the (Σ_n -)collection and (Σ_n -)separation schemes where j is allowed to be included in a formula as a class predicate.

Remark

- For the purposes of this talk we will assume that all elementary embeddings are non-trivial.

Basic Properties

Theorem (Folklore)

If $N \subseteq M$ are models of ZFC^- and $j : M \to N$ is a Σ_1-elementary embedding then there exists an ordinal κ such that $j(\kappa) > \kappa$.

Definition

$j : M \to N$ is cofinal if for any $y \in N$ there is an $x \in M$ such that $y \in j(x)$.

Example

If $M, N \models \mathrm{ZFC}$ then any embedding is cofinal because $y \in (V_{j(\alpha)})^N = j((V_\alpha)^M)$ for some α.

Definable Embeddings

Theorem (Gaifman)

Suppose that M, N are models of ZF^- and $j : M \to N$ is a cofinal, Σ_0 -elementary embedding. Then j is fully elementary.

Theorem (Suzuki)

Assume that $V \models \mathrm{ZF}^-$. Then there is no cofinal elementary embedding $j : V \to V$ which is definable from parameters.

Remark

Collection is necessary for Gaifman's theorem to hold.

Question

Are there definable, cofinal elementary embeddings $j : V \to V$ where $V \models \mathrm{ZF(C)}-$?

Cofinal Embeddings

Theorem (M.)

There is no cofinal Σ_0-elementary embedding $j : V \to V$ such that $V \models \mathrm{ZFC}^-_j$ and $V_{crit(j)} \in V$.

Sketch

- Suppose for a contradiction the $j : V \to V$ was a cofinal Σ_0-elementary embedding.
- Let κ denote the critical point.
- Let $\lambda := sup\{j^n(\kappa) : n \in \omega\}$.
- Note that $V_\kappa \in V$ implies $V_\lambda \in V$.
- There are two cases:
    - **Case 1:** λ^+ exists. (This is essentially Woodin's proof of the Kunen's inconsistency)
    - **Case 2:** For all $x \in V$, there is an injection $f : x \to \lambda$.

Cofinal Embeddings

Theorem (M.)

There is no cofinal Σ_0-elementary embedding $j : V \to V$ such that $V \models \mathrm{ZFC}_j^-$ and $V_{crit(j)} \in V$.

Sketch

Case 2: For all $x \in V$, there is an injection $f : x \to \lambda$.
In this case, any set x can be coded as the Mostowski collapse of some $C_x \subseteq \lambda \times \lambda$. Then

$$\begin{aligned} j(trcl(\{x\})) &= j(coll(C_x)) = coll(j(C_x)) \\ &= coll(\bigcup_{\alpha<\lambda} j(C_x \cap V_\alpha)) \\ &= coll(\bigcup_{\alpha<\lambda} j \restriction V_\lambda(C_x \cap V_\alpha)) \end{aligned}$$

Therefore j is definable from the parameter $j \restriction V_\lambda$. But this contradicts there being no definable elementary embedding. □

Removing the Assumption $V_{crit(j)} \in V$

Theorem

Let $V \models$ "$\mathrm{ZFC}^- + DC_\mu$ for all cardinals μ" and let $\mathcal{C}$ be a proper class. Then for any ordinal α there is a set $b \subseteq \mathcal{C}$ and a surjection $f : b \to \alpha$.

Lemma

Suppose that $V, M \models$ "$\mathrm{ZFC}^- + \mathrm{DC}_\mu$ for all cardinals μ" and $j : V \to M$ is a non-trivial elementary embedding with critical point κ. Then for any $\alpha \in \kappa + 1$, $V_\alpha \in V$.

Question

The proof of the theorem starts by adding a bijection between V and ORD and seems to require both well-ordering and collection. So is it provable in ZFC^- or $\mathrm{ZFC}-$?