Choiceless Elementary Embeddings of Sets: Problems

v. 5, September 11, 2022

This is a list of problems that we will consider discussing during the workshop. Please suggest additional problems in and around these areas, but please do so before September 1, 2022. Any unnamed problem is proposed by the local participants.

In the end we will select a few problems and try and put some focus on them during the meeting. Please let us know if a particular problem is particularly attractive to you and we will try to be democratic about the selection at the end.

Do note that this list of open problems will end up in the public domain. If you are close to solving a certain problem, it might be best to not suggest it.

(1) Weakly critical cardinals are just weakly compact cardinals in their embedding characterisation. Some of the characterisations are obviously weaker than others, e.g. the tree property with inaccessibility can be phrased as a statement about well-ordered sets which is incredibly weak, compared to existence of embeddings. Is there a way to phrase the tree property (e.g. by using generalised V_κ-trees) which is equivalent to κ being weakly critical? What about some compactness theorems for a particular logic?

(2) Finding a nontrivial condition allowing us to not only lift an embedding to symmetric extensions, but also preserve its closure and strength.

(3) (**Hayut**) Let κ be a Löwenheim–Skolem cardinal. Let us assume that every tree of height κ and width of each level $< V_\kappa$, has a cofinal branch. Can we conclude that κ is weakly critical? (Maybe we need to assume that κ is the limit of Löwenheim–Skolem cardinals?) Can we show that the assumption of being a LS cardinal is necessary?

(4) (**Hayut**) Similarly, can we find a combinatorial statement characterizing supercompactness? I'm thinking about Magidor's characterization using the ineffable tree property, but we probably also need to assume that κ is a Löwenheim–Skolem cardinal.

(5) (**Hayut**) Is there any ultrafilter type characterization of weakly critical, or critical cardinal? In particular, since we don't have any type of Łoś theorem, this might be more related to some type of $L_{\kappa,\kappa}$-compactness (maybe L_{V_κ,V_κ}?).

(6) (**Hayut**) Let κ be 1-extendible, meaning that there is an elementary embedding $j\colon V_{\kappa+1} \to V_{\lambda+1}$. Is there a critical cardinal below κ? I suspect that the answer is negative, and this should be (almost) the most extreme case possible. We can replace "1-extendible" with "strong" or "super-strong" and still get a meaningful question.

(7) (**Hayut**) Is it consistent that the first inaccessible is measurable?

(8) (**Goldberg**) Consistency strength related problems:
   (a) Is the existence of an elementary embedding from $V_{\lambda+3}$ to itself consistent relative to a large cardinal axiom compatible with AC?
   (b) Assume the Axiom of Choice and let λ be a cardinal such that Woodin's $\mathcal{E}_0(\lambda)$ exists. Is there an inner model of DC_λ containing $V_{\lambda+1}$ in which there is an elementary embedding from $V_{\lambda+3}$ to $V_{\lambda+3}$?
   (c) Does the existence of an elementary embedding from $V_{\lambda+3}$ to $V_{\lambda+3}$ plus DC_{ω_1} imply the consistency of the existence of an elementary embedding from $V_{\lambda+3}$ to $V_{\lambda+3}$ plus DC?
   (d) Does a Reinhardt cardinal (in the context of NBG without choice) imply the consistency of ZFC + I_0?
   (e) Is ZF + a rank Berkeley cardinal equiconsistent with NBG + a Reinhardt cardinal?

(9) (**Goldberg**) Combinatorics of rank Berkeley cardinals: Assume there is a rank Berkeley cardinal.
   (a) Must some successor cardinal be singular?
   (b) Must some double successor cardinal be regular?
   (c) Must there be a proper class of Mahlo cardinals?
   (d) For sufficiently large λ, is $\lambda^{+M} < \lambda^+$ in any inner model $M \models \mathsf{ZFC}$?
   (e) For sufficiently large ordinals λ, does $\mathrm{HOD} \models 2^\lambda = \lambda^+$?
   (f) For sufficiently large odd ordinals α, is $\aleph^*(V_{\alpha+1})$ the cardinal successor of $\aleph^*(V_\alpha)$?
   (g) Must HOD contain a proper class of Woodin cardinals? A proper class of cardinals κ such that κ is $\kappa^{+\mathrm{HOD}}$-supercompact in HOD?

(h) Must HOD satisfy "There are no supercompact cardinals?"

(10) (**Goldberg**) Assume there is a rank Berkeley cardinal λ. Is there an inner model with a rank Berkeley cardinal in which AD holds? In which the Ketonen order is linear? In which the Wadge order is linear on subsets of $V_{\lambda+1}$? In which there is a proper class of strong partition cardinals? A proper class of Löwenheim–Skolem cardinals?

(11) (**Schlutzenberg**) Can $j\colon V \to V$ be definable from some class of ordinals? In particular, from $j \upharpoonright \mathrm{Ord}$?

(12) (**Schlutzenberg**) Say κ is V-critical if there is an elementary $k\colon V \to M$ with M transitive and $\mathrm{crit}(k) = \kappa$.
 (a) Is V-criticality first-order definable?
 (b) Assume $j\colon V \to V$. Is V-criticality first-order definable? Is there an embedding $k\colon V : M$ with $\mathrm{crit}(k) < \mathrm{crit}(j)$?

 (V-criticality is definable if there is a proper class of weak LS cardinals, and in particular if there is a super-Reinhardt cardinal.)

(13) (**Schlutzenberg**) Suppose $V = L(V_\delta)$ and $\mathrm{cf}(\delta) = \omega$. Can there be $j\colon V_\delta \to V_\delta$ with $\kappa_\omega(j) < \delta$? Or can there be j which is generic over $L(V_\delta)$ without adding sets of rank $< \delta$?

(14) (**Schlutzenberg**) Suppose $j\colon V \to V$ and let $\lambda = \kappa_\omega(j)$. Is λ weakly compact in HOD? Etc. (It is greatly Mahlo in HOD.)

(15) (**Schlutzenberg**) Suppose G is set-generic and $j\colon V[G] \to V[G]$. Is there $k\colon V \to V$? Likewise, can a Berkeley cardinal be added by (small?) forcing? And if κ is super-Reinhardt in $V[G]$, is κ also super-Reinhardt in V? (In the last case, it is known that there is some super-Reinhardt in V. There are some other partial results.)

(16) (**Schlutzenberg**) Can there be $j\colon V[G] \to V$ with G set-generic over V? (If so, then $V[G]$ and V have the same sets of ordinals.) What is its consistency strength?

(17) (**Schlutzenberg**) Does $\mathsf{ZF} + I_{0,\lambda} + \mathrm{cf}(\aleph^*(V_{\lambda+1})^{L(V_{\lambda+1})} = \omega)$ imply $V_{\lambda+1}^{\#}$? What if we assume ZFC?

(18) (**Schlutzenberg**) (From "Periodicity in the cumulative hierarchy") Can λ be even and $j\colon V_{\lambda+2} \to V_{\lambda+2}$ be Σ_n-elementary, where $0 < n < \omega$, with j definable from parameters over $V_{\lambda+2}$?

(19) (**Schlutzenberg**) Suppose $j\colon V \to V$ and $\gamma \geq \kappa_\omega(j)$ and $G \subseteq \mathrm{Col}(\omega, V_\gamma)$ is V-generic. Does $V[G] \models \mathsf{AD}$?

(20) (**Schlutzenberg**) Does $\mathsf{ZF} + \mathsf{DC} + j\colon V \to V$ have a stronger consistency strength than $\mathsf{ZF} + j\colon V \to V$? (See Question 8(c) as well.)

(21) (**Schlutzenberg**) Suppose $j\colon V \to V$. Is there some $\gamma > \kappa_\omega(j)$ such that $V_\gamma \models \mathsf{ZF}$? (See Question 8(e) as well.)

(22) (**Schlutzenberg**) Are there useful forcing techniques for extending (V, j) with $j\colon V \to V$ to $(V[G], j^+)$ with $j^+\colon V[G] \to V[G]$ and $j \subseteq j^+$ where G is V-generic and the forcing is larger than $\kappa_\omega(j)$?

(23) (**Matthews**) Suppose $j\colon V \to V$, can we prove that there is a regular cardinal $\gamma > \kappa_\omega(j)$?

(24) (**Jeon**) Does ZF + "there is a Reinhardt cardinal" prove the consistency of ZF_j^- + "there is a cofinal Reinhardt embedding"?

(25) (**Jeon**) If there is a Reinhardt cardinal, then is there a regular $\gamma > \kappa_\omega(j)$ such that $H(\gamma)$, defined as $\{x \mid \gamma \not<^* |\mathrm{trcl}(x)|\}$, is a model of ZF^-?