Lecture Notes: Axiomatic Set Theory

Asaf Karagila

Last Update:
September 17, 2023

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Chapter 1

Introduction

1.1 Why do we need axioms?

In modern mathematics, axioms are given to define an object. The axioms of a group define the notion of a group, the axioms of a Banach space define what it means for something to be a Banach space. In these examples, however, we are not interested in the objects residing within the objects we define, but rather the properties of the objects we defined.

Sets formalize the notion of a collection of mathematical objects. This is a primitive and foundational notion, whose roots lie in the roots of counting. We cannot count a collection of objects, without first defining this collection.

Set theory was developed in the late 19th century, and along with logic, it was proposed as a way to formalize mathematics into a uniform language. This meant that we are interested both in the structure in which sets exist, as well as the properties of the sets that exist there. One of the naive properties expected of sets in the late 19th century was that every well-defined collection is in fact a set.

Theorem 1.1 (Russell's paradox). *Not every well-defined collection defines a set.*

Proof. Let X be the collection of $\{A \mid A \text{ is a set and } A \notin A\}$. If X is a set, then either $X \in X$, in which case the defining property requires that $X \notin X$; and if $X \notin X$, then the defining property of X requires that $X \in X$. In either case we arrive to the conclusion that $X \in X$ and $X \notin X$. This is a contradiction, so X cannot be a set. $\square$

Zermelo first gave an axiomatization for what properties sets *should have.* These axioms were later developed by Fraenkel, Skolem and cemented by von Neumann in his Ph.D. dissertation.

Why does the choice of axioms even affect us? Well, proofs do not live in vacuum. After the foundational crises began to resolve themselves, people became increasingly more aware to the importance of formal definitions and the necessity of axioms. Cantor famously tried to prove (and on occasion disprove) the continuum hypothesis. Later Gödel and Cohen proved that if set theory is consistent, then both the continuum hypothesis and its negation are consistent with set theory. Later in the course we will see how Gödel proved the consistency of the continuum hypothesis. It turns out that the choice of axioms could have implications on problems in mathematics outside of set theory. Here are two examples for such questions.

Question (Sierpiński sets). Recall that $A \subseteq \mathbb{R}$ is a null set if for every $\varepsilon > 0$ there is a sequence of intervals (a_n, b_n) for $n < \omega$ such that $A \subseteq \bigcup_n (a_n, b_n)$ and $\sum_n (b_n - a_n) < \varepsilon$.

Is there an uncountable $X \subseteq \mathbb{R}$, such that for every null set A, $A \cap X$ is countable?

Question (Productivity of ccc spaces). We say that a topological space X satisfies the *countable chain condition*, or that X is ccc, if every family of pairwise disjoint open sets is countable.

Let X and Y be compact Hausdorff spaces which are ccc. Is $X \times Y$ ccc as well?

1.2 Classes and sets

The objects of the universe of set theory are called sets, so when we say that something exist we mean to say that it is a set. But the universe of set theory is sometimes just a set inside a larger universe. The language of set theory has only one extralogical symbol,[1] the binary relation symbol $\in$.

If M is a set and E is a binary relation on M, we could ask whether or not the structure $\langle M, E\rangle$ satisfies the axioms of set theory, as these are just first-order axioms. We can talk about subsets of M, as collections of "M-sets". We say that $A \subseteq M$ is a *class of* M if there is some formula in the language of set theory $\varphi(x, p_1, \ldots, p_n)$ and there are parameters $p_1, \ldots, p_n \in M$ such that $A = \{x \in M \mid \langle M, E\rangle \models \varphi(x, p_1, \ldots, p_n)\}$.[2] Some classes correspond to a set of M, for example, $\{x \in M \mid x \neq x\}$ corresponds to the empty set of M; but some classes do not correspond to sets, as we saw with Russell's paradox. These are called *proper classes*. If A is a class which corresponds to a set, we will "confuse" A with the set it defines.

Exercise 1.1. Show that every set is a class.

We can conservatively extend the language by adding symbols for various definable objects, predicates and functions, and that will ease our writing. For example $a \subseteq b$ is a shorthand for $\forall x(x \in a \to x \in b)$. We will also have a few logical abbreviations.

- If $\varphi(u)$ is a formula, $(\exists u \in x)\varphi(u)$ is a shorthand for $\exists u(u \in x \land \varphi(u)$; $(\forall u \in x)\varphi(u)$ is a shorthand for $\forall u(u \in x \to \varphi(u))$.
- If $\varphi(u)$ is a formula, $\exists! u\varphi(u)$ is a shorthand for $\exists u(\varphi(u) \land \forall x(\varphi(x) \to x = u))$.
- We will denote by $a \subseteq b$ the formula $\forall x(x \in a \to x \in b)$.
- We will denote by $\varnothing$ the class $\{x \mid x \neq x\}$.
- We will write $\{a_1, \ldots, a_n\}$ for the class $\{x \mid x = a_1 \lor \ldots x = a_n\}$.
- We will denote by $\mathcal{P}(a)$ the class $\{u \mid u \subseteq a\}$.
- We will denote by $\bigcup a$ the class $\{u \mid (\exists b \in a) u \in b\}$.
- We will write $a \cup b$ for the class $\bigcup\{a, b\}$.
- We will denote by $\bigcap a$ the class $\{u \mid (\forall b \in a) u \in b\}$.
- We will write $a \cap b$ for the class $\bigcap\{a, b\}$.

We will use these symbols, and define others as we go along, as freely as we would like, understanding that we can always translate every statement into a statement involving only $\in$. We will often use the above abbreviations when a is a class, which we will understand as a different class, definable from the definition of a. For example, if a is the class defined by $\varphi(u)$, then $\mathcal{P}(a)$ is the class $\{x \mid (\forall y \in x)\varphi(y)\}$.

[1] We take $=$ to be a symbol of the underlying logic, although we can do without this.

[2] There are places where *every* subset of M is a class, but we will only use the term class to denote a definable collection.

1.3 The axioms of set theory

These are the axioms of the set theory commonly called the Zermelo–Fraenkel axioms, and denoted by ZF. Some of these might not make a lot of sense right now, and we will have to justify them in one way or another.

Extensionality: Two set are equal if and only if they have the same elements,

$$\forall x \forall y (x = y \leftrightarrow x \subseteq y \wedge y \subseteq x).$$

Empty set: The empty set exists,

$$\exists x \forall y (y \notin x).$$

Union: Every set has a union set,

$$\forall x \exists y (y = \bigcup x)).$$

Power: Every set has a power set,

$$\forall x \exists y (y = \mathcal{P}(x)).$$

Foundation: The $\in$ relation is well-founded,

$$\forall x (x \neq \varnothing \to (\exists y \in x)(x \cap y = \varnothing)).$$

Infinity: There is an infinite set,

$$\exists x (\varnothing \in x \wedge (\forall z \in x)(z \cup \{z\} \in x)).$$

Separation: If $\varphi(u, p_1, \ldots, p_n)$ is a formula in the language of set theory, then for every choice of parameters $p_1, \ldots, p_n$ and every set x there is a subset y composed of those elements satisfying φ,

$$\forall p_1, \ldots, p_n \forall x \exists y \forall u (u \in y \leftrightarrow u \in x \wedge \varphi(u, p_1, \ldots, p_n)).$$

Replacement: If $\varphi(u, v, p_1, \ldots, p_n)$ is a formula in the language of set theory, for every choice of parameters $p_1, \ldots, p_n$, if for some x we can prove that for $u \in x$ there exists exactly one v such that $\varphi(u, v, p_1, \ldots, p_n)$ holds, namely that φ defines a function on x, then there is y which is the range of this function,

$$\begin{aligned}\forall p_1, \ldots, p_n \forall x ((\forall u \in x)(\exists! v \varphi(u, v, p_1, \ldots, p_n))& \\ \to \exists y \forall v (v \in y \leftrightarrow (\exists u \in x)\varphi(u, v, p_1, \ldots, p_n))).&\end{aligned}$$

The first four axioms are self-explanatory. We will gracefully ignore the axiom of foundation for the time being, and return to it later. The axiom of infinity postulates the existence of a set with a certain property. We will later see that under a reasonable definition of "finite", any such set cannot be finite. The axioms of Separation and Replacement are in fact schemata, and not a single axiom. As we are working with first-order logic, we cannot quantify over arbitrary collection of objects; we can only quantify over objects. These schemata tell us that for each formula we add an axiom which has a particular syntactic structure which we can identify.

The axiom schema of Separation tells us that while not every well-defined collection defines a set, it is certainly the case that every definable subcollection of an existing set defines a set on its own. The axiom schema of Replacement tells us that if we can define a function, then the range of that function applied to any set x is also a set. Namely, if we can define a rule for replacing the elements of a set$x$, then the collection of "replaced elements" is a set as well.

Sometimes we will be interested in theories that we obtain by removing some of the axioms from ZF. For example, we will prove that if ZF without Foundation is consistent, then ZF is consistent as well. Let us name some of the subtheories of ZF:

- Z is ZF without Replacement.
- ZF^- is ZF without Power Set.
- ZF_0 is ZF without Foundation.

Theorem 1.2. *For every x, $x \notin x$.*

Proof. If x is a set, then by Exercise 1.3 $\{x, x\} = \{x\}$ is also a set. It follows that $x \cap \{x\} \neq \varnothing$, in contradiction to Foundation. □

Exercise 1.2. Show that the axioms Empty Set and Separation are redundant.

Exercise 1.3. Show that if x, y are sets, then $\{x, y\}$ is a set.

Exercise 1.4 (*). For every x and y, if there is a sequence $x_0 = x$, $x_n = y$ and $x_{k+1} \in x_k$ for $0 \leq k < n$, then $x \notin y$.

Exercise 1.5. Show that if $x \neq \varnothing$, then $\bigcap x$ is a set.

Exercise 1.6. For every x, $\mathcal{P}(x) \nsubseteq x$.

Exercise 1.7 (*) (Hilbert's paradox). Show that there is no set S with the properties: (a) If $x \in S$, then $\mathcal{P}(x) \in S$; and (b) if $T \subseteq S$, then $\bigcup T \in S$. (Hint: Consider $\bigcup S$, and reduce the two properties to obtain a contradiction to the previous exercise.)

Exercise 1.8. Recall that we can encode the ordered pair $\langle a, b\rangle$ as $\{\{a\}, \{a, b\}\}$. Show that if a and b are sets, then $\langle a, b\rangle$ is a set, and that $a \times b = \{\langle u, v\rangle \mid u \in a \wedge v \in b\}$ is also a set.

Exercise 1.9. Write an explicit formula in the language of set theory stating that y is a linear ordering of the set x.

Exercise 1.10. Write a formula $\varphi(x)$ in the language of set theory stating that x is an injective function.

Exercise 1.11 (**). We say that $\varphi(x, y, z, p)$ satisfies the ordered pair property for a parameter p, if we can prove without Replacement $\forall x \forall y \exists! z \varphi(x, y, z, p)$. Let $a \times_\varphi b$ denote the Cartesian product defined with φ as defining ordered pairs, namely $\{z \mid (\exists x \in a)(\exists y \in b)\varphi(x, y, z, p)\}$.

Suppose that for every φ satisfying the ordered pair property, $a \times_\varphi b$ exists. Prove that Replacement holds.

Exercise 1.12 (**). Suppose that Replacement holds only for parameter-free formulas. Prove that the full schema of Replacement holds.

Chapter 2

Ordinals, recursion and induction

Definition 2.1. A class A is *transitive* if for every $b \in A$, $b \subseteq A$.

Exercise 2.1. If A is a class such that for every $a \in A$, a is transitive, then $\bigcup A$ and $\bigcap A$ are also transitive classes.

Exercise 2.2. If A is transitive set, then $\mathcal{P}(A)$ is transitive, $A \cup \{A\}$ is also a transitive set.

Definition 2.2. Let A be a class, and let R be a relation on A.

1. We say that R is a *well-founded relation* if for every $b \subseteq A$, if b is non-empty, then there is some $x \in b$ such that for all $y \in b$, $\langle y, x \rangle \notin R$. In order words, every non-empty subset of A has an R-minimal element.
2. We say that R is *extensional* if for every $a, b \in A$, $a = b$ if and only if $\forall x(\langle x, a \rangle \in R \leftrightarrow \langle x, b \rangle \in R)$.
3. We say that R is *set-like* if for every $a \in A$, the class $\{b \in A \mid \langle b, a \rangle \in R\}$ is a set.

The Axiom of Foundation states that $\in$ is a well-founded relation; the Axiom of Extensionality states that $\in$ is in fact an extensional relation. Note that if A is a set, then every relation on A is indeed set-like.

For the remainder of the course, a *partial order* will be an irreflexive and transitive relation. A partial order $<$ of a class A is total (or linear) if for every $a, b \in A$ one of the mutually exclusive statements hold: $a = b$ or $a < b$ or $b < a$. Finally, a *well-order* is a well-founded linear order.

Definition 2.3. If A and B are partially ordered classes, an *embedding* is a function $F \colon A \to B$, such that $F(a) <_B F(a')$ if and only if $a <_A a'$. A surjective embedding is called an *isomorphism*.

Remark. Note that the definition makes sense for an arbitrary relation, and not necessarily a partial order.

Exercise 2.3. Show that an embedding of partial orders is injective.

Exercise 2.4. Given two well-ordered sets $(X, <_X)$ and $(Y, <_Y)$, we can either embed X into an initial segment of Y or embed Y into an initial segment of X. Moreover such embedding is unique.

2.1 Ordinals

We say that a set x is an *ordinal* if it is a transitive set such that $\in$ is a well-founded and linear ordering of x. We will always use Greek letters to denote ordinals, with the exception of finite ordinals which we will denote by Latin letters such as k, n, m and so on.

Proposition 2.4. *α is an ordinal if and only if α is a transitive set linearly ordered by $\in$.*

Proof. If α is an ordinal then by definition it is a transitive set which is well-ordered by $\in$, and in particular it is linearly ordered. If α is a transitive set which is linearly ordered by $\in$, then by the Axiom of Foundation, $\in$ is well-founded and therefore $\in$ is a well-order of α. ☐

Proposition 2.5. *α is an ordinal if and only if it is a transitive set of transitive sets.*

Proof. Suppose that α is an ordinal, let $x \in \alpha$, $y \in x$ and $z \in y$. By transitivity of α we have that $y \in \alpha$ and therefore $z \in \alpha$. Since $\in$ is a well-ordering of α it is a transitive relation on α, so $z \in x$. Therefore x is transitive.

In the other direction, suppose that α is a transitive set of transitive sets, then if $\in$ is not a well-order of α, by Foundation it means that $\in$ is not a linear order of α, namely there are $x, y \in \alpha$ such that $x \notin y$ and $y \notin x$. Let A be the set $\{x \in \alpha \mid (\exists y \in \alpha) y \neq x \land x \notin y \land y \notin x\}$, then by Foundation there is some $x \in A$ such that $x \cap A = \varnothing$, fix such x. Then $A(x) = \{y \in \alpha \mid x \neq y \land x \notin y \land y \notin x\}$ is non-empty, and by Foundation there is some $y \in A(x)$ such that $y \cap A(x) = \varnothing$. Therefore every $z \in y$ satisfies either $z \in x$ or $x \in z$. By transitivity of y, if $z \in y$ and $x \in z$ we get that $x \in y$, so $y \notin A(x)$; and therefore $z \in x$ for every $z \in y$. By the $\in$-minimality of x in A, if $z \in x$ it follows that for every $y \in \alpha$ either $z = y$ or $z \in y$ or $y \in z$; but by transitivity of x if $y \in z$, then $y \in x$ which is impossible, so $z \in y$. Therefore for every $z \in x$ we get that $z \in y$ which means that we proved that $y \subseteq x$ and $x \subseteq y$, so $x = y$. ☐

Remark. Both these proofs rely heavily on the Axiom of Foundation, and for a good reason. It is consistent, for example, that in the absence of Foundation there exist x such that $x = \{x\}$, in which case x is a transitive set of transitive sets; or that there exists an infinite sequence x_n such that $x_n = \{x_k \mid k > n\}$, in which case x_0 is a transitive set linearly ordered by $\in$, but it is not an ordinal.

We shall denote the class of the ordinals by Ord. If α and β are ordinals, we will write $\alpha < \beta$ to denote that $\alpha \in \beta$, and $\alpha \leq \beta$ to denote that $\alpha \in \beta$ or $\alpha = \beta$ which translates to $\alpha \subseteq \beta$.

Exercise 2.5. If A is a set of ordinals, then $\bigcup A$ is an ordinal. Moreover, $\bigcup A$ is the least ordinal α such that for all $\beta \in A$, $\beta \leq \alpha$. In other words, $\bigcup A = \sup A$.

Exercise 2.6. Let A be a non-empty class of ordinals, then $\bigcap A$ is an ordinal α and for every $\beta \in A$, $\alpha \leq \beta$. In other words, $\bigcap A = \min A$.

Exercise 2.7. The class of ordinals is transitive and well-ordered by $\in$. Therefore the class of all the ordinals is a proper class.

Definition 2.6. If α is an ordinal, we say that α is a *successor ordinal* if there exists $\beta \in \mathrm{Ord}$ such that $\alpha = \beta \cup \{\beta\}$, and we will write $\alpha = S(\beta)$. If α is a non-empty ordinal which is not a successor we say that α is a *limit ordinal.* We shall write Lim to denote the class of limit ordinals.

Let ω denote the class $\{\alpha \in \mathrm{Ord} \mid \alpha \notin \mathrm{Lim} \land \forall \beta < \alpha, \beta \notin \mathrm{Lim}\}$. We will denote by 0 the ordinal $\varnothing$ and for every natural number n, we will identify the nth successor of 0 as n. So $1 = S(0)$ and so on.

Exercise 2.8. Prove in ZF without Infinity that the Axiom of Infinity holds if and only if ω is a set.

2.2 Transfinite induction and recursion

Theorem 2.7 (Transfinite Induction). *Suppose that A is a class of ordinals such that whenever $\beta \subseteq A$, $\beta \in A$. Then $A = \mathrm{Ord}$.*

Proof. Note that $\varnothing \in A$, since $\varnothing \subseteq A$. Let β be an ordinal, by Separation $\beta \setminus A$ is a subset of β. It $\beta \setminus A$ is empty, then $\beta \subseteq A$ and therefore $\beta \in A$. Otherwise, there is a least $\gamma \in \beta \setminus A$. By virtue of being minimal, if $\xi \in \gamma$, then $\xi \in A$. Therefore $\gamma \subseteq A$, so $\gamma \in A$. It follows that $\beta \setminus A$ is indeed the empty set, so $\beta \in A$ for all $\beta \in \mathrm{Ord}$. □

Theorem 2.8 (Transfinite Recursion). *Suppose that G is a class function defined on all sets, then there is a unique class function F with domain* Ord *such that for every α, $F(\alpha) = G(F \restriction \alpha)$.*

Proof. We say that f is an α-approximation if f is defined on α, and for all $\xi < \alpha$, $f(\xi) = G(f \restriction \xi)$. By induction, if f is an α-approximation, then f is unique; and again by induction we can prove that for every α, there exists an α-approximation. Therefore define $F(\alpha) = x$ if and only if f is the unique $\alpha + 1$-approximation and $x = f(\alpha)$.

The uniqueness of F is proved similarly by induction: if F' is another function with the same property, then $\{\alpha \in \mathrm{Ord} \mid F(\alpha) = F'(\alpha)\} = \mathrm{Ord}$ by transfinite induction. □

Exercise 2.9 (**). Prove in Z that Replacement is equivalent to the Transfinite Recursion theorem.

Definition 2.9 (Ordinal arithmetic). Let α and β be ordinals. We define by recursion the following operations.

Addition:

1. $\alpha + 0 = \alpha$.
2. $\alpha + S(\beta) = S(\alpha + \beta)$.
3. $\alpha + \beta = \sup\{\alpha + \gamma \mid \gamma < \beta\}$ for $\beta \in \mathrm{Lim}$.

Multiplication:

1. $\alpha \cdot 0 = \alpha$.
2. $\alpha \cdot S(\beta) = \alpha \cdot \beta + \alpha$.
3. $\alpha \cdot \beta = \sup\{\alpha \cdot \gamma \mid \gamma < \beta\}$ for $\beta \in \mathrm{Lim}$.

Exponentiation:

1. $\alpha^0 = 1$.
2. $\alpha^{S(\beta)} = \alpha^\beta \cdot \alpha$.
3. $\alpha^\beta = \sup\{\alpha^\gamma \mid \gamma < \beta\}$ for $\beta \in \mathrm{Lim}$.

Exercise 2.10. Show that ordinal addition and multiplication are associative, and that $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$.

Exercise 2.11. Show that if α, β and γ are ordinals, then $\alpha^\beta \cdot \alpha^\gamma = \alpha^{\beta+\gamma}$ and $(\alpha^\beta)^\gamma = \alpha^{\beta \cdot \gamma}$.

2.3 Transitive classes

Definition 2.10. Let a be a set, *the transitive closure of* a is the smallest transitive set x such that $a \subseteq x$. We denote this set by $\operatorname{tcl}(a)$.

Theorem 2.11. *For every set a, $\operatorname{tcl}(a)$ exists.*

Proof. We define by induction, $F(0) = a$ and $F(n+1) = \bigcup F(n)$ for $n < \omega$, by Replacement if $\{F(n) \mid n < \omega\}$ is a set, define $F(\alpha) = \bigcup\{F(n) \mid n < \omega\}$ for all $\alpha \geq \omega$. We claim that $F(\omega)$ is $\operatorname{tcl}(a)$.

To see that $F(\omega)$ is transitive, suppose that $x \in F(\omega)$, then there is some $n < \omega$ such that $x \in F(n)$ and therefore $x \subseteq F(n+1)$ and so $x \subseteq F(\omega)$. Moreover, since $F(0) = a$ we automatically have that $a \subseteq F(\omega)$.

Suppose that x is a transitive set such that $a \subseteq x$, we will show that for all $n < \omega$, $F(n) \subseteq x$ and therefore $F(\omega) \subseteq x$. For $n = 0$ this is just the assumption that $a \subseteq x$. Suppose that $F(n) \subseteq x$, then for every $y \in F(n+1)$ there is some $u \in F(n)$ such that $y \in u$ by the definition of $F(n+1)$ as $\bigcup F(n)$. By the assumption that x is transitive and that $F(n) \subseteq x$ we get that $y \in x$ and therefore $u \subseteq x$, so $y \in x$ as well. □

Theorem 2.12 (Generalized Induction). *Let R be a well-founded and set-like relation on a class A, and let $B \subseteq A$ such that whenever $\{b \in A \mid b \mathrel{R} a\} \subseteq B$, then $a \in B$ as well. Then $A = B$.*

Proof. Suppose that $B \subseteq A$ is a class with the above property. Let $a \in A$, define by recursion $F(0) = \{a\}$ and $F(n+1) = \{b \in A \mid \exists y \in F(n) \wedge b \mathrel{R} y \wedge b \notin B\}$ for $n < \omega$; finally, $F(\alpha) = \bigcup\{F(\beta) \mid \beta < \alpha\}$ for $\alpha \geq \omega$. By the assumption that R is set-like, F is a well-defined function, so $F(\omega)$ is a subset of A. By well-foundedness, if $F(\omega)$ is non-empty, then there is an R-minimal element b there. So for some $n < \omega$, $b \in F(n)$. This means that there is no $y \in F(\omega)$ such that $y \mathrel{R} b$ and $y \notin B$. But this means exactly that $b \in B$ which is a contradiction to the assumption that $b \in F(\omega)$, except if $n = 0$ and $b = a$. But then it means that $F(n+1) = \varnothing$ for $n > 0$, so $a \in B$ by the defining property of B. □

It shouldn't come as a great surprise that the idea of the proof is not very different from the one of the basic transfinite induction on the ordinals. This leads us to a theorem and proof similar to the transfinite recursion theorem.

Theorem 2.13 (Generalized Recursion). *Let R be a well-founded and set-like relation on A, and suppose that G is a function whose domain is $\{\langle a, x\rangle \mid a \in A\}$. Then there is a unique function F whose domain is A and $F(a) = G(a, F \restriction a)$ (here $F \restriction a$ is the restriction of F to $\{b \in A \mid b \mathrel{R} a\}$).* □

Theorem 2.14. *Suppose that R is a well-founded and set-like relation on A. Then there exists a unique function $\operatorname{rank}_R \colon A \to \mathrm{Ord}$ such that $\operatorname{rank}_R(a) = \sup\{\operatorname{rank}_R(b) + 1 \mid b \mathrel{R} a\}$.* □

Exercise 2.12. Show that R is a well-founded relation on A if and only if there exists a function $F \colon A \to \mathrm{Ord}$ such that whenever $a \mathrel{R} b$, $F(a) < F(b)$.

Under the Axiom of Foundation, $\in$ is a well-founded relation, and it is certainly set-like. This leads us to the specific case of the generalized induction and recursion.

Theorem 2.15 ($\in$-Induction). *If A is a class such that $x \subseteq A \to x \in A$, then $\forall x(x \in A)$.* □

Theorem 2.16 ($\in$-Recursion). *Suppose that $G(x)$ is a function defined for all x, then there is a unique function F such that $F(x) = G(\{F(y) \mid y \in x\})$.* □

Theorem 2.17 (Mostowski's Collapse Lemma). *Suppose that R is an extensional, well-founded and set-like relation on A. Then there exists a unique transitive class A' and a unique isomorphism $\pi\colon A \to A'$ such that $a\ R\ b$ if and only if $\pi(a) \in \pi(b)$. In particular, if R was $\in$ and A is transitive, then $A = A'$ and $\pi(a) = a$ for all a.*

Proof. Define by recursion, $\pi(a) = \{\pi(b) \mid b\ R\ a\}$. □

Exercise 2.13. Complete the proof of Mostowski's Collapse Lemma.

Exercise 2.14. Use the collapse lemma to prove that every well-ordered set is isomorphic to a unique ordinal.

Exercise 2.15. There is no function f whose domain is ω, and for all $n < \omega$, $f(n+1) \in f(n)$.

Chapter 3

The relative consistency of the Axiom of Foundation

So far we have taken the Axiom of Foundation for granted. And while the previous chapter should have given us sufficient motivation, we still would like to know that if ZF_0 does not prove any false statement, then ZF will not prove false statement either.

In this section we only assume ZF_0. The definition of ordinals, mind you, stays the same, although the proofs of the equivalent definitions will no longer work. Transfinite induction and recursion also stay the same, although $\in$-induction fails.

Definition 3.1. We say that a is a well-founded set, if $\{\langle x, y\rangle \in a \times a \mid x \in y\}$ is a well-founded relation on a.

Exercise 3.1. The following are equivalent over ZF_0:

1. The Axiom of Foundation.
2. Every set is well-founded.
3. Every transitive set is well-founded.
4. Every set is a subset of a well-founded set.

Definition 3.2. The *von Neumann hierarchy* is defined by recursion on the ordinals:

1. $V_0 = \varnothing$.
2. $V_{\alpha+1} = \mathcal{P}(V_\alpha)$.
3. $V_\alpha = \bigcup\{V_\beta \mid \beta < \alpha\}$ for $\alpha \in \mathrm{Lim}$.

We define $V = \bigcup\{V_\alpha \mid \alpha \in \mathrm{Ord}\}$ and call V the *von Neumann universe*.

Note that if we write $x \in V$, what we write is really that $\exists\alpha(x \in V_\alpha)$.

Exercise 3.2. Show that for every α, V_α is a transitive set and conclude that V is a transitive class.

Exercise 3.3. Show that for every α, V_α is a well-founded set.

Exercise 3.4. Show that if $\alpha < \beta$, then $V_\alpha \subseteq V_\beta$.

Definition 3.3 (Relativization). Suppose that $\theta(x, \bar{p})$ is a formula in the language of set theory. We define the relativization of a formula φ to θ and $\bar{p}$ by recursion on the structure of φ:

- If φ is atomic, e.g. $x \in y$ or $x = y$, we define $(x \in y)^{(\theta,\bar{p})}$ as $x \in y \wedge \theta(x, \bar{p}) \wedge \theta(y, \bar{p})$, and similarly $(x = y)^{(\theta,\bar{p})}$ as $x = y \wedge \theta(x, \bar{p}) \wedge \theta(y, \bar{p})$.
- If $\varphi = \varphi_1 * \varphi_2$ for some connective, we define $\varphi^{(\theta,\bar{p})}$ as $(\varphi_0)^{(\theta,\bar{p})} * (\varphi_2)^{(\theta,\bar{p})}$.
- If $\varphi = \neg\psi$, we define $\varphi^{(\theta,\bar{p})}$ as $\neg(\psi^{(\theta,\bar{p})})$.
- If φ is $\exists x\psi$, we define $\varphi^{(\theta,\bar{p})}$ as $\exists x(\theta(x, \bar{p}) \wedge \psi^{(\theta,\bar{p})})$.
- If φ is $\forall x\psi$, we define $\varphi^{(\theta,\bar{p})}$ as $\forall x(\theta(x, \bar{p}) \rightarrow \psi^{(\theta,\bar{p})})$.

If θ has no parameters, we will omit them, and write φ^θ. Moreover, if we denote by M the class defined by θ (and $\bar{p}$), we will write φ^M for the relativization of φ to θ.

Theorem 3.4. *If φ is an axiom of* ZF*, then φ^V holds. In other words, V satisfies* ZF*, that is* ZF_0 *and the Axiom of Foundation.*

Proof. Extensionality is easy to verify, and Infinity holds since $V_\omega \in V$ and it is a witness for the existence of an inductive set. Power set and Union can be proved by transfinite induction: if $x \in V_\alpha$, then $\bigcup x \in V_\alpha$, and $\mathcal{P}(x) \in V_{\alpha+1}$.[1]

For readability, we will prove Replacement without parameters. Suppose that $\varphi^V(u, v)$ is a formula such that for $x \in V$ it holds that $(\forall u \in x)\exists!(v \in V)\varphi^V(u, v)$. We want to show that there is some $y \in V$ such that

$$y = \{v \in V \mid (\exists u \in x)\varphi(u, v)\}.$$

Define $\psi(u, v)$ as $v \in V \wedge \varphi^V(u, v)$. Then by the assumption, $(\forall u \in x)\exists! v\psi(u, v)$. Therefore, by Replacement we have that $y = \{v \mid (\exists u \in x)\psi(u, v)\}$ is a set in the universe. It remains to show that $y \in V$. Note that $y \subseteq V$, the function $f(v) = \min\{\alpha \mid v \in V_\alpha\}$ is a well-defined function on y, and therefore there is a set of ordinals A such that $\alpha \in A$ if and only if $\alpha = f(v)$ for some $v \in y$. Let $\alpha = \sup A$, then for every $v \in y$ we get that $v \in V_\alpha$, and therefore $y \subseteq V_\alpha$, which means that $y \in V_{\alpha+1}$, so $y \in V$ as wanted.

Finally, the Axiom of Foundation holds because every V_α is a well-founded set, and every $x \in V$ satisfies that $x \subseteq V_\alpha$ for some α. Therefore, in V every set is a subset of a well-founded set and Foundation holds. □

Proposition 3.5. *For every α, $\alpha \subseteq V_\alpha$ and there is no $\beta < \alpha$ such that $\alpha \subseteq V_\beta$.*

Proof. We prove this by induction α. For $\alpha = 0$ this is true vacuously. Suppose that the assumption holds for all $\beta < \alpha$. It follows that if $\beta < \alpha$, then $\beta \subseteq V_\beta$, so $\beta \in V_{\beta+1} \subseteq V_\alpha$. Therefore $\alpha \subseteq V_\alpha$. Suppose that β was the least such that $\alpha \subseteq V_\beta$. If $\beta < \alpha$, then $\beta + 1 \subseteq V_\beta$, which is a contradiction to the induction hypothesis. □

Exercise 3.5 (*). $a \in V$ if and only if $\operatorname{tcl}(a)$ is well-founded.

Exercise 3.6. The Axiom of Foundation is equivalent to the statement that every set lies in V.

In other words, if we started from ZF (rather than ZF_0), then $V = \{x \mid x = x\}$. In other words, constructing V in a model of ZF gives us the model again. In *other* other words, every model of ZF is its own V.

[1] In fact, we get more here: V "computes" power sets and unions correctly.

Exercise 3.7. Give an alternative proof to the following statement: Every $x \in V$ has a transitive closure.

Exercise 3.8 (*). Show that $\in$-Induction is equivalent to the Axiom of Foundation.

Exercise 3.9. Show that W is a transitive class satisfying Foundation, then $W \subseteq V$.

Exercise 3.10. Formulate and prove an analogous theorem for Theorem 3.4 for ZF_0 without Infinity. Moreover, prove that in V as you defined it, Infinity holds if and only if it held in the outset of the theorem.

Exercise 3.11. Show that $\langle V_\omega, \in \rangle$ is a model for ZF without Infinity. Therefore ZF_0 proves the consistency of ZF without Infinity.

Exercise 3.12. Show that if δ is a limit ordinal, then V_δ satisfies all the axioms of ZF without Replacement.

Proposition 3.6. *$V_{\omega+\omega}$ does not satisfy Replacement.*

Proof. Let $\varphi(x, y)$ be the formula stating: x and y are ordinals and $y = \omega + x$. Then φ defines a function on ω in $V_{\omega+\omega}$. However the image of this function is the set $\{\omega + n \mid n < \omega\}$ which does not lie in $V_{\omega+\omega}$. Therefore Replacement fails. $\square$

Remark. One of the consequences of Gödel's second incompleteness theorem is that ZF does not prove its own consistency. Therefore ZF cannot prove that there exists a set M and a relation E such that $\langle M, E \rangle$ satisfy all the axioms of ZF. The last two exercises prove, therefore, that assuming Infinity or Replacement increases the power of our theory.

Remark. We have seen that the Axiom of Foundation is consistent. But maybe it is outright provable? It turns out that the answer is negative, but due to time constraints we will not see this in details. This idea has even been extended to "Anti-Foundation Axioms" which posits the existence of non-well founded sets in various ways.

From here on end, we shall denote by V the universe of set theory in which we are working.

Chapter 4

Cardinals and their arithmetic

4.1 The definition of cardinals

As you may recall from the basic set theory course, we are interested in "measuring" the size of sets. For finite sets, we can just count their elements, but generally we need to devise a way that will work for infinite sets as well.

Definition 4.1. We say that two sets x and y are *equipotent* if there exists a bijection $f\colon x \to y$.

Theorem 4.2 (Cantor–Bernstein theorem). *If x is equipotent with a subset of y, and y is equipotent with a subset of x, then x and y are equipotent.* □

Theorem 4.3 (Cantor's theorem). *There is no surjection from x onto $\mathcal{P}(x)$.* □

Equipotence gives rise to an equivalence relation on the sets, but this equivalence relation is such that with the exception of $\varnothing$, the class of sets equipotent with x is always a proper class. We would like to find objects which can be used to represent the equipotency classes.

Definition 4.4 (Scott's trick). Suppose that E is an equivalence relation on V. Let x be any set, and let α be the least ordinal such that for some $y \in V_\alpha$, $\langle x, y\rangle \in E$; we define the *partial equivalence class* x/E to be $\{y \in V_\alpha \mid \langle x, y\rangle \in E\}$.

Proposition 4.5. *Let E be an equivalence relation on V, then $x\ E\ y$ if and only if $x/E = y/E$.* □

Exercise 4.1. Show that Scott's trick always gives rise to sets namely x/E is a set, and show that $\{x/E \mid x \in V\}$ is a class as well.

We can use Scott's trick to define the cardinals in ZF. However, if we define $|x|$ to be the Scott cardinal for x, namely x/E where E denotes the equipotence relation, then we will not have the property that $\big||x|\big| = |x|$. This is somewhat upsetting, since we do expect a finite set to have a cardinal number which is somewhat related to its actual size.

We digress from this discussion to recall a few facts about well-ordered sets.

Definition 4.6. We say that a set x is *well-orderable* if it is equipotent with an ordinal. Equivalently, if there is some linear order $<$ on x which is a well-order.

Theorem 4.7. *If x is well-orderable, then there is a least ordinal α equipotent with x.*

Proof. By assumption, the class $\{\alpha \mid \alpha \text{ is equipotent with } x\}$ is non-empty. □

Definition 4.8. We say that an ordinal α is an initial ordinal, if there is no $\beta < \alpha$ such that α and β are equipotent.

Exercise 4.2. Show that a well-ordering is isomorphic to an initial ordinal, if and only if every proper initial segment has a strictly smaller cardinality.

Theorem 4.9. *If $\alpha \leq \omega$, then α is an initial ordinal.*

Proof. We prove by induction. For 0 this is trivial, as is the case $n = 1$ (for the only smaller ordinal is empty, and there is no bijection between a non-empty set and an empty set); assume for $n \geq 1$, then if there is a bijection $f\colon n+1 \to n$, then there is one such that $f(n) = n-1$. Therefore by restricting f to n we obtain a bijection between n and $n-1$, contrary to the induction hypothesis.

The case ω follows directly, if $n < \omega$ is equipotent with ω, then by the Cantor–Bernstein theorem, $n+1$ and n are equipotent which is a contradiction. □

Definition 4.10. We say that x is *finite* if it is equipotent with a finite ordinal. By the previous theorem, it is a unique ordinal. We say that x is *countable*, if it is equipotent with a subset of ω.

Exercise 4.3. The ordinal $\omega + \omega$ is equipotent with ω, therefore not every limit ordinal is an initial ordinal.

Definition 4.11. Let x be a set. We define the *cardinal of* x to be either the unique initial ordinal equipotent with x in the case that x is well-orderable, or the Scott cardinal of x in case x cannot be well-ordered. We denote this as $|x|$.

Now we have that $|x| = |y|$ if and only if x and y are equipotent. Moreover, if x can be well-ordered, then $|x|$ is the least ordinal to which x is equipotent, and in particular if x is finite then $|x|$ faithfully represents the number of elements in x. We will write $|x| \leq |y|$ to denote that x is equipotent with a subset of y, and $|x| < |y|$ to mean that $|x| \leq |y|$ and $|x| \neq |y|$.

Definition 4.12 (Cardinal arithmetic). Suppose that x and y are two sets.

Addition We define $|x| + |y|$ to be $|x \times \{0\} \cup y \times \{1\}|$.

Multiplication We define $|x| \cdot |y|$ to be $|x \times y|$.

Exponentiation We define $|x|^{|y|}$ to be $|\{f\colon y \to x\}|$.

Exercise 4.4. Show that cardinal addition is commutative and associative. Moreover, show that if n, m are finite ordinals, then ordinal and cardinal arithmetic coincide.

Exercise 4.5. Show that $|x| \cdot (|y| + |z|) = (|x| \cdot |y|) + (|x| \cdot |z|)$; $|x|^{|y|} \cdot |x|^{|z|} = |x|^{|y|+|z|}$; and $(|x|^{|y|})^{|z|} = |x|^{|y| \cdot |z|}$.

Exercise 4.6. Show that $|x| + |x| = |x| \cdot 2$ (by induction conclude for every finite ordinal in place of 2), and $|x| \cdot |x| = |x|^2$ (again, conclude by induction for every finite ordinal in place of 2).

Exercise 4.7. Show that for every x, there is some y such that $|x| \leq |y|$, and $|y|^2 = |y|$.

Exercise 4.8. If α is an infinite ordinal, then $|\alpha| + |1| = |\alpha + 1| = |\alpha|$.

4.2 The Aleph numbers

Theorem 4.13 (Hartogs' theorem). *If x is a set, then there is some ordinal α such that $|\alpha| \nleq |x|$.*

Proof. Let $\mathcal{W}$ denote the set $\{A \subseteq \mathcal{P}(x) \mid \langle A, \subsetneq\rangle$ is a well-ordered set$\}$, consider the function defined on $\mathcal{W}$ such that $F(A) = \alpha$ if and only if $\langle A, \subsetneq\rangle \cong \alpha$, this is a well-defined function, since a well-ordered set is isomorphic to a unique ordinal. Note that if an ordinal β is in the range of F, then every $\gamma < \beta$ is also in the range of F: if A has order type β, then it has an initial segment which have order type γ. Therefore $\operatorname{rng} F$ is an ordinal, denote it by α. If $f\colon \alpha \to x$ is an injection, define $a_\beta = \operatorname{rng} f \restriction \beta$ for $\beta \leq \alpha$. We get that $A = \{a_\beta \mid \beta \leq \alpha\}$ is well-ordered by strict inclusion, and therefore $A \in \mathcal{W}$. But at the same time A is isomorphic to α, and therefore $\alpha \in \operatorname{rng} F$ which is a contradiction since $\operatorname{rng} F = \alpha$ and $\alpha \notin \alpha$. Therefore $|\alpha| \nleq |x|$. □

Exercise 4.9. Show that if α is the least ordinal which does not inject into a set x, then α is an initial ordinal (i.e. a cardinal).

Definition 4.14 (Hartogs' number). Let x be a set, we write $\aleph(x)$ as the least initial ordinal α such that $|\alpha| \nleq |x|$.

Definition 4.15. We define by recursion, $\omega_0 = \omega$; $\omega_{\alpha+1} = \aleph(\omega_\alpha)$ and if ω_β were defined for all $\beta < \alpha$ for $\alpha \in \mathrm{Lim}$, then $\omega_\alpha = \sup\{\omega_\beta \mid \beta < \alpha\}$.

Note that by Exercise 4.8 every ω_α is a limit ordinal.

Exercise 4.10. $\eta \geq \omega$ is an initial ordinal if and only if there is some α such that $\eta = \omega_\alpha$.

In order to discern cardinality from order type, and to make it easier to understand the context in which arithmetic operations are interpreted, we write $\aleph_\alpha$ as the cardinal of ω_α. While these are formally the same object, we will understand $\omega_2 + \omega_1$ as ordinal addition, whereas $\aleph_2 + \aleph_1$ as cardinal addition.

Theorem 4.16. *For every α, $\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$.*

Proof. We will define a well-ordering on $\mathrm{Ord} \times \mathrm{Ord}$ and show that restricting it to $\omega_\alpha \times \omega_\alpha$ has order type ω_α. This will provide us with a bijection between the two sets, thus proving the theorem.

We define the following ordering on $\mathrm{Ord} \times \mathrm{Ord}$:

$$\langle \beta, \gamma\rangle <_{gp} \langle \beta', \gamma'\rangle \iff \begin{cases} \max\{\beta, \gamma\} < \max\{\beta', \gamma'\} & \text{or} \\ \max\{\beta, \gamma\} = \max\{\beta', \gamma'\} \wedge \beta < \beta' & \text{or} \\ \max\{\beta, \gamma\} = \max\{\beta', \gamma'\} \wedge \beta = \beta' \wedge \gamma < \gamma' \end{cases}$$

Easily this is an irreflexive order, and verifying transitivity and linearity is straightforward. To see that this is a well-ordering, suppose that A is a non-empty subset of $\mathrm{Ord} \times \mathrm{Ord}$. The set $\{\max\{\beta, \gamma\} \mid \langle \beta, \gamma\rangle \in A\}$ is a non-empty set of ordinals, let $A' \subseteq A$ be those pairs in A mapped to its minimum. Among the pairs in A', let β be the least ordinal appearing in the left coordinate of an ordered pair in A'; and let γ be the least ordinal such that $\langle \beta, \gamma\rangle \in A'$. It is not hard to check that $\langle \beta, \gamma\rangle$ is the minimum element of A.

Let $\downarrow(\beta, \gamma)$ denote the set $\{\langle \varepsilon, \delta\rangle \mid \langle \varepsilon, \delta\rangle <_{gp} \langle \beta, \gamma\rangle\}$ this is a set, since every such $\langle \varepsilon, \delta\rangle$ must satisfy that $\max\{\epsilon, \delta\} < \max\{\beta, \gamma\} + 1$. By the fact that $<_{gp}$ is a well-ordering, every proper initial segment of the form $\downarrow(\beta, \gamma)$. Moreover, note that if η is any ordinal, then $\eta \times \eta = \downarrow(0, \eta)$.

We will now prove by induction that $\downarrow(0,\omega_\alpha)$ has the same order type as ω_α. For $\alpha = 0$, we get that every proper initial segment is a subset of $\downarrow(0,m)$ for some $m < \omega$. Since the underlying set of $\downarrow(0,m)$ is $m \times m$, which is finite, we get that $<_{gp}$ up to $\omega \times \omega$ is of order type ω (note that this is the initial segment $\downarrow(0,\omega)$ as remarked above).

Suppose that α is infinite and for all $\alpha' < \alpha$ the order type of $\downarrow(0,\omega_{\alpha'})$ is $\omega_{\alpha'}$. Let $\eta < \omega_\alpha$, and we may assume $\eta \geq \omega$, since $\omega_\alpha \times \omega_\alpha = \bigcup\{\eta \times \eta \mid \eta < \omega_\alpha\}$ it is enough to show that $\downarrow(0,\eta)$ has cardinality strictly less than $\aleph_\alpha$. By the fact that $\eta < \omega_\alpha$, $|\eta| < \aleph_\alpha$, then $|\eta| = \aleph_{\alpha'}$ for some $\alpha' < \alpha$. By the induction hypothesis, then, $|\eta \times \eta| = |\eta|$, and therefore of $\downarrow(0,\eta)$ has cardinality $\aleph_{\alpha'}$. In particular, every proper initial segment of $\downarrow(0,\omega_\alpha)$ has cardinality strictly less than $\aleph_\alpha$, so it means that $\omega_\alpha \times \omega_\alpha = \downarrow(0,\omega_\alpha)$ is isomorphic to ω_α as wanted. $\square$

Corollary 4.17. *If $\alpha \leq \beta$, then $\aleph_\alpha + \aleph_\beta = \aleph_\alpha \cdot \aleph_\beta = \aleph_\beta$.*

Exercise 4.11. Let α and β be ordinals such that at least one of them is infinite. Then $\alpha+\beta, \alpha\cdot\beta$ and α^β all have the same cardinality as $\max\{|\alpha|, |\beta|\}$.

Exercise 4.12. There is no set x such that $|\mathcal{P}(x)| = \aleph_0$.

Let us define the following relation on cardinals $|x| \leq^* |y|$ if and only if there is a surjective function from a subset of y onto x (alternatively, either $x = \varnothing$, or there is a surjection from y onto x). This relation is reflexive and transitive, but not provably anti-symmetric.

Exercise 4.13 (Lindenbaum's theorem). Prove that for every x there is some ordinal α such that $|\alpha| \not\leq^* |x|$. Let $\aleph^*(x)$ denote the least such ordinal, show that $\aleph^*(x)$ is a cardinal and that $\aleph(x) \leq \aleph^*(x)$. We shall refer to $\aleph^*(x)$ as the *Lindenbaum number of x*.

Exercise 4.14. Show that $\aleph(x) < \aleph(\mathcal{P}(\mathcal{P}(\mathcal{P}(x))))$.

Exercise 4.15. Show that if $|x| \leq^* |y|$, then $|\mathcal{P}(x)| \leq |\mathcal{P}(y)|$.

Exercise 4.16 (**) (Specker trees). For a set x, let us define the *Specker tree* on x, $\mathbf{S}(x)$ by recursion: $\mathbf{S}(x)$ is $\{|x|\} \cup \bigcup\{\mathbf{S}(y) \mid |\mathcal{P}(y)| = |x|\}$, with $|y| <_{\mathbf{S}} |y'|$ if and only if $|y'| = |\mathcal{P}(y)|$. Prove that $\mathbf{S}(x)$ is a set, that $<_{\mathbf{S}}$ is a well-founded partial order on $\mathbf{S}(x)$. (Hint: Use the previous exercise to justify that $<_{\mathbf{S}}$ is well-founded.)

4.3 Finiteness

We defined a set to be finite if it was in bijection with a finite ordinal, and that is certainly one way of defining finiteness. But we can examine what other properties finite ordinals have which "familiar infinite sets" do not, and try to extrapolate these for our definitions.

Definition 4.18 (Finiteness). Let x be a set.

1. We say that x is *finite* if it can be put in bijection with a finite ordinal.
2. We say that x is *amorphous* if it cannot be written as the disjoint union of two infinite sets.
3. We say that x is *Tarski-finite* if every $\subseteq$-chain in $\mathcal{P}(x)$ is finite.
4. We say that x is *strongly $*$-finite* if there is no surjection from x onto ω.
5. We say that x is *$*$-finite* if there is no surjection from x onto $x \cup \{x\}$.

6. We say that x is *Dedekind-finite* if every injection from x into x is a bijection.

Remark. The fourth definition is sometimes called by the terrible name "weakly Dedekind-finite" (which is probably due to the fact that its negation is a weakening of Dedekind-infinite sets), and the term Tarski-finite is sometimes squandered on a definition equivalent to (true) finiteness.

Exercise 4.17 (*). Show that the above definitions form a hierarchy. Namely, if x satisfies a definition, it must satisfy all those that follow it.

Exercise 4.18. Show that if x is well-ordered, then being Dedekind-finite implies being finite.

Exercise 4.19 (*). Show that the union and product of two strongly $*$-finite sets is again a strongly $*$-finite set.

Exercise 4.20. Show that x is Dedekind-finite if and only if $\aleph(x) \leq \aleph_0$.

Exercise 4.21 (*). Show that if there exists an infinite Dedekind-finite set x, then there is a Dedekind-finite set y which is not $*$-finite. (Hint: show that the set $I(x)$ of all finite injective sequences from x is Dedekind-finite, and then show that there is a surjection from $I(x) \setminus \{\varnothing\}$ onto $I(x)$.)

Theorem 4.19 (Kuratowski). *x is strongly $*$-finite if and only if $\mathcal{P}(x)$ is Dedekind-finite. In other words,*

$$\aleph_0 \leq^* |x| \iff \aleph_0 \leq |\mathcal{P}(x)|.$$

Proof. In the one direction, by Exercise 4.15 if $\aleph_0 \leq^* |x|$, then $\aleph_0 < 2^{\aleph_0} \leq |\mathcal{P}(x)|$. So if x is not strongly $*$-finite, $\mathcal{P}(x)$ is Dedekind-infinite.

In the other direction, suppose that $\mathcal{P}(x)$ is Dedekind-infinite and a_n is a sequence of sets in $\mathcal{P}(x)$. Our goal is to show that there is a sequence b_n of pairwise disjoint and non-empty sets, in which case we can define the following function:

$$f(y) = \begin{cases} n & a \in b_n \\ 0 & a \notin \bigcup\{b_n \mid n < \omega\} \end{cases}$$

which is easily a surjection onto ω.

If the a_n's are pairwise disjoint, we finished the proof. And if there is a $\subseteq$-decreasing subsequence (without loss of generality, the sequence itself) we can define $b_{n+1} = a_n \setminus a_{n+1}$ to obtain this. So we may assume that every $\subseteq$-decreasing subsequence is finite. We define b_n by induction. Let $s_0 = \{a_n \mid n < \omega\}$, and for simplicity we may also assume $x = a_0$. Suppose that b_m was defined for $m < n$, and suppose that

$$\left\{a_k \setminus \bigcup\{b_m \mid m < n\} \mid k \geq n\right\}$$

is infinite. Define n^* to be the least k, if it exists, such that both a_k and $x \setminus a_k$ do not cover $\bigcup\{b_m \mid m < n\}$. If n^* exists define $b_n = a_{n^*} \setminus \bigcup\{b_m \mid m < n\}$ in the case where $\{a_k \setminus (\bigcup\{b_m \mid m < n\} \cup a_{n^*} \mid k \geq n^*\}$ is infinite; or $b_n = x \setminus (a_{n^*} \setminus \bigcup\{b_m \mid m < n\})$ (in which case the set defined similarly must have infinite many elements).

In case that n^* does not exist, define $s_{j+1} = \{a_k \setminus \bigcup\{b_m \mid m < n\} \mid k \geq n\}$ and restart the process described above. If for some j, we managed to define b_n for all $n < \omega$ from s_j, then we finished because we have a sequence of pairwise disjoint non-empty sets. Otherwise, for every j we got stuck, then taking each finite sequence of b_n's but by the definition of s_{j+1}, the next finite sequence is pairwise disjoint from it. Therefore in either case we end up with a sequence of pairwise disjoint sets, as wanted. $\square$

The idea behind the proof, ultimately, is at each step take either a subset of some a_n or a subset of its complement, which is never empty, and the union of everything we have thus far—including the new set—will not cover everything. If we happened to run into a dead-end, we refine the sequence of sets and continue the construction from those refined sets. In either case, this produces a sequence of pairwise disjoint sets, from which we can define a surjection as wanted.

Chapter 5

Absoluteness and reflection

5.1 Absoluteness

Definition 5.1. If φ is a formula in the language of set theory, we say that it is a *bounded formula* if every quantifier appearing in φ is of the form $(\exists x \in y)$ or $(\forall x \in y)$.

Definition 5.2 (The Levy Hierarchy). Let φ be a formula in the language of set theory.

1. φ is a Σ_0 or Π_0 if it is a bounded formula.
2. φ is a Σ_{n+1} if there is a Π_n formula ψ such that $\varphi = \exists x \psi$.
3. φ is a Π_{n+1} if there is a Σ_n formula ψ such that $\varphi = \forall x \psi$.

We say that φ is a Σ_n^{ZF} (Π_n^{ZF}) if ZF proves that φ is equivalent to a Σ_n (Π_n) formula, and we say that φ is Δ_n^{ZF} if it is both Σ_n^{ZF} and Π_n^{ZF}.

From this point onward, we will omit the ZF superscript, and write just Σ_n, Π_n and Δ_n.

Exercise 5.1. Σ_n and Π_n are closed under conjunction, disjunction and bounded quantification; if $n > 0$ then Σ_n formulas are closed under existential quantifiers and Π_n are closed under universal quantifiers. Finally, Π_n formula is the negation of a Σ_n formula (and therefore vice versa).

Exercise 5.2. Show that following statements are Δ_0: "x is an ordinal", "x is transitive", "x is a function", "x is a finite ordinal", "x is ω", "x is a function and y is in the domain of x".

Exercise 5.3. Show that "x is a finite set" is a Δ_1 statement.

Theorem 5.3. *Suppose that $\varphi(u_1, \ldots, u_n)$ is a Δ_0 formula. Then for every $x_1, \ldots, x_n$ and for every transitive class A it holds that $\langle A, \in\rangle \models \varphi(x_1, \ldots, x_n)$ if and only if $x_1, \ldots, x_n \in A$ and $\varphi(x_1, \ldots, x_n)$ holds in V.*

Remark. From here on end, we will write $A \models \varphi$ to mean that $\langle A, \in\rangle \models \varphi$.

Proof. We prove this by structural induction on φ. For φ an atomic formula this is obviously true, and the proof for connectives is as usual. If φ is $(\forall y \in x)\psi$ and $x \in A$, then by transitivity $x \subseteq A$. Therefore $A \models \psi(y)$ if and only if $\psi(y)$ holds in V, so $A \models \varphi(x)$ if and only if $\varphi(x)$ holds in V. The proof for $(\exists y \in x)\psi$ is similar. $\square$

Definition 5.4. We say that φ is *downwards absolute* if whenever A is a transitive class such that $A \models \varphi$, and B is a transitive subclass of A (possibly a set), then $B \models \varphi$. Similarly, φ is *upwards absolute* if whenever A is a transitive class such that $A \models \varphi$, then whenever B is a transitive class with $A \subseteq B$, then $B \models \varphi$. If φ is both upwards and downwards absolute, we just say it is absolute.

The above theorem, then, states that Δ_0 formulas are absolute between any two transitive [sets or] classes which include the relevant assignments.

Theorem 5.5. *Every Σ_1 formula is upwards absolute and every Π_1 formula is downwards absolute. Consequently, every Δ_1 formula is absolute.*

Proof. Let $\exists x\varphi(x)$ be a Σ_1 formula with φ a Δ_0 formula. Assume that $A \models \exists x\varphi(x)$ and B is a transitive superclass of A. Then there is some $a \in A$ such that $A \models \varphi(a)$, by the previous theorem $B \models \varphi(a)$ and therefore $B \models \exists x\varphi(x)$.

The proof for the Π_1 case is similar: if $A \models \forall x\psi(x)$ with ψ a Δ_0 formula, and $B \subseteq A$ is a transitive class, then for every $b \in B$ we get that $A \models \psi(b)$, so $B \models \psi(b)$. Therefore $B \models \forall x\psi(x)$. The consequence for Δ_1 now follows. $\square$

Exercise 5.4. Show that φ is upwards absolute if and only if $\neg\varphi$ is downwards absolute.

Definition 5.6 (Elementary submodel). Suppose that A is a structure in a language $\mathfrak{L}$. We say that B is an *elementary substructure* of A if it is a substructure, and the following holds: For every formula $\varphi(u_1, \dots, u_n)$ in $\mathfrak{L}$, and for every $b_1, \dots, b_n$: $A \models \varphi(b_1, \dots, b_n)$ if and only if $B \models \varphi(b_1, \dots, b_n)$.

We denote this by $B \preccurlyeq A$. If $\mathfrak{L}$ is the language of set theory, we write $\prec_{\Sigma_1}, \prec_{\Pi_n}$, etc. to mean that the above definition holds for the specified class of formulas.

As we have seen, every transitive class is a Δ_1-elementary submodel of V.

Theorem 5.7 (Tarski–Vaught criterion). *Let A be a structure in the language $\mathfrak{L}$ and let B be a substructure of A. Then $B \preccurlyeq A$ if and only if for every formula $\varphi(x, u_1, \dots, u_n)$ and $b_1, \dots, b_n \in B$, if $A \models \exists x\varphi(x, b_1, \dots, b_n)$ then there is $b \in B$ such that $A \models \varphi(b, b_1, \dots, b_n)$.* $\square$

Theorem 5.8. *Suppose $\{M_\alpha \mid \alpha < \beta\}$ is a sequence of structures in some fixed language $\mathfrak{L}$, and for $\alpha < \alpha'$, $M_\alpha \preccurlyeq M_{\alpha'}$. Then M_β defined as $\bigcup\{M_\alpha \mid \alpha < \beta\}$ is an $\mathfrak{L}$-structure and for all $\alpha < \beta$, $M_\alpha \preccurlyeq M_\beta$.* $\square$

Theorem 5.9. *If M is a well-orderable structure, and $A \subseteq M$, then there is an elementary submodel $N \preccurlyeq M$ such that $A \subseteq N$ and $|N| = |A| + \aleph_0$.* $\square$

Exercise 5.5. Verify that "x is countable" is a Σ_1 formula.

Definition 5.10. We say that x is *Σ_n-definable* (*Π_n-definable*) in a transitive class A, if there is a Σ_n (Π_n) formula $\varphi(u)$ such that $A \models \varphi(u) \leftrightarrow u = x$. If we allow parameters in the formula, in which case we say that x is Σ_n-definable (Π_n-definable) in $p_1, \dots, p_n$ (and require them to be in A).

Exercise 5.6. Show that "α is an initial ordinal" is a Π_1 formula. Show that ω_1 is Π_2-definable.

Exercise 5.7. If $M \preccurlyeq N$ then every member of N which is definable, is an element of M.

Theorem 5.11. *Let $H(\omega_1)$ denote the set $\{x \mid \mathrm{tcl}(x)$ is countable$\}$. If M is a countable elementary submodel of $H(\omega_1)$, then M is transitive.*

Proof. Suppose that M is a countable elementary submodel of $H(\omega_1)$. Let $x \in M$, then in $H(\omega_1)$ there exists a bijection between x and a subset of ω. By elementarity the same must hold in M. However, ω is Δ_0-definable, so $\omega \in M$ and $\omega \subseteq M$ by similar arguments. Let $f \in M$ be an injective function $f : x \to \omega$, then every member of the range of f is in M and therefore every member of the domain of M must also be in M. In other words, $x \subseteq M$. $\square$

Exercise 5.8 (*). Let $H(\omega_2)$ denote the set $\{x \mid |\operatorname{tcl}(x)| \leq \aleph_1\}$ and assume that M is a countable elementary submodel of $H(\omega_2)$. Prove that M cannot be transitive, and use Mostowski's collapse lemma to prove that ω_1 is not Σ_1-definable. (Hint: Assume by contradiction, collapse M and use Theorem 5.5.)

5.2 Reflection

We saw in the previous section that the V_α's which are transitive satisfy that they are Δ_1-elementary submodels of V. But what about formulas which are not Δ_1? Can we find a way to reflect them in some transitive set?

Theorem 5.12 (The Reflection Theorem). *Let $\varphi(u_1, \dots, u_n)$ be a formula in the language of set theory. Then* ZF *proves that for every α there exists $\beta > \alpha$ such that for all $x_1, \dots, x_n \in V_\beta$, $\varphi(x_1, \dots, x_n) \leftrightarrow \varphi^{V_\beta}(x_1, \dots, x_n)$.*

In this case, we say that V_β *reflects* φ. Note that φ^{V_β} holds if and only if $V_\beta \models \varphi$. We will first prove two lemmas; and for readability we will assume that φ has one free variable.

Lemma 5.13. *Let $\varphi(x, u)$ be a formula in the language of set theory, then for every α there is some $\beta \geq \alpha$ such that for all $a \in V_\beta$, $\exists x \varphi(x, a) \to (\exists x \in V_\beta)\varphi(x, a)$.*

Proof of Lemma 5.13. We define by recursion a sequence β_n. Take $\beta_0 = \alpha$. Suppose that β_n was defined, then the function f be defined on V_{β_n} as follows $f(a) = \min\{\gamma \mid \exists x \varphi(x, a) \to (\exists x \in V_\gamma)\varphi(x, a)\}$, by Replacement $\operatorname{rng} f$ is a set of ordinals, let $\beta_{n+1} = \sup \operatorname{rng} f$. Finally, let $\beta = \sup\{\beta_n \mid n < \omega\}$.

If $a \in V_\beta$, then there is some $n < \omega$ such that $a \in V_{\beta_n}$, then if there exists x such that $\varphi(x, a)$, then by definition there is such x in $V_{\beta_{n+1}}$ and therefore in V_β. $\square$

Lemma 5.14. *Suppose that $\{\alpha_n \mid n < \omega\}$ is a set of ordinals such that for each n, V_{α_n} reflects φ. Let $\alpha = \sup\{\alpha_n \mid n < \omega\}$, then V_α reflects φ.*

Proof of Lemma 5.14. We prove this by induction on the structure of φ. If φ is atomic, then absoluteness implies that every V_γ reflects φ. Connectives and negations are easily verified using truth tables.

Suppose that φ has the form $\exists x \psi(x, u)$ and let $a \in V_\alpha$. There is some n such that $a \in V_{\alpha_n}$, and therefore $V \models \exists x \psi(x, a)$ if and only if $(\exists x \in V_{\alpha_n})\psi^{V_{\alpha_n}}(x, a)$, by the induction hypothesis V_α reflects ψ and therefore $V \models \exists x \psi(x, a)$ if and only if $(\exists x \in V_\alpha)\psi^{V_\alpha}(x, a)$. $\square$

Proof of Theorem 5.12. We prove this by induction on the structure of φ. For atomic formulas this follows from absoluteness. Negation and connectives follow by verifying truth tables. For $\varphi(u)$ defined as $\exists x \psi(x, u)$, we recursively define an intertwined sequence: $\beta_0 = \alpha + 1$; for odd indices, β_{2n+1} is the least ordinal obtained from Lemma 5.13 such that $V_{\beta_{2n+1}}$ is closed under $\exists x \psi$; for even indices, we take β_{2n+2} to be the least ordinal above β_{2n+1} such that $V_{\beta_{2n+2}}$ reflects ψ, such ordinal exists by the inductive hypothesis on ψ.

Let β be $\sup\{\beta_n \mid n < \omega\}$, and let $a \in V_\beta$. If $V \models \exists x \psi(x, a)$, then by the same argument as Lemma 5.13 we get that $(\exists x \in V_\beta)\psi(x, a)$, but since V_β is the limit of points which reflects ψ, this is the same as saying that $(\exists x \in V_\beta)\psi^{V_\beta}(x, a)$, or $\varphi^{V_\beta}(a)$. In the other direction, if $\varphi^{V_\beta}(a)$ holds, then there is some $x \in V_\beta$ such that $\psi^{V_\beta}(x, a)$ holds, but again by reflection $\psi(x, a)$ holds so $\exists x\psi(x, a)$ holds. Therefore V_β reflects φ as wanted. $\square$

Remark. For $n > 0$ we can prove there exists a Σ_n-truth predicate. Reflecting it means that there is a proper class of ordinals which are Σ_n-elementary submodels of V; and moreover that ZF cannot be given a finite axiomatization, as we could reflect such finite list of axioms and obtain that ZF proves its own consistency. This would be a contradiction to Gödel's second incompleteness theorem.

Question: why do compactness and reflection not prove the consistency of ZF?

Exercise 5.9 (*). The axiom schema of Replacement is equivalent to the Reflection theorem over Z with the assumption that every set belongs to some V_α.

Exercise 5.10 (*). Suppose that $\{D_\alpha \mid \alpha \in \mathrm{Ord}\}$ is a sequence of transitive sets such that ZF proves that: $D_\alpha \subseteq D_\beta$ for all $\alpha < \beta$; for $\alpha \in \mathrm{Lim}$, $D_\alpha = \bigcup\{D_\beta \mid \beta < \alpha\}$; and every set lies inside some D_α. Then for every φ in the language of set theory, ZF proves that for every α there is some $\beta > \alpha$ for which $\varphi \leftrightarrow \varphi^{D_\beta}$.

Namely, every continuous filtration of V satisfies Reflection.

Exercise 5.11. x is Σ_n-definable if and only if there is an ordinal α such that x is Σ_n-definable in V_α.

Exercise 5.12. Let φ be a formula in the language of set theory. Show that the class of ordinals β such that V_β reflects φ is closed. Namely, if $\{\beta < \delta \mid V_\beta \text{ reflects } \varphi\}$ is unbounded in δ, then V_δ reflects φ.

Chapter 6

The Axiom of Choice

6.1 The Axiom of Choice

Definition 6.1. We say that a function f is a *choice function* if for all $x \in \operatorname{dom} f$, $f(x) \in x$. We say that x *admits a choice function* if there is a choice function on $x \setminus \{\varnothing\}$.

Exercise 6.1. Show that "f is a choice function" is a Δ_0 statement, and "x admits a choice function" is a Σ_1 statement.

Theorem 6.2. *If x is finite, then x admits a choice function.*

Proof. Suppose that $|x| = n$, then for $n = 0$ the statement is trivially true. Suppose that for $|x| = n$ there is a choice function, let $|x \cup \{x_n\}|$ be a set of size $n + 1$, then $|x|$ admits a choice function f, and either $x_n = \varnothing$ in which case f is a choice function on $x \cup \{x_n\}$, or x_n is non-empty, in which case let $y_n \in x_n$ be some element and then $f \cup \{\langle x_n, y_n\rangle\}$ is a choice function on $x \cup \{x_n\}$. $\square$

Remark. The proof seems as though it should work for $|x| = \aleph_0$. However, when removing one element from a set of size $\aleph_0$, we remain with a set of size $\aleph_0$ so there is no way we can appeal to an induction hypothesis. It is true, however, that if x and y admit a choice function, then $x \cup y$ admits a choice function as well.

Definition 6.3. *The Axiom of Choice* is the axiom stating that for every x admits a choice function.

Definition 6.4. If X is a set, $\prod X$ is the set of all choice functions that X admits.

Exercise 6.2. Suppose that $X = \{x_i \mid i \in I\}$, then $\{f\colon I \to \bigcup X \mid f(i) \in x_i\}$ is non-empty if and only if $\prod X$ is non-empty.

Theorem 6.5 (Zermelo's theorem). *The Axiom of Choice holds if and only if for every x is well-orderable.* $\square$

In other words, the Axiom of Choice is equivalent to stating that every cardinal is an ordinal.

Exercise 6.3. Let α be an initial ordinal. Show that if $x \leq \alpha$ or $x \leq^* \alpha$, then x is well-orderable.

Theorem 6.6. *The following are equivalent:*

1. AC.

2. ***(Zorn's lemma)*** *If $\langle P, <\rangle$ is a partial order where every chain $C \subseteq P$ has an upper bound, then there is a maximal element in P.*

3. ***(Weak Zorn's lemma)*** *If $\langle P, <\rangle$ is a partial order where every well-ordered chain $C \subseteq P$ has an upper bound, then there is a maximal element in P.*

4. ***(Hausdorff's Maximality Principle)*** *If $\langle P, <\rangle$ is a partial order, then there is a maximal chain $C \subseteq P$.*

5. *For every x, $\mathcal{P}(x)$ admits a choice function.*

6. *For every x whose members are pairwise disjoint with $\varnothing \notin x$, there is C such that for every $y \in x$, $C \cap y$ is a singleton.*

7. *$\leq$ is a linear ordering of the cardinals.*

8. *$\leq^*$ is a linear ordering of the cardinals.* □

Exercise 6.4. If $\langle P, \leq\rangle$ is a partial order satisfying the conditions of the Zorn's lemma, if P is well-orderable, then P has a maximal element.

Exercise 6.5 (*). The Axiom of Choice is equivalent to the statement "There is some $n < \omega$, such that given any n cardinals, at least two of them are comparable by $\leq$ or by $\leq^*$".

Remark. If we change the above to be "Given any infinitely many cardinals, two of them are comparable [in either order]" is not known to imply the axiom of choice, or even the fact that any Dedekind-finite set is finite, which means we do not even know if this is equivalent to "Given any countably infinite set of cardinals, two of them are comparable".

Theorem 6.7. AC *is equivalent to the statement "If x is well-orderable, then $\mathcal{P}(x)$ is well-orderable".*

Proof. One direction follows from Zermelo's theorem. We shall prove the second direction. It is enough, of course, to show that every V_α is well-orderable.

We prove by induction that for every α, V_α can be well-ordered. We proved that V_ω is countable, so for $\alpha \leq \omega$ the claim is true. If V_α can be well-ordered, by the assumption $\mathcal{P}(V_\alpha) = V_{\alpha+1}$ can be well-ordered.

Suppose that α is a limit ordinal, and for every $\beta < \alpha$, V_β can be well-ordered. Let λ be $\aleph(V_\alpha)$, then $\lambda \times \lambda$ can be well-ordered, and therefore $\mathcal{P}(\lambda \times \lambda)$ can be well-ordered, fix a well-ordering $\lhd$ of $\mathcal{P}(\lambda \times \lambda)$. Since every $\beta < \alpha$ satisfies that V_β can be well-ordered, it has to be the case that $|V_\beta| < \lambda$. Therefore, for every $\beta < \alpha$ there is some $R_\beta \subseteq \lambda \times \lambda$ such that R_β is an extensional and well-founded relation on its domain, and the Mostowski collapse of $\langle \operatorname{dom} R_\beta, R_\beta\rangle$ is exactly V_β. Moreover, by the fact that $\lhd$ is a well-ordering of $\mathcal{P}(\lambda \times \lambda)$ we can choose the least such relation. Note that the isomorphism between R_β and V_β is unique, so once R_β was chosen (using $\lhd$) there is no choice when we identify between V_β and $\operatorname{dom} R_\beta$. Let $\pi_\beta \colon V_\beta \to \operatorname{dom} R_\beta$ be that unique isomorphism.

Finally, we define a well-ordering on V_α as follows: $x \prec y$ if and only if $\operatorname{rank}(x) < \operatorname{rank}(y)$,[1] or $\operatorname{rank}(x) = \operatorname{rank}(y) = \beta$ and $\pi_\beta(x) < \pi_\beta(y)$. Easily, $\prec$ is a well-order of V_α as wanted. □

Exercise 6.6. The naive proof that V_α is well-orderable, without considering well-ordering of previous steps, fails. Why? Moreover, the proof *does not* imply that there is a class well-ordering of V. Why?

[1] Note that $\operatorname{rank}(x) = \alpha$ if and only if $\alpha = \min\{\beta \mid x \in V_{\beta+1}\} = \min\{\beta \mid x \subseteq V_\beta\}$ is the rank function which exists from Replacement and Foundation.

Exercise 6.7. Show that if AC holds, then for every $\delta \in \mathrm{Lim}$, $V_\delta \models \mathsf{AC}$.

Theorem 6.8 (Tarski). *The Axiom of Choice holds if and only if for every infinite x, $|x|^2 = |x|$.*

Proof. In the one direction, if the Axiom of Choice holds and x is infinite, then $|x| = \aleph_\alpha$ for some α, and therefore $|x|^2 = \aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha = |x|$.

In the other direction, let λ be $\aleph(x)$, and without loss of generality $x \cap \lambda = \varnothing$. Then the following holds:

$$|x| + \lambda = (|x| + \lambda)^2 = |x|^2 + \lambda^2 + |x| \cdot \lambda \cdot 2 = |x| + \lambda + |x| \cdot \lambda.$$

Therefore it has to be the case that $|x| \cdot \lambda \leq |x| + \lambda$, and since the other inequality is trivial we get $|x| + \lambda = |x| \cdot \lambda$. Using Lemma 6.9, we get that $|x|$ and λ are comparable, but by taking $\lambda = \aleph(x)$ we get that it is necessarily the case that $|x| \leq \lambda$, so x can be well-ordered. Since every finite set is well-orderable by definition, we have concluded that every set is well-orderable, so the Axiom of Choice holds. $\square$

Lemma 6.9. *If a is a set and λ is an initial ordinal, then $|a| + \lambda = |a| \cdot \lambda$ implies that $|a|$ is comparable with λ.*

Proof. Without loss of generality we can assume that $a \cap \lambda = \varnothing$ and that λ is infinite (if λ is finite then it is comparable with $|a|$ already). Let $f \colon a \times \lambda \to a \cup \lambda$ be a bijection. If there exists some $b \in a$ such that $f``\{b\} \times \lambda \subseteq a$,[2] then f defines an injection from λ into a by $\alpha \mapsto f(b, \alpha)$. Assume otherwise, then for every $b \in a$ there is some α such that $f(b, \alpha) \in \lambda$. This defines an injection from a into $\lambda \times \lambda$ given by $b \mapsto \langle \alpha, f(b, \alpha) \rangle$ where α is the least ordinal for which $f(b, \alpha) \in \lambda$. Since λ is infinite, $|\lambda \times \lambda| = \lambda$, and therefore $|a| \leq |\lambda|$. $\square$

Exercise 6.8 (*) (Abraham's Lemma). If there is a surjection $f \colon a \cup b \to a \times b$ then either $|b| \leq^* |a|$ or $|a| \leq^* |b|$.

Theorem 6.10. *Suppose that for every infinite set G there is a binary operation $\odot$ such that $\langle G, \odot \rangle$ is a group. Then the Axiom of Choice holds.*

Proof. Let x be an infinite set, and let λ be an ordinal such that $\lambda \geq \aleph(x)$. Without loss of generality, $x \cap \lambda = \varnothing$. Let $G = x \cup \lambda$ and fix $\odot$ to be a binary operation on G which makes it into a group. Note that by the group axioms, if $a \odot b = c \odot b$, then $a = c$.

If there is some $y \in x$ such that for all $\alpha < \lambda$, $\alpha \odot y \in x$, then $\alpha \mapsto \alpha \odot y$ is an injective function from λ into x. By the choice of λ, this is of course impossible. Therefore for every $y \in x$ there is some $\alpha < \lambda$ such that $\alpha \odot y \in \lambda$. Then $y \mapsto \langle \alpha, \beta \rangle$ such that α is the least ordinal for which $\alpha \odot y = \beta$ is an injective function from x into $\lambda \times \lambda$. Therefore x is well-orderable, and therefore the axiom of choice holds. $\square$

6.2 Weak version of the Axiom of Choice

Definition 6.11. Let x, y be sets, we shall write $[x]^{|y|}$ as $\{a \subseteq x \mid |a| = |y|\}$, $[x]^{<|y|}$ and $[x]^{\leq|y|}$ are defined similarly by replacing $|a| = |y|$ by the suitable inequality.

Definition 6.12. Let $\mathsf{AC}_x^y(z)$ denote the statement: Every $a \subseteq [z]^{\leq|y|}$ such that $|a| \leq |x|$ admits a choice function. If we omit one (or more) of the parameters, then we replace it by a universal quantifier. So $\mathsf{AC}_{\aleph_0}$ is the same as $\forall x \forall y \mathsf{AC}_{\aleph_0}^{|y|}(x)$, or "Every countable family admits a choice function".

We also define versions with $\leq |x|$ and $< |x|$ for the parameters x and y above.

[2] We will use $f``X$ to denote the direct image of X, namely $\{f(x) \mid x \in X\}$.

Exercise 6.9. For all $\alpha < \beta$, $\mathsf{AC}_{\aleph_\beta} \to \mathsf{AC}_{\aleph_\alpha}$.

Exercise 6.10 (*). $\mathsf{AC}_{\aleph_0}$ implies that every Dedekind-finite set is finite.

Theorem 6.13. *$\forall\alpha$ $\mathsf{AC}_{\aleph_\alpha}$ if and only if for every x, $\aleph(x) = \aleph^*(x)$.*

Proof. Suppose that every well-orderable family admits a choice function. We saw that $\aleph(x) \leq \aleph^*(x)$; suppose that $\eta < \aleph^*(x)$, then there is a surjection $f\colon x \to \eta$, but now $A_\alpha = \{y \in x \mid f(y) = \alpha\}$ is a family of non-empty sets, and by assumption admits a choice function. It is not hard to verify that such a choice function implies that $\eta < \aleph(x)$ and so equality follows.

In the other direction, suppose that $\aleph(x) = \aleph^*(x)$ for every x. We will prove by induction that $\mathsf{AC}_{\aleph_\alpha}$ holds. Suppose that α is an ordinal, such that $\mathsf{AC}_{<\aleph_\alpha}$ holds. Note that for $\alpha = 0$ this is a theorem of ZF. Let $\{a_\eta \mid \eta < \omega_\alpha\}$ be a family of non-empty sets. For each $\gamma < \omega_\alpha$, let x_γ be the set $\prod_{\eta<\gamma} a_\eta$, then by our induction hypothesis x_γ is non-empty. We define by induction two sequences:

- $\lambda_\gamma = \aleph\left(\bigcup\{D_\eta \mid \eta < \gamma\}\right) + \sup\{\aleph(\lambda_\eta) \mid \eta < \gamma\}$.
- $D_\gamma = x_\gamma \times \lambda_\gamma$.

Finally, let $\lambda = \sup\{\lambda_\gamma \mid \gamma < \omega_\alpha\}$ and $D = \bigcup\{D_\gamma \mid \gamma < \omega_\alpha\}$. There is a natural surjection from D onto λ, so $\lambda < \aleph^*(D)$ and therefore $\lambda < \aleph(D)$. Fix an injection $F\colon \lambda \to D$, then F cannot be injective into any fixed D_γ, as $\lambda \geq \aleph(D_\gamma)$. This means that for every $\gamma < \omega_\alpha$ there is some $\beta < \lambda$ such that $F(\beta) \in D_{\gamma'}$ for some $\gamma' > \gamma$. Define β_γ to be the least such β (this might not be an injective assignment).

Finally, note that if $F(\beta_\gamma) = \langle f_\gamma, \xi_\gamma\rangle$, then f_γ is a choice function whose domain includes a_γ. Which therefore defines a choice function from the entire family: $h(a_\gamma) = f_\gamma(a_\gamma)$. □

Exercise 6.11 (*). $\mathsf{AC}_{\aleph_0}$ holds if and only if for every countable family $\{a_n \mid n < \omega\}$ there is an infinite $I \subseteq \omega$ such that $\{a_i \mid i \in I\}$ admits a choice function.

Theorem 6.14. *$\mathsf{AC}_{\aleph_0}$ holds if and only if whenever X is a metric space and $A \subseteq X$, then $x \in \overline{A}$ if and only if there is a sequence $\langle a_n \mid n < \omega\rangle \subseteq A$ such that $\lim a_n = x$.*

Proof. Let $A = \{A_n \mid n < \omega\}$ be a family of non-empty sets. Without loss of generality, $A_n \cap A_m = \varnothing$ for $n \neq m$. Let $X = \bigcup A \cup \{\infty\}$, with ∞ some set not in $\bigcup A$. We define the following metric on X:

$$d(x,y) = \begin{cases} 0 & x = y \\ \left|\frac{1}{n} - \frac{1}{m}\right| & x \in A_n, y \in A_m, n \neq m \\ \frac{1}{n} & x, y \in A_n \\ \frac{1}{n} & x \in A_n, y = \infty \text{ or } x = \infty, y \in A_n \end{cases}$$

We leave the reader with the task of verifying the definitions of a metric on X. Moreover, A is dense in X: if $\varepsilon > 0$, then for some $n < \omega$, $\frac{1}{n} < \varepsilon$, then $A \cap B_\varepsilon(\infty)$ contains A_m for all $m > n$.

By the assumption, there is some sequence $x_n \in X$ such that $x_n \in A$ and $\lim x_n = \infty$. This defines a choice function on $\{A_m \mid \exists n(x_n \in A_m)\}$ by taking the least such x_n for each m. Moreover, this family of sets has to be infinite, otherwise $\inf\{d(x_n, \infty) \mid n < \omega\} > \frac{1}{m}$ for some m, in contradiction to the convergence assumption. By Exercise 6.11 we get $\mathsf{AC}_{\aleph_0}$ as wanted.

In the other direction, assume $\mathsf{AC}_{\aleph_0}$ holds and X is an arbitrary metric space. Let $A \subseteq X$ be some non-empty subset, and $a \in \overline{A}$. Then for every $n < \omega$, $A \cap B_{\frac{1}{n}}(a)$ is non-empty. Using $\mathsf{AC}_{\aleph_0}$ we can choose some $x_n \in A$ such that $d(a, x_n) < 1/n$, and it is clear now that $\lim x_n = a$. □

Definition 6.15. We say that $\langle T, <_T\rangle$ is a *tree* if $<_T$ is a well-founded partial order on T, such that for every $t \in T$, $\{s \in T \mid s <_T t\}$ is linearly ordered by $<_T$. We write T_α as the elements of T whose rank in $<_T$ is α, and the height of T is $\sup\{\alpha + 1 \mid T_\alpha \neq \varnothing\}$. We say that $B \subseteq T$ is a *branch* if it is a maximal chain; and it is a *cofinal branch* if it is a branch such that for every α, $B \cap T_\alpha \neq \varnothing$.

Definition 6.16. Let $\mathsf{DC}_{\aleph_\alpha}(y)$ denote the statement that whenever $T \subseteq y$ is a tree of height ω_α in which every chain of length less than ω_α has an upper bound not in the chain, then T has a cofinal branch. If we omit y, we do not restrict T to being a subset of any particular y; and $\mathsf{DC}_{<\aleph_\alpha}$ denotes $\forall\beta(\beta < \alpha \to \mathsf{DC}_{\aleph_\beta})$. If we omit the $\aleph_\alpha$ subscript we will **always** mean $\mathsf{DC}_{\aleph_0}$.

Exercise 6.12. Show that if $|X| = \aleph_\alpha$, then $\mathsf{DC}_{\aleph_\alpha}(X)$ holds. In other words, if T is a well-orderable tree satisfying the assumptions of $\mathsf{DC}_{\aleph_\alpha}$, then T has a cofinal branch.

Exercise 6.13. For every α and x, $\mathsf{DC}_{\aleph_\alpha}(x) \to \mathsf{AC}_{\aleph_\alpha}(x)$.

Exercise 6.14 (**). $\forall\alpha\, \mathsf{AC}_{\aleph_\alpha} \to \mathsf{DC}$.

Remark. It was proved by Azriel Levy that the above implication cannot be extended even to $\mathsf{DC}_{\aleph_1}$.

Theorem 6.17. *The following are equivalent:*

1. DC.
2. *Every structure in a countable language has an elementary submodel of size* $\aleph_0$.
3. *For every* $\alpha > \omega$, V_α *has a countable elementary submodel.*

The implication of (1) $\implies$ (2) is beyond the scope of this course, but it can be proved by tracing the usual proof of the Downwards Löwenheim-Skolem theorem and checking that we only need DC to prove it for the countable case.

Proof. Clearly (2) $\implies$ (3). It remains to show that (3) $\implies$ (1). Let α be an arbitrary infinite ordinal, we will show that if $T \in V_\alpha$ is a tree satisfying the conditions of DC, then T has a branch. Consider the structure $\langle V_\alpha, \in, T, <_T\rangle$ with T and $<_T$ being constant symbols with the axioms satisfying that $<_T$ is a tree order on T (as far as V_α is concerned) and T has height ω and no maximal elements.

Let $M \prec V_\alpha$ be a countable elementary submodel. Then $\langle T, <_T\rangle \in M$ and $M \models$ "$\langle T, <_T\rangle$ is a tree satisfying the conditions of DC". Let T' be $M \cap T$. Then T' is a countable subtree of T, moreover by elementarity, if $t \in T'$, then $V_\alpha \models$ "t is not maximal" and therefore M satisfies the same, so t is not maximal in T'. In other words, T' has height at least ω and no maximal elements. But if $t \in T$, then it is impossible that t lies in T_ω, since $T_\omega = \varnothing$, so $T_\omega \cap M = \varnothing$. Therefore T' is a countable subtree of T which also satisfies the assumptions of DC.

By Exercise 6.12 we get that T' has a branch B. But now B is also a branch in T, since it meets every T_n, thus proving DC. $\square$

Definition 6.18. We say that $\mathsf{W}_{|x|}$ holds if every cardinal is comparable with $|x|$. As before, $\mathsf{W}_{<|x|}$ means that for every y for which $|y| < |x|$, $\mathsf{W}_{|y|}$ holds.

Exercise 6.15. Show that if $\mathsf{W}_{|x|}$ holds, then $|x|$ is an initial ordinal. Namely, x can be well-ordered.

Exercise 6.16. $W_{\aleph_0}$ holds if and only if every Dedekind-finite set is finite.

Exercise 6.17. Suppose that α is a limit ordinal such that for some $\eta < \alpha$, there is a sequence $\{\alpha_\gamma \mid \gamma < \eta\}$ for which $\sup\{\alpha_\gamma \mid \gamma < \eta\} = \alpha$. Then $\mathsf{DC}_{<\aleph_\alpha}$ implies $\mathsf{DC}_{\aleph_\alpha}$, $\mathsf{AC}_{<\aleph_\alpha}$ implies $\mathsf{AC}_{\aleph_\alpha}$, and if $\mathsf{AC}_{\aleph_\beta}$ holds and $W_{<\aleph_\alpha}$ holds, then $\mathsf{W}_{\aleph_\alpha}$ holds.

Definition 6.19. Let x be a non-empty set. We say that $\mathcal{F}$ is a *filter* on x, if $\mathcal{F} \subseteq \mathcal{P}(x)$ satisfying the following properties:

1. $x \in \mathcal{F}$,
2. if $a, b \in \mathcal{F}$, then $a \cap b \in \mathcal{F}$,
3. if $a \in \mathcal{F}$ and $a \subseteq b \subseteq x$, then $b \in \mathcal{F}$.

We will usually require that $\varnothing \notin \mathcal{F}$. In the case that $\varnothing \in \mathcal{F}$ we say that $\mathcal{F}$ is the *improper filter*, and we will always mention explicitly when we allow the improper filter in a statement.

Exercise 6.18. Let x be a non-empty set, then $\mathcal{F}_{fin} = \{a \subseteq x \mid x \setminus a \text{ is finite}\}$ is a filter on x and it is the improper filter if and only if x is finite.

Definition 6.20. Let $\mathcal{F}$ be a filter on x. We say that $\mathcal{F}$ is *principal* if $\bigcap \mathcal{F} \in \mathcal{F}$. We say that $\mathcal{F}$ is an *ultrafilter* if for every $a \subseteq x$ either $a \in \mathcal{F}$ or $x \setminus a \in \mathcal{F}$.

Exercise 6.19. If $\mathcal{F}$ is a filter on x and $a \subseteq x$, then there is a filter $\mathcal{F}'$ such that $\mathcal{F} \cup \{a\} \subseteq \mathcal{F}'$ if and only if for every $b \in \mathcal{F}$, $a \cap b$ is non-empty.

Exercise 6.20. The intersection of filters is a filter, and the union of a $\subseteq$-increasing sequence of filters is also a filter. Deduce that if $\mathcal{F}$ is a filter on x, with $a \subseteq x$ such that $a \cap b \neq \varnothing$ for all $b \in \mathcal{F}$, then there is a smallest filter which contains $\mathcal{F}$ and a.

Exercise 6.21. $\mathcal{F}$ is an ultrafilter x if and only if there is no filter $\mathcal{F}'$ on x such that $\mathcal{F} \subsetneq \mathcal{F}'$.

Theorem 6.21. *If $\mathcal{P}(x)$ is well-orderable, then every filter on x can be extended to an ultrafilter.*

Proof. Fix a filter $\mathcal{F}$ on x. Enumerate $\mathcal{P}(x)$ as $\{a_\alpha \mid \alpha < \eta\}$, and by recursion define a $\subseteq$-increasing sequence of filters: $\mathcal{F}_0 = \mathcal{F}$, and $\mathcal{F}_\alpha + 1$ is the smallest filter such that $\mathcal{F}_\alpha \cup \{a_\alpha\}$ if there is such filter, or $\mathcal{F}_\alpha$ otherwise. Then $\mathcal{F}_\eta$ is the increasing union of filters on x, therefore itself is a filter on x, and given any $a \subseteq x$, there is some α such that $a = a_\alpha$, and therefore either $a_\alpha \in \mathcal{F}_{\alpha+1}$ or its complement is there. Therefore $\mathcal{F}_\eta$ is indeed an ultrafilter. $\square$

Exercise 6.22 (*). If every filter can be extended to an ultrafilter, then every set can be linearly ordered.

Exercise 6.23 (*). If A is an infinite amorphous set, then A cannot be linearly ordered.

Definition 6.22. *The Partition Principle*, abbreviated as PP states that $|x| \leq^* |y|$ if and only if $|x| \leq |y|$. In other words, if there is a surjective function from y onto x, then there is an injective function from x into y.

Exercise 6.24. Show that PP implies that for every x, $\aleph(x) = \aleph^*(x)$.

Exercise 6.25. Show that PP implies that $\leq^*$ is anti-symmetric. Namely, there is a Cantor–Bernstein theorem for the $\leq^*$ relation on the cardinal. Use this to prove there are no infinite Dedekind-finite sets.

Remark. It is open whether or not PP implies the Axiom of Choice. As of 2016, this is the oldest open problem in set theory.

Chapter 7

Sets of Ordinals

In this chapter we work in ZFC, that is ZF + AC. Unless stated otherwise.

7.1 Cofinality

Definition 7.1. Let α be an ordinal. $A \subseteq \alpha$ is *cofinal* (in α) if $\sup A = \alpha$. The *cofinality* of α is the least ordinal δ such that there is a cofinal $A \subseteq \alpha$ such that $\operatorname{otp}(A) = \delta$. We denote this as $\operatorname{cf}(\alpha) = \delta$.

Using the Mostowski collapse, or rather its inverse, $\operatorname{cf}(\alpha) = \delta$ if and only if δ is the least ordinal such that there is an increasing function from δ into α whose range is cofinal in α.

Definition 7.2. We say that α is a *regular* ordinal if $\operatorname{cf}(\alpha) = \alpha$, and otherwise it is a *singular* ordinal.

Exercise 7.1. $\operatorname{cf}(\operatorname{cf}(\alpha)) = \operatorname{cf}(\alpha)$. Namely, $\operatorname{cf}(\alpha)$ is always regular.

Exercise 7.2. α is regular if and only if for all $A \subseteq \alpha$, if $|A| < |\alpha|$ then $\sup A < \alpha$.

Exercise 7.3 (*). If α is regular, then α is a cardinal. But not every cardinal is regular.

As the consequence of these two exercises, $\operatorname{cf}(\alpha)$ is always an infinite cardinal. We will sometimes be interested in this cardinal in the context of cardinal arithmetic, so we will write things like $\aleph_\alpha^{\operatorname{cf}(\aleph_\alpha)}$ to hint that we are interested in the cardinal arithmetic of these sets, rather than their ordinal arithmetic.

Exercise 7.4. If there is a function $f \colon \delta \to \alpha$ which is not decreasing and $\operatorname{rng} f$ is cofinal in α, then $\operatorname{cf}(\alpha) = \operatorname{cf}(\delta)$. In other terms, if $A \subseteq \alpha$ is cofinal, then $\operatorname{cf}(\operatorname{otp}(A)) = \operatorname{cf}(\alpha)$.

Theorem 7.3. *Let α be an ordinal. If α is not a limit ordinal, then ω_α is regular; if α is a limit ordinal, then $\operatorname{cf}(\omega_\alpha) = \operatorname{cf}(\alpha)$.*

Proof. For $\alpha = 0$ we get $\omega_0 = \omega$, and of course that every finite set of finite ordinals is bounded below ω. If α is a limit ordinal then $\{\omega_\beta \mid \beta < \alpha\}$ is a cofinal subset of ω_α, so by the previous exercise $\operatorname{cf}(\alpha) = \operatorname{cf}(\omega_\alpha)$.

Finally, if $\alpha = \beta + 1$ and $A \subseteq \omega_\alpha$ has order type $\eta < \omega_\alpha$, then $|A| \leq \aleph_\beta$ and for every $\gamma \in A$, $\gamma < \omega_\alpha$, $|\gamma| \leq \aleph_\beta$. Therefore we can choose suitable injections and prove that $|\sup A| \leq \aleph_\beta \cdot \aleph_\beta = \aleph_\beta$. And by the very definition of ω_α as $\omega_{\beta+1}$ we get that $\sup A < \omega_\alpha$. $\square$

Remark. It is consistent with ZF that ω_1, and indeed that every limit ordinal, has cofinality ω.

Definition 7.4. We define $H(\omega_\alpha)$ to be the set $\{x \mid |\operatorname{tcl}(x)| < \aleph_\alpha\}$.

Exercise 7.5. Show that $H(\omega_\alpha)$ is a continuous filtration of V, and conclude that it satisfies a Reflection theorem.

Exercise 7.6. If $\omega_\alpha > \omega$ is regular, then $H(\omega_\alpha)$ satisfies ZFC$^-$, namely ZFC without Power Set.

7.2 Some cardinal arithmetic

As we have seen before, for $\aleph$ numbers the basic cardinal arithmetic is fairly simple:

$$\aleph_\alpha + \aleph_\beta = \aleph_\alpha \cdot \aleph_\beta = \aleph_{\max\{\alpha,\beta\}}.$$

Using the axiom of choice, we can make infinite arithmetic well-defined. The reason choice is needed is that when we want to ensure two infinite unions have the same cardinality, we need to choose bijections between the sets we unify. If there are finitely many, this is not an issue, but for infinitely many this can become problematic.

Definition 7.5. We define $\sum_{i\in I} |a_i|$ as $|\bigcup\{\{i\} \times a_i \mid i \in I\}|$ and $\prod_{i\in I} |a_i|$ as $|\prod_{i\in I} a_i|$.

Exercise 7.7. The definitions of infinite addition and multiplication are well-defined. Moreover, if $|a_i| = |a|$ for all i, then $\sum_{i\in I} |a_i| = |I| \cdot |a|$ and $\prod_{i\in I} |a_i| = |a|^{|I|}$.

Exercise 7.8. $|a|^{\sum_{i\in I} |b_i|} = \prod_{i\in I} |a|^{|b_i|}$.

Exercise 7.9 (*). For every sequence of sets if $|I| \geq \aleph_0$, then $\sum_{i\in I} |a_i| = |I| \cdot \sup\{|a_i| \mid i \in I\}$.

Exercise 7.10. $\operatorname{cf}(\omega_\alpha) = \delta$ if and only if for all $|I| < \delta$, and for all $i \in I$, $|A_i| < \aleph_\alpha$, $\sum_{i\in I} |A_i| < \aleph_\alpha$.

Proposition 7.6. *If $\alpha \leq \beta$, then $\aleph_\alpha^{\aleph_\beta} = 2^{\aleph_\beta}$.*

Proof.

$$2^{\aleph_\beta} \leq \aleph_\alpha^{\aleph_\beta} \leq \left(2^{\aleph_\beta}\right)^{\aleph_\beta} = 2^{\aleph_\beta \cdot \aleph_\beta} = 2^{\aleph_\beta}. \qquad \square$$

Theorem 7.7 (Hausdorff's formula). *For all α and β, $\aleph_{\alpha+1}^{\aleph_\beta} = \aleph_{\alpha+1} \cdot \aleph_\alpha^{\aleph_\beta}$.*

Proof. If $\alpha \leq \beta$, then

$$\aleph_\alpha^{\aleph_\beta} \leq \aleph_{\alpha+1}^{\aleph_\beta} \leq \left(2^{\aleph_\alpha}\right)^{\aleph_\beta} \leq 2^{\aleph_\alpha \cdot \aleph_\beta} = 2^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta}.$$

If $\alpha > \beta$, then every function from ω_β to $\omega_{\alpha+1}$ is bounded, since $\omega_{\alpha+1}$ is regular. For every $\eta < \omega_{\alpha+1}$ there are at most $\aleph_\alpha^{\aleph_\beta}$ functions from ω_β into η, and there are $\aleph_{\alpha+1}$ such η's and so the calculation follows. $\square$

Theorem 7.8 (König's lemma). *If for all $i \in I$, $\lambda_i < \kappa_i$ are cardinals. Then*

$$\sum_{i\in I} \lambda_i < \prod_{i\in I} \kappa_i.$$

Proof. For each $i \in I$, let B_i be a set of size κ_i and $A_i \subseteq B_i$ a subset of size λ_i, we may further assume that for $i \neq j$, $B_i \cap B_j = \varnothing$. Take any $F\colon \bigcup\{A_i \mid i \in I\} \to \prod_{i\in I} B_i$, then for every i, the set $X_i = \{F(a)(i) \mid a \in A_i\}$ has cardinality at most λ_i. Therefore $\{B_i \setminus X_i \mid i \in I\}$ is a family of non-empty sets. Let f be a choice function from this family, then for every $i \in I$, there is no $a \in A_i$ such that $F(a) = f$ as the two must differ on i. Therefore F is not surjective, and the conclusion follows. $\square$

Remark. It is consistent with ZF that $\mathbb{R}$ is a countable union of countable sets. In such situation König's lemma fails as we can take $\lambda_n = \aleph_0$ and $\kappa_n = 2^{\aleph_0}$ (which is not well-orderable!). Of course the problem is deeper there: infinite summation and products of cardinals are not well-defined.

Corollary 7.9. $\kappa < \kappa^{\mathrm{cf}(\kappa)}$.

Proof. If κ is some $\aleph_{\alpha+1}$ or ω, then this is just a consequence of Cantor's theorem and basic cardinal arithmetic. Otherwise, κ is a limit cardinal. Let $\{\lambda_i \mid i < \mathrm{cf}(\kappa)\}$ be a strictly increasing sequence of cardinals such that $\sup\{\lambda_i \mid i < \mathrm{cf}(\kappa)\} = \kappa$. Let $\kappa_i = \lambda_{i+1}$, then for all $i < \mathrm{cf}(\kappa)$, $\lambda_i < \kappa_i$. Therefore

$$\kappa = \sum_{i<\mathrm{cf}(\kappa)} \lambda_i < \prod_{i<\mathrm{cf}(\kappa)} \kappa_i \leq \prod_{i<\mathrm{cf}(\kappa)} \kappa = \kappa^{\mathrm{cf}(\kappa)}. \qquad \square$$

Corollary 7.10. $\mathrm{cf}(2^{\aleph_0}) > \aleph_0$. *In particular* $\aleph_\omega \neq 2^{\aleph_0}$. $\square$

Remark. It was shown by Cohen and Solovay that this is in fact the only restriction on the continuum in ZFC.

Exercise 7.11. Prove that $\aleph_\omega^{\aleph_1} = \aleph_\omega^{\aleph_0} \cdot 2^{\aleph_1}$.

Definition 7.11. Let κ and λ be cardinals, we define the *weak power* as

$$\kappa^{<\lambda} = \sup\{\kappa^\mu \mid \mu < \lambda\}.[1]$$

Definition 7.12. We say that an infinite cardinal κ is a *strong limit cardinal* if for all $\lambda < \kappa$, $2^\lambda < \kappa$.

Exercise 7.12. Show that if κ is a strong limit cardinal, then it is a limit cardinal.

Definition 7.13. We say that κ is a *weakly inaccessible cardinal* if it is a regular limit cardinal. We say that κ is a (strongly) *inaccessible cardinal* if it is a strong limit and weakly inaccessible cardinal.

Exercise 7.13. Show that if $\kappa^{<\kappa} = \kappa$, then κ is a regular cardinal. Show that if κ is inaccessible, then $\kappa^{<\kappa} = \kappa$.

Exercise 7.14. Show that if κ is a strong limit cardinal such that $\kappa = \aleph_\kappa$, then $|V_\kappa| = \kappa$.

Exercise 7.15. Show that if κ is inaccessible, then $V_\kappa \models \mathrm{ZFC}$. In particular, show that ZFC does not prove the existence of inaccessible cardinals.

7.3 Clubs and stationary sets

Definition 7.14. Let α be a limit ordinal, and let $C \subseteq \alpha$. We say that C is a *closed set* if whenever $\beta < \alpha$ and $\sup(C \cap \beta) = \beta$, then $\beta \in C$. We say that C is unbounded if $\sup C = \alpha$. If C is both closed and unbounded we say that it is a club set.[2]

Definition 7.15. We say that $S \subseteq \alpha$ is a *stationary set* if whenever $C \subseteq \alpha$ is a club, then $S \cap C \neq \varnothing$.

Exercise 7.16. If $S \subseteq \alpha$ is a stationary set, then S is unbounded.

[1] Here μ is a cardinal, of course, but we can replace it by $\kappa^{|\mu|}$.

[2] Club is an abbreviation for "CLosed and UnBounded". In some places this is abbreviated as "cub" instead.

Exercise 7.17. Show that if C is a club in a limit ordinal α such that $\mathrm{cf}(\alpha) > \omega$, then $C \cap \mathrm{Lim}$ is a club in α as well. And show that if λ is a regular cardinal, then $S^\kappa_\lambda = \{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \lambda\}$ is stationary. In particular, $S^{\omega_2}_\omega$ and $S^{\omega_2}_{\omega_1}$ are two disjoint stationary subsets of ω_2.

For the remainder of the section, we will always assume that κ is a regular uncountable cardinal. Whenever we say club or stationary set without qualifications, we will mean as a subset of κ.

Definition 7.16. We say that a function $f\colon \kappa \to \kappa$ is a *normal function* if it is increasing and continuous. Namely, $f(\alpha) < f(\beta)$ whenever $\alpha < \beta$ and if δ is a limit ordinal, then $f(\delta) = \sup\{f(\alpha) \mid \alpha < \delta\}$.

Both the term "continuous" and "closed" that we use here are justified topologically when considering an ordinal as a topological space, using the order topology.

Remark. This definition also makes sense in the context of class functions from Ord to itself. For example, ordinal arithmetic, as well as the function $\alpha \mapsto \omega_\alpha$.

Exercise 7.18. C is a club if and only if there is a normal function f such that $C = \operatorname{rng} f$.

Theorem 7.17. *S is stationary if and only if for every normal function f, there is some $\alpha \in S$ such that $f(\alpha) = \alpha$.*

We first prove the following lemma.

Lemma 7.18. *If f is a normal function, then $\{\alpha < \kappa \mid f(\alpha) = \alpha\}$ is a club.*

Proof of Lemma 7.18. The fact that the set is closed is easy. To see it is unbounded, take any $\alpha_0 < \kappa$, and define $\alpha_{n+1} = f(\alpha_n)$ and $\alpha = \sup\{\alpha_n \mid n < \omega\}$. Then

$$f(\alpha) = \sup\{f(\alpha_n) \mid n < \omega\} = \sup\{\alpha_{n+1} \mid n < \omega\} = \alpha.$$

Therefore there is some $\alpha \geq \alpha_0$ such that $f(\alpha) = \alpha$. □

Proof of Theorem 7.17. If S is stationary, then by the lemma, the set of fixed points is a club so its intersection with S is non-empty. In the other direction, if S is a non-stationary set, then there is some normal function f such that $\operatorname{rng} f \cap S = \varnothing$, and in particular S does not contain any fixed points for f. □

Exercise 7.19. Prove there is a cardinal μ such that $\mu = \aleph_\mu$. Moreover, show that there is one which is in fact a strong limit cardinal.

Proposition 7.19. *Suppose $\mathrm{cf}(\alpha) = \kappa > \omega$, then there is a continuous function $f\colon \kappa \to \alpha$ whose range is cofinal.*

Proof. Let $g\colon \kappa \to \alpha$ be a function witnessing that $\mathrm{cf}(\alpha) = \kappa$. Define f by recursion:

$$f(\gamma) = \sup\{g(\beta) + 1, f(\beta) + 1 \mid \beta < \gamma\}.$$ □

Note that for $\mathrm{cf}(\alpha) = \omega$, the above proposition is trivial, since any cofinal ω sequence is automatically a continuous function from ω into α. But what this means for the case where $\mathrm{cf}(\alpha) > \omega$ is that we can translate statements about clubs in α to statements about clubs in $\mathrm{cf}(\alpha)$.

Exercise 7.20. Suppose that $\mathcal{L}$ is a countable first-order language, and let M be a structure in $\mathcal{L}$ whose universe is κ, an uncountable regular cardinal. Then there is a club C such that for all $\alpha \in C$ the substructure of M, M_α whose universe is α, satisfies $M_\alpha \prec M$.

7.4 The Club filter

Proposition 7.20. *The intersection of two clubs is a club.*

Proof. Suppose that C and D are clubs, if $\eta < \kappa$ such that $\sup(C \cap D \cap \eta) = \eta$, then in particular $\sup(C \cap \eta) = \eta = \sup(D \cap \eta)$. Therefore $\eta \in C$ and $\eta \in D$, so $C \cap D$ is closed.

Suppose that $\eta < \kappa$ is any ordinal, then we construct $\alpha_0 = \eta$, α_{2n+1} is the least ordinal in C such that $\alpha_{2n} < \alpha_{2n+1}$ and α_{2n+2} is the least ordinal in D such that $\alpha_{2n+1} < \alpha_{2n+2}$. These ordinals exist since neither C nor D is bounded. Let $\alpha = \sup\{\alpha_n \mid n < \omega\}$, then $\alpha \in C \cap D$. Therefore $C \cap D$ is unbounded as wanted. $\square$

Exercise 7.21. If S is a stationary set and C is a club, then $S \cap C$ is stationary.

Theorem 7.21. *If $\gamma < \kappa$, and $\{C_\alpha \mid \alpha < \kappa\}$ is a family of clubs, then $C = \bigcap\{C_\alpha \mid \alpha < \gamma\}$ is also a club.*

Proof. Suppose that $\eta < \kappa$ is an ordinal, such that $C \cap \eta$ is unbounded in η. Then for every $\alpha < \gamma$, $C_\alpha \cap \eta$ is unbounded in η. Therefore $\eta \in C_\alpha$ for all $\alpha < \gamma$, so $\eta \in C$.

Suppose that $\eta < \kappa$. Similar to the previous proof, we construct an increasing sequence of order-type $\gamma \cdot \omega$ such that $c_{\gamma \cdot n+\alpha} \in C_\alpha$, and $c_0 > \eta$. If $\beta = \sup\{c_{\gamma\cdot n+\alpha} \mid n < \omega, \alpha < \gamma\}$, then easily $\beta \in C_\alpha$ for all α and therefore C is indeed unbounded. $\square$

Corollary 7.22. *Let $\mathcal{F}$ be the filter generated by all the club sets, namely $A \in \mathcal{F}$ if and only if A contains a club. Then $\mathcal{F}$ is closed under $<\kappa$-intersections. Such filter is called a κ-complete filter.*

At the same time, it is clear that the intersection of κ clubs need not be a club itself, just consider $C_\alpha = \kappa \setminus \alpha$ to see that $\bigcap\{C_\alpha \mid \alpha < \kappa\} = \varnothing$. However, we can somewhat correct for this problem.

Definition 7.23. Let $\gamma \leq \kappa$, the *diagonal intersection* of $\{C_\alpha \mid \alpha < \gamma\}$ is the following set:

$$\triangle\{C_\alpha \mid \alpha < \gamma\} = \{\beta < \kappa \mid \beta \in \bigcap\{C_\alpha \mid \alpha < \beta\}\}.$$

Easily, if $\gamma < \kappa$, then the diagonal intersection is just the intersection, at least above γ itself. But for $\gamma = \kappa$ this is no longer true. Nevertheless, the following theorem shows that the situation is still under control.

Theorem 7.24. *Suppose that C_α is a club for $\alpha < \kappa$, then $C = \triangle\{C_\alpha \mid \alpha < \kappa\}$ is a club.*

Proof. Suppose that $\eta < \kappa$ such that $C \cap \eta$ is unbounded. Then for every $\alpha < \eta$, the set $\{\beta \in C \mid \alpha < \beta < \eta\}$ is a subset of C_α, therefore $C_\alpha \cap \eta$ is unbounded for every $\alpha < \eta$. As each C_α is a club, $\eta \in C_\alpha$ for all $\alpha < \eta$ and so $\eta \in C$.

Suppose that $\eta < \kappa$ is any ordinal, pick $\alpha_0 \in C_0$ such that $\alpha_0 > \eta$. Suppose α_n was chosen, take α_{n+1} to be an ordinal in $\bigcap\{C_\beta \mid \beta < \alpha_n\}$ such that $\alpha_{n+1} > \alpha_n$. Let $\alpha = \sup\{\alpha_n \mid n < \omega\}$. Then for every $\beta < \alpha$, $C_\beta \cap \alpha$ is unbounded below α, since for all large enough n, $\alpha_n \in C_\beta$. Therefore $\alpha \in C$, and therefore C is unbounded. $\square$

Corollary 7.25 (Fodor's lemma). *Suppose that S is a stationary set and $f\colon S \to \kappa$ is regressive, i.e. $f(\alpha) < \alpha$ for all $\alpha \in S$. Then there is some β such that $\{\alpha \mid f(\alpha) = \beta\}$ is stationary.*

Proof. Suppose not, then for every $\alpha < \kappa$, there is some C_α which is a club and disjoint from $\{\beta \mid f(\beta) = \alpha\}$. Let $C = \triangle\{C_\alpha \mid \alpha < \kappa\}$, by the theorem, C is a club and in particular non-empty. But if $\alpha \in C \cap S$, then for all $\beta < \alpha$, $f(\alpha) \neq \beta$. This is impossible since f was regressive. $\square$

Definition 7.26. We say that a filter $\mathcal{F}$ on κ is *normal* if whenever f is a regressive function on κ, it is constant on some S such that $\kappa \setminus S \notin \mathcal{F}$.

Exercise 7.22. Let κ be a regular uncountable cardinal. If $\mathcal{F}$ is a normal filter such that for every α, $\kappa \setminus \alpha \in \mathcal{F}$, then $\mathcal{F}$ contains the club filter.

Exercise 7.23 (Solovay's theorem) (**). If S is a stationary subset of κ, then there is a partition of S into $\{S_\alpha \mid \alpha < \kappa\}$ such that S_α is stationary for all α.

Exercise 7.24. (*) Suppose that a train has $\omega_1 + 1$ stations. It embarks from station 0 empty. When it stops at station α, if it has any passengers, one of them will get off. Then countably many new passengers will get on the train, and it continues to the next station. How many passengers are on the train when it reaches its final destination, station ω_1?

Proposition 7.27. *Suppose that $\kappa > \aleph_0$ is weakly inaccessible, then $\kappa = \aleph_\kappa$.*

Proof. Let α be such that $\kappa = \aleph_\alpha$. Since α is a limit ordinal, we have that $\operatorname{cf}(\kappa) = \operatorname{cf}(\alpha) = \kappa$. So $\alpha \geq \kappa$, but since there is a club in κ of order type α, namely $\{\omega_\eta \mid \eta < \alpha\}$, we get equality. $\square$

Exercise 7.25. Show that if $\kappa > \aleph_0$ is weakly inaccessible, then the set $\{\alpha \mid \alpha = \aleph_\alpha\}$ is a club below κ. Show that if κ is a strongly inaccessible cardinal, then the set $\{\alpha \mid \alpha \text{ is a strong limit cardinal}\}$ is also a club below κ.

Theorem 7.28. *Let κ be an ordinal such that $\operatorname{cf}(\kappa) > \aleph_0$. Suppose that $\{\alpha \mid \operatorname{cf}(\alpha) = \alpha\}$ is a stationary set of κ. Then κ is a weakly inaccessible cardinal, and it is not the first weakly inaccessible cardinal.*

Proof. First note that κ is a cardinal, since it is a limit of cardinals; and it is in fact a limit cardinal, since otherwise κ is the successor of some λ, and then $\{\alpha \mid \lambda < \alpha < \kappa\}$ does not contain any cardinals, but it is a club in κ.

Since κ is a limit cardinal, the set of cardinals below κ is a club, therefore the set of limit cardinals is a club, and by the assumption, it contains a regular cardinal—in fact many regular cardinals—which is to say that there is some weakly inaccessible cardinal below κ. Finally, if $\delta = \operatorname{cf}(\kappa) < \kappa$, then there is a function $f\colon \delta \to \kappa$ which is continuous and unbounded, so $\operatorname{rng} f$ is a club. Look at the club $C = \operatorname{rng}(f \restriction \mathrm{Lim})$, if $\alpha \in C$, then $\operatorname{cf}(\alpha) < \delta$. Therefore every regular cardinal in C must be at most δ, so the set of regular cardinals is not stationary after all. $\square$

Definition 7.29. An uncountable cardinal with the property that regular cardinals (equivalently, inaccessible cardinals) below it form a stationary set is called a *weakly Mahlo cardinal*.

Chapter 8

Inner models of ZF

8.1 Inner models

Definition 8.1. We say that a class M is an *inner model* if it is transitive, $\mathrm{Ord} \subseteq M$ and for every axiom φ of ZF, φ^M holds.

Definition 8.2. A class M is called *almost universal* if whenever x is a set, and $x \subseteq M$, then there is some $y \in M$ such that $x \subseteq y$.

Proposition 8.3. *If M is almost universal, then M is a proper class.*

Proof. Suppose otherwise, then $M \subseteq M$, and therefore for some $y \in M$ we have that $M \subseteq y$. This means that $y \in y$, a contradiction to Foundation. □

Definition 8.4. Bounded Separation, or Δ_0-Separation, is the schema of Separation restricted only for Δ_0 formulas. Similar definitions can be made for Replacement as well as more complex classes of formulas (e.g. Σ_1-Replacement).

Theorem 8.5. *If M is a transitive class which is almost universal and satisfies Δ_0-Separation, then M is an inner model of V.*

Proof. First we claim that $\mathrm{Ord} \subseteq M$, to see this let α be such that $\alpha \subseteq M$, then by almost universality there is some $y \in M$ such that $\alpha \subseteq y$. By Δ_0-Separation, and the fact that Ord is definable by a Δ_0 formula, $y \cap \mathrm{Ord} \in M$. As $\alpha \subseteq y$, either $y \cap \mathrm{Ord} = \alpha$ in which case $\alpha \in M$ or there is some $\gamma \in y \cap \mathrm{Ord}$ such that $\alpha < \gamma$, and then by the transitivity of M we get that $\alpha \in M$.

We start verifying the axioms: Extensionality, Empty Set, Infinity and Foundation follow from the fact that M is a transitive class and $\omega \in M$.

Next, we claim: If $x \in M$, then $\mathcal{P}^M(x) = \mathcal{P}(x) \cap M \in M$. Recall $\mathcal{P}^M(x) = \{u \in M \mid u \subseteq x\}$, so clearly $\mathcal{P}^M(x) = \mathcal{P}(x) \cap M$. Suppose now that $x \in M$, then by almost universality there is some $y \in M$ such that $\mathcal{P}^M(x) \subseteq y$. Consider the Δ_0 formula, $u \subseteq x$ (recall this is a shorthand for $\forall v(v \in u \to v \in x)$), then $y' = \{u \in y \mid u \subseteq x\} \in M$ as it was obtained by Δ_0-Separation from y, using x as a parameter. But as $\mathcal{P}^M(x) \subseteq y$, it means that $y' = \mathcal{P}^M(x)$. Therefore M satisfies the Power Set axiom.

It remains to prove that Replacement holds, which will imply that Separation holds as well. For this we first prove that for all α, $V_\alpha^M = M \cap V_\alpha$: for $\alpha = 0$ this is just $\varnothing$; for successor steps this holds from the Power Set axiom in M:

$$V_{\alpha+1}^M = \mathcal{P}^M(V_\alpha^M) = \mathcal{P}(V_\alpha^M) \cap M = \mathcal{P}(V_\alpha \cap M) \cap M = \mathcal{P}(V_\alpha) \cap M = V_{\alpha+1} \cap M;$$

and for limit cases this follows from the fact that $\bigcup\{V_\beta \cap M \mid \beta < \alpha\} = \bigcup\{V_\beta \mid \beta < \alpha\} \cap M$.

Let $\varphi(u, v, \bar{p})$ be a formula such that for some $\bar{p}, x \in M$, $M \models (\forall u \in x)\exists! v\varphi(u, v, \bar{p})$. Then this means that $V \models \Big((\forall u \in x)\exists! v\varphi(u, v, \bar{p})\Big)^M$. By the Reflection theorem there is some β large enough such that $\bar{p}, x \in V_\beta$ and $V \models \left(\Big((\forall u \in x)\exists! v\varphi(u, v, \bar{p})\Big)^M\right)^{V_\beta}$.

It is not hard to check that if A and B are transitive classes, then $\left(\psi^A\right)^B$ is equivalent to $\psi^{A\cap B}$. Therefore, the Reflection theorem gives us that $V \models \Big((\forall u \in x)\exists! v\varphi(u, v, \bar{p})\Big)^{V_\beta^M}$. But being a Δ_0 sentence where all the parameters ($\bar{p}, x$ and V_β^M) are in M, removing the $\exists! v$ gives us the following Δ_0 formula:

$$\psi(v, x, \bar{p}, X) = x \in X \wedge \bar{p} \in X \wedge (\exists u \in x)\varphi^X(u, v, \bar{p}).$$

Placing V_β^M as X gives us the set $\{v \mid (\exists u \in x)\varphi(u, v, \bar{p})\} \in M$ as wanted. $\square$

Proposition 8.6. *If $\mathcal{P}^M(x) = \mathcal{P}(x)$ for all $x \in M$, then $M = V$.*

Proof. Since $M \subseteq V$, it is enough to prove that $V \subseteq M$, and for that it is enough to verify that for all α, $V_\alpha \subseteq M$. We will show that $V_\alpha = V_\alpha^M$, which is certainly enough.

For $\alpha = 0$, this is trivial. At successor steps,

$$V_{\alpha+1}^M = \mathcal{P}^M(V_\alpha^M) = \mathcal{P}^M(V_\alpha) = \mathcal{P}(V_\alpha) = V_{\alpha+1}.$$

Finally, for limit steps, $V_\alpha^M = \bigcup\{V_\beta^M \mid \beta < \alpha\} = \bigcup\{V_\beta \mid \beta < \alpha\} = V_\alpha$. $\square$

Proposition 8.7. *If α is a cardinal in V, it is a cardinal in M. And $\omega_1^M \leq \omega_1$.*

Proof. Recall that $\varphi(\alpha)$ stating that α is cardinal is a Π_1 formula, since $\alpha \in M$, and M is transitive, it is downwards absolute. In the other direction, if $\alpha < \omega_1^M$, then $M \models \alpha$ is countable. But as we saw being a countable set is a Σ_1 property, which is therefore upwards absolute, so α is countable in V. So the least uncountable ordinal of M cannot be larger than the least uncountable ordinal in V. $\square$

Theorem 8.8 (Balcar–Vopěnka). *Suppose that M, N are two inner models such that for every α, $\mathcal{P}^M(\alpha) = \mathcal{P}^N(\alpha)$ and $M \models \mathsf{AC}$, then $M = N$.*

Proof. First note that having the same sets of ordinals means also having the same sets of pairs of ordinals. And so on. This is because we can define a bijection between pairs of ordinals and ordinals (using Theorem 4.16).

First we will show that $M \subseteq N$. For $x \in M$, first fix a bijection between $\operatorname{tcl}(\{x\})$ and some ordinal α. This bijection induces a binary relation E on α which codes the $\in$ relation on $\operatorname{tcl}(\{x\})$. By the above remark, $E \in N$. Now $\langle \alpha, E\rangle$ is a set-like, extensional and well-founded structure, so we can collapse it. But its Mostowski collapse must be equal to $\operatorname{tcl}(\{x\})$. Therefore $\operatorname{tcl}(\{x\}) \in N$, and so $x \in N$.

In the other direction, we prove by $\in$-induction that every $x \in N$ also lies in M. Let $x \in N$ such that $x \subseteq M$, let $y \in M$ such that $x \subseteq y$ (e.g. V_α^M for a suitable α). Let $f \colon y \to \beta$ be some bijection in M, then it is also in N. But now $f"x$ is a set of ordinals in N, and therefore it lies in M. Since both f and $f"x$ are in M, it follows that $x \in M$ as well. $\square$

Remark. The Axiom of Choice plays a crucial role in this proof. It is consistent that there are two models of ZF with the same sets of ordinals, but not with the same sets of sets of ordinals.

Theorem 8.9. *Suppose that F is a function defined by recursion from a function G which was Σ_n for $n \geq 1$, then F is Σ_n as well.*

Proof. Note that being an ordinal is a Δ_0 formula, and $F(\alpha) = y$ if and only if there is a function f whose domain is α coding the construction of $F(\alpha)$. □

Corollary 8.10. *Suppose that $\mathcal{L}$ is a first order language, then the formula $\varphi(x, \mathcal{L})$ stating that x is a term, a formula, or a sentence in the language $\mathcal{L}$ is Δ_0 with parameters $\mathcal{L}$. In particular being a formula is absolute for infinite transitive classes. Moreover, if A is a structure for $\mathcal{L}$ and σ is an assignment, then $A \models_\sigma x$ is a Δ_0 formula with parameters A and σ.*

Proof. If a transitive class (or set) is infinite, then it contains all the finite ordinals. Note that ω is Δ_0-definable (it is either the set of ordinals, or an element). The proof above works for any recursive definition such as being a term, etc. □

We can agree that if $\mathcal{L}$ is a countable language, then we can code it using finite ordinals. This means that terms, etc. are just elements of V_ω, being recursively constructed as sequences of sequences of sequences, etc.

Remark. This is an important place to make the distinction between the meta-theory and the theory. Namely, when we write $V \models \varphi$, this is a statement made in the meta-theory, whereas when A is an $\mathcal{L}$ structure and ψ some sentence in $\mathcal{L}$, then $A \models \psi$ is a statement about specific sets made *inside* V. This issue was also present in the Reflection theorem, where we make the move from the formulas in our meta-theory to formulas inside V, and the problem is that the meta-theory formulas (as well as the satisfaction relation) are not objects of V, instead they are objects of the meta-theory.

We can do that, however, because we can faithfully "recreate" the formal logic of the meta-theory inside the theory. While it is possible that V disagrees with its meta-theory on what are the natural numbers, which may cause an excess of formulas, inference rules, and other objects which are effectively coded by formulas, we still get a faithful copy of the meta-language inside V.

This means that if M is a transitive class such that $V_\omega \subseteq M$, and $\mathcal{L}, A \in M$ with A being an $\mathcal{L}$ structure, then $M \models$ "B is a definable subset of A" if and only if B is a definable subset of A.

8.2 Gödel's constructible universe

Definition 8.11. Let M be a transitive set,

$$\operatorname{Def}(M) = \{B \subseteq M \mid B \text{ is definable with parameters in } \langle M, \in \rangle\}.$$

Exercise 8.1. If M is finite, then $\operatorname{Def}(M) = \mathcal{P}(M)$.

Exercise 8.2. If M is infinite and well-orderable, then $|M| = |\operatorname{Def}(M)|$.

Exercise 8.3. If M is transitive, then $\operatorname{Def}(M)$ is transitive, and $M \in \operatorname{Def}(M)$.

Theorem 8.12. *If M is a transitive set, then $\operatorname{Def}(M)$ is the smallest transitive set such that $M \in \operatorname{Def}(M)$ and $\operatorname{Def}(M)$ satisfies Δ_0-Separation.*

Proof. It is clear that any transitive N satisfying Δ_0-Separation with $M \in N$ will also include very definable subset of M, so it is enough to show that $\operatorname{Def}(M)$ indeed satisfies this property. Suppose now that $\varphi(x, p)$ is a Δ_0 formula and let $A \in \operatorname{Def}(M)$ and $p \in \operatorname{Def}(M)$.[1]

[1] We want to show the proof for the case where parameters are allowed, as it gives better insight, but one parameter is plenty.

Then there is some definition (over M) for A, say $\varphi_A(x, \bar{q})$ with $\bar{q} \in M$, and a definition $\varphi_p(x, \bar{q})$ for p, also with $\bar{q} \in M$. Note that we can assume that the parameters are the same by allowing repetition and ignoring unneeded parameters. We prove by induction on the complexity of φ that $\{x \in A \mid \varphi(x, p)\} \in \mathrm{Def}(M)$.

- Suppose that φ is atomic, then it has the form $x \in p$ or $p \in x$ or $x = p$. All three are easily translated to formulas defining subsets of M.
- For negation, conjunction, disjunction and implication this is just complement, intersection, union and subsets, and $\mathrm{Def}(M)$ is clearly closed under all of these.
- Finally, for quantifiers we have $(\forall u \in p)\varphi(u, x)$ or $(\exists u \in p)\varphi(u, x, p)$ or $(\forall u \in x)\varphi(u, x, p)$ or $(\exists u \in x)\varphi(u, x, p)$. Let $\psi(u, x, \bar{y})$ denote the formula which defines the set definable from $\varphi(u, x, p)$ (note that here x acts as a parameter). First we consider the case with $\exists u \in p$:

  $$\{x \in A \mid (\exists u \in p)\varphi(u, x)\} = \left\{x \in M \;\middle|\; \varphi_A^M(x, \bar{q}) \wedge (\exists u(\varphi_p(u, \bar{q}) \wedge \psi^M(u, x, \bar{y}))\right\},$$

  clearly this results in a definable subset of M, and the case for $\forall u \in p$ is similar. The cases with p as a parameter of φ are proven using the induction hypothesis.

  $$\{x \in A \mid (\exists u \in x)\varphi(u, p)\} = \left\{x \in M \;\middle|\; \varphi_A^M(x, \bar{q}) \wedge (\exists u \in x)\psi^M(u, x, \bar{y})\right\}.$$

  Again, the case for $\forall u \in x$ is proved similarly. □

Exercise 8.4 (**). The function $Y = \mathrm{Def}(X)$ is a Δ_1 function.

Definition 8.13. We define by recursion Gödel's Constructible hierarchy.

1. $L_0 = \varnothing$.
2. $L_{\alpha+1} = \mathrm{Def}(L_\alpha)$.
3. $L_\alpha = \bigcup\{L_\beta \mid \beta < \alpha\}$ for $\alpha \in \mathrm{Lim}$.

Let L be $\bigcup\{L_\alpha \mid \alpha \in \mathrm{Ord}\}$. By $x \in L$ we mean $\exists\alpha(x \in L_\alpha)$, and $V = L$ to mean that $\forall x(x \in L)$.

Exercise 8.5. The function $\alpha \mapsto L_\alpha$ is a Δ_1 function. And for all $\alpha \geq \omega$, L_α satisfies Δ_0-Separation.

Theorem 8.14. *L satisfies* ZF*.*

Proof. It is enough to verify that the conditions of Theorem 8.5 hold. Easily $\mathrm{Ord} \subseteq L$, and by the existence of the constructible hierarchy, L is an almost universal class. It remains to check that L satisfies Δ_0-Separation.

For readability purposes, we will prove Δ_0-Separation for formulas without parameters. Suppose that $\varphi(u)$ is a Δ_0 formula, and let $x \in L$. Let $y = \{u \in x \mid \varphi(u)\}$, then $y \subseteq L$, so there is some α such that $x \in L_\alpha$ and $y \subseteq L_\alpha$. As L_α is transitive and φ is a Δ_0 formula, $\varphi(u)$ holds if and only if $L_\alpha \models \varphi(u)$. But this means that y is definable over L_α using x as a parameter, so $y \in L_{\alpha+1}$. Therefore L satisfies all the axioms of ZF. □

Theorem 8.15. *If M is an inner model of V, then $L^M = L$.*

Proof. By induction we will show that $L_\alpha^M = L_\alpha$. For $\alpha = 0$ this is clearly true; as is the case for $\alpha \in \mathrm{Lim}$:

$$L_\alpha^M = \bigcup\{L_\beta^M \mid \beta < \alpha\} = \bigcup\{L_\beta \mid \beta < \alpha\} = L_\alpha.$$

Suppose that $\alpha = \beta + 1$ and $L_\beta^M = L_\beta$, then by the absoluteness of the Def function, we get that $\mathrm{Def}^M(L_\beta) = \mathrm{Def}(L_\beta)$. □

Corollary 8.16. *The following are quick and important corollaries from the theorem.*

1. $L^L = L$.

2. $L \models V = L$.

3. *L is the smallest inner model.*

4. *If an inner model M satisfies $V = L$, then $M = L$.*

Exercise 8.6. If M is a transitive set satisfying Δ_0-Separation, and $V_\omega \in M$, then $L^M = L_{M \cap \mathrm{Ord}}$.

Exercise 8.7. For all $\alpha \in \mathrm{Lim}$, $(L_\alpha)^L = L_\alpha = L^{L_\alpha}$. So if $\alpha \in \mathrm{Lim}$, then $L_\alpha \models V = L$.

Exercise 8.8. Show that for an unbounded class of ordinals α, $V_\alpha \neq L_\alpha$. Prove that if $V = L$ holds, then there is a closed and unbounded class of ordinals for which $V_\alpha = L_\alpha$.

8.3 The properties of L

We wish to investigate the construction of L and the properties of the sets inside L, known as *constructible sets*.

Definition 8.17. Let x be a set in L. We define $\mathrm{rank}_L(x) = \alpha$ if $x \in L_{\alpha+1}$ but $x \notin L_\alpha$.

Exercise 8.9. Show that the formula $\varphi(x, \alpha)$ meaning that $\mathrm{rank}_L(x) = \alpha$ is a Δ_1 formula.

Theorem 8.18. AC^L.

Proof. We first define by induction a well-ordering $\prec_\alpha$ on L_α. We begin by fixing a well-ordering of the formulas in the language of set theory of order type ω, and let φ_n denote the nth formula in the enumeration. For $x \in L_{\alpha+1}$, let $n(x)$ denote the least n such that φ_n can be used to define x over L_α.

For $\alpha + 1$ we define, $x \prec_{\alpha+1} y$ if and only if $n(x) < n(y)$, or $n(x) = n(y)$ and the parameters used to define x appear in the lexicographic ordering induced by $\prec_\alpha$ before the parameters used to define y.

For $\alpha \in \mathrm{Lim}$, define $x \prec_\alpha y$ if and only if $\mathrm{rank}_L(x) < \mathrm{rank}_L(y)$, or $\mathrm{rank}_L(x) = \mathrm{rank}_L(y) = \beta$ and $x \prec_\beta y$.

Finally, define $x <_L y$ as we define $\prec_\alpha$ for the limit case. This is a definable well-ordering of L which defines a bijection between L and Ord, and therefore every set can be well-ordered, so AC holds in L. □

Theorem 8.19 (Gödel's Condensation Lemma). *If α is a limit ordinal and $M \prec L_\alpha$, then there is some β such that $M \cong L_\beta$.*

Proof. Since M is a well-founded structure, we can collapse it to a transitive set, N. We claim that $N = L_\beta$. Since L_α satisfies Δ_0-Separation, so must N. Therefore $L^N = L_\beta$ for $\beta = N \cap \mathrm{Ord}$. But $L_\alpha \models V = L$, so $N \models V = L$ as well, and therefore $N = L_\beta$. □

Exercise 8.10. If α is infinite, then $|L_\alpha| = |\alpha|$.

Recall Cantor's Continuum Hypothesis (CH), is the statement $2^{\aleph_0} = \aleph_1$.

Theorem 8.20. CH^L.

Proof. Assume $V = L$. Suppose that $A \subseteq \omega$, then there is some $\alpha \in \mathrm{Lim}$ such that $A \in L_\alpha$. Let M be a countable elementary submodel of L_α such that $A \in M$, and let $\pi\colon M \to L_\beta$ be the Mostowski collapse of M. Then $\pi(A) = A$, and β is a countable ordinal. Therefore if $A \subseteq \omega$, then $A \in L_{\omega_1}$. In particular, there are at most $|L_{\omega_1}| = \aleph_1$ subsets of ω. □

Exercise 8.11 (*). Show that $L \models \mathsf{GCH}$, namely for every α, $2^{\aleph_\alpha} = \aleph_{\alpha+1}$.

Exercise 8.12 (*). Show that $H(\omega_\alpha^L)^L = L_{\omega_\alpha^L}$.

We finish this section with a remark about generalizations of L:

1. If A is any set, then $L(A)$ is defined the same way as L, only L_0 is now $\mathrm{tcl}(\{A\})$. It can be shown that $L(A)$ is the smallest inner model in which A is an element; and choice holds there if and only if there is a definable well-ordering of A.
2. If A is any set, then $L[A]$ is defined by augmenting the first-order structure over which we take Def to include a predicate interpreted as A. Namely, $L_0 = \varnothing$, and $L_{\alpha+1}$ is the set of all definable sets in the structure $\langle L_\alpha, \in, A \cap L_\alpha[A]\rangle$. We can show that $L[A]$ is the smallest inner model of ZFC satisfying $A \cap M \in M$.

Both of these models have many uses throughout set theory. And one can show that if $A \subseteq L$, then $L(A) = L[A]$.

8.4 Ordinal definable sets

Definition 8.21. We say that x is *ordinal definable* if there is a formula in the language of set theory φ such that $x = \{u \mid \varphi(u, \alpha_1, \dots, \alpha_n)\}$ where $\alpha_1, \dots, \alpha_n \in \mathrm{Ord}$.

Theorem 8.22. *$\{x \mid x$ is ordinal definable$\}$ is a class. We shall denote it by by* OD.

Proof. If x is ordinal definable and $\varphi(u, \alpha)$ is the formula defining it, by the reflection theorem there is some β large enough such that $x, \alpha \in V_\beta$ and V_β reflects the fact that φ defines x (with the parameter α). Therefore if $x \in \mathrm{OD}$, then there is some β and a formula in the (internal) language of set theory $\psi(u, \alpha)$ such that $V_\beta \models x = \{u \mid \psi(u, \alpha)\}$. On the other hand, if x is ordinal definable in some V_β by $\varphi(u, \alpha)$, then there is a formula $\psi(u, \alpha, \beta)$ stating that there exists y which is a transitive set, closed under the power set operation, and its ordinals are β—i.e. $y = V_\beta$—such that in y, x is ordinal definable using some formula and α as a parameter. So indeed x is in OD. □

Definition 8.23. We denote by HOD the class $\{x \mid \mathrm{tcl}(\{x\}) \subseteq \mathrm{OD}\}$, that is the class of all sets which are *hereditarily ordinal definable*.

Exercise 8.13 (*). $\mathrm{OD} = \bigcup\{\mathrm{Def}(\{V_\beta \mid \beta < \alpha\}) \mid \alpha \in \mathrm{Ord}\}$.

Exercise 8.14. Show that HOD is an inner model.

Exercise 8.15 (*). Show there is a definable function from Ord onto HOD.

Remark. The model HOD is not robust as the model L. It can be that $\mathrm{HOD}^{\mathrm{HOD}} \neq \mathrm{HOD}$, for example, or that M is an inner model, but $\mathrm{HOD}^V \subsetneq \mathrm{HOD}^M$. It is always the case that if V is a model of ZFC, then there is a larger model W such that $V = \mathrm{HOD}^W$. And many other strange properties which do not happen with the case of L. Interestingly, HOD is compatible with the failure of CH.

Chapter 9

Some combinatorics on ω_1

9.1 Aronszajn trees

Definition 9.1. We say that a tree T is an *Aronszajn tree* if it has height ω_1, every level of T is countable, and there are no cofinal branches.

To avoid trivialities, we only consider *normal trees*, meaning given $\alpha < \omega_1$ and $t \in T$, there is an element in T_α comparable with t, and every node has at least two successors.

Theorem 9.2. *There exists an Aronszajn tree.*

Proof. Let T^* be the tree whose nodes are order embeddings of countable ordinals into bounded sets of $\mathbb{Q}$, ordered by end-extension. Then T^* has height ω_1, no cofinal branch; but almost all the levels of T^* are in fact uncountable. We will refine T^* to an Aronszajn tree.

We define by recursion the levels of T. Suppose that the levels T_β were defined for all $\beta < \alpha$, and that the following condition holds:

$$\forall \gamma < \beta \, \forall x \in T_\gamma \, \forall q > \sup x \; \exists y \in T_\beta : x < y \wedge \sup y \leq q. \tag{*}$$

The condition states, in other words, that every embedding of γ into $\mathbb{Q}$, and any strict upper bound of that embedding, can be extended to an embedding of β into $\mathbb{Q}$ with the same upper bound. So T_β is rich enough to have witnesses for extensions of embeddings into arbitrarily small intervals.

Let $T_0 = \{\varnothing\}$, the only thing it can be. And if T_α was defined and $t \in T_\alpha$, we define its successors in $T_{\alpha+1}$ to be $t^\frown q$ for all $q > \sup t$. Take $T_{\alpha+1}$ to be the set of all these successors, for all $t \in T_\alpha$, then $T_{\alpha+1}$ is a countable union of countable sets, and easily $(*)$ continues to hold.

Suppose that α is a limit ordinal, we need to decide which branches of the possible branches we can add to T_α will be taken. For every $x \in \bigcup\{T_\beta \mid \beta < \alpha\}$, and every $q > \sup x$, we can construct recursively a chain $\{x_n \mid n < \omega\}$ such that $x < x_n$, $\sup x_n < q$, and $\{\operatorname{dom} x_n \mid n < \omega\}$ is cofinal in α. For every $x \in \bigcup\{T_\beta \mid \beta < \alpha\}$ and every $q > \sup x$, choose a sequence like that, and define T_α as the union of all these chosen sequences. It is easy to see that $(*)$ still holds, and that T_α is still countable. $\square$

Exercise 9.1. Let T be the above tree as constructed in L. If there is a cofinal branch in T, then $\omega_1^L < \omega_1$.

Exercise 9.2. In T as constructed above there is an uncountable antichain.

9.2 Diamond and Suslin trees

Definition 9.3. We say that T is a *Suslin tree* if it has height ω_1, but every antichain is countable.

Exercise 9.3. Show that a Suslin tree is an Aronszajn tree. In other words, if every antichain is countable, then every chain is countable.

We want to prove that there is a Suslin tree. However, ZFC cannot prove that a Suslin tree exists. We need additional assumptions, one such assumption is the following axiom.

Definition 9.4. A $\diamondsuit$-sequence is a sequence $\langle A_\alpha \mid \alpha < \omega_1 \rangle$ such that:

1. $A_\alpha \subseteq \alpha$.
2. For every $A \subseteq \omega_1$, the set $\{\alpha \mid A \cap \alpha = A_\alpha\}$ is stationary.

The axiom $\diamondsuit$ asserts that there exists a $\diamondsuit$-sequence.

Proposition 9.5. *$\diamondsuit$ implies* CH.

Proof. If A is a subset of ω, then there is some $\alpha > \omega$ such that $A = A \cap \omega_1 = A_\alpha$. This defines an injection from $\mathcal{P}(\omega)$ into ω_1, and therefore CH holds. $\square$

Theorem 9.6. *$\diamondsuit$ holds in L.*

Proof. We work in L, and construct recursively for $\alpha < \omega_1$ a sequence of pairs $\langle A_\alpha, C_\alpha \rangle$ such that $A_\alpha, C_\alpha \subseteq \alpha$, and C_α is closed and unbounded in α. For $\alpha = 0$ we can only take $A_\alpha = C_\alpha = \varnothing$. For $\alpha + 1$, let $C_{\alpha+1} = A_{\alpha+1} = \alpha + 1$. For a limit α, define the pair as follows:

> $\langle A_\alpha, C_\alpha \rangle$ is the least pair in $<_L$ such that $A_\alpha, C_\alpha \subseteq \alpha$ with C_α a club in α, and for all $\beta \in C_\alpha$, $A_\alpha \cap \beta \neq A_\beta$. If no such pair exists, take $A_\alpha = C_\alpha = \alpha$.

We claim that $\langle A_\alpha \mid \alpha < \omega_1 \rangle$ is a $\diamondsuit$-sequence. Assume otherwise, and let $\langle A, C \rangle$ be the $<_L$-least pair such that $A \subseteq \omega_1$ and $C \subseteq \omega_1$ is a club such that for all $\alpha \in C$, $A \cap \alpha \neq A_\alpha$. Since the sequence, A and C were all definable from $<_L$, both our sequence and $\langle A, C \rangle$ are elements of L_{ω_2}, by condensation arguments. Let M be a countable elementary submodel of L_{ω_2},[1] then by the virtue of definability, $\langle A_\alpha \mid \alpha < \omega_1 \rangle$ and $\langle A, C \rangle$ are both elements of M. Let L_γ be the transitive collapse of M and let $\pi \colon M \to L_\gamma$ denote the isomorphism. Note that $\omega_1 \cap M$ is necessarily an ordinal δ, and that $\delta = \omega_1^{L_\gamma}$, namely $\pi(\omega_1) = \delta$; so $\pi(A) = A \cap \delta$ and $\pi(C) = C \cap \delta$.

Moreover, since for $\alpha < \delta$, $\pi(\alpha) = \alpha$, it follows that $\pi(A_\alpha) = A_\alpha$. Therefore we get that $\pi(\langle A_\alpha \mid \alpha < \omega_1 \rangle) = \langle A_\alpha \mid \alpha < \delta \rangle$. Now L_γ satisfies that $\langle A \cap \delta, C \cap \delta \rangle$ is the $<_L$-least pair satisfying that $C \cap \delta$ is a club in δ and for all $\beta \in C \cap \delta$, $A \cap \delta \cap \beta = A \cap \beta \neq A_\beta$. By elementaritiy, this is true in L_{ω_2}, and therefore in L itself. But this means that $A_\delta = A \cap \delta$. On the other hand, $\delta \in C$, since C is a club in ω_1 and unbounded below δ. And this is a contradiction. $\square$

Theorem 9.7. *If $\diamondsuit$ holds, then there is a Suslin tree.*

Proof. Fix a $\diamondsuit$-sequence $\langle A_\alpha \mid \alpha < \omega_1 \rangle$. We will construct a tree T by recursion, and for simplicity we will assume the underlying set of the tree is ω_1 itself, and that for every α, $\bigcup\{T_\beta \mid \beta < \alpha\}$ is an ordinal (it matters very little which, but we can assume that it is $\alpha \cdot \omega$).

The root of the tree, of course, is $\{0\}$. Suppose that we constructed T_α, let $T_{\alpha+1}$ contain some countably many ordinals in such way that every node in T_α has at least two successors.

[1] M is necessarily not transitive, why?

Let α be a limit ordinal, and $T_{<\alpha} = \bigcup\{T_\beta \mid \beta < \alpha\}$ defined. If A_α is a maximal antichain in $T_{<\alpha}$, let T_α be a suitable extension which preserves the maximality of A_α; namely, every node in T_α lies above an element of A_α. Otherwise, pick any suitable countable level, such that every $x \in T_{<\alpha}$ has an extension in T_α. In either case we can use Lemma 9.8.

Let $T = \bigcup\{T_\alpha \mid \alpha < \omega_1\}$. We claim that T is a Suslin tree. Suppose that A is a maximal antichain in T, then $\langle T, <_T, A\rangle$ is a first-order structure whose domain is ω_1. Therefore there is a club $C \subseteq \omega_1$ such that for $\alpha \in C$, $T \cap \alpha = T_{<\alpha} = \alpha$ and $A \cap \alpha$ is a maximal antichain in $T_{<\alpha}$. Using the $\diamondsuit$-sequence, there is a stationary subset S such that for $\alpha \in S$, $A \cap \alpha = A_\alpha$. Pick $\alpha \in S \cap C$, then $A \cap \alpha = A_\alpha$ is a maximal antichain in $T_{<\alpha}$. But since we chose T_α to be such that $A \cap \alpha$ is still a maximal antichain in $T_{<\alpha+1}$, we get that if $t \in T$, then t is comparable with an element from T_α and therefore comparable with an element of $A \cap \alpha$. This means that $A \cap \alpha$ is in fact maximal in T, so $A = A_\alpha$ and therefore countable. □

Lemma 9.8. *Suppose that α is a countable limit ordinal and $\langle T, <_T\rangle$ is a countable tree of height α which is normal. If $A \subseteq T$ is a maximal antichain, then we can extend T by adding one more countable level such that A remains a maximal antichain.*

Proof. The added level must be obtained by realizing a point at the end of a cofinal branch through T. For every $t \in T$ such that there is some $a \in A$ for which $a <_T t$; for every such t, choose a cofinal branch—which exists due to the normality assumption—and realize it.[2] Then T_α that was added is the realization of only countably many branches; every point in T_α extends a point which extends some $a \in A$, so A is still maximal; and the extended tree is still normal for obvious reasons. □

[2]Namely, add an upper bound to that cofinal branch.

Chapter 10

Coda: Games and determinacy

Definition 10.1. Suppose that $A \subseteq \omega^\omega$, we define the game $G(A)$ to be the game where two players take turns choosing natural numbers for ω turns. This defines a sequence $x \in \omega^\omega$ such that Player I played $x(2n)$ and Player II played $x(2n+1)$. We say that Player I won if $x \in A$, and otherwise Player II won.

Player I	$x(0)$		$x(2)$		$x(4)$	$\dots$
Player II		$x(1)$		$x(3)$		$\dots$

We call the sequence x the *outcome* of the game. And we will use x_{I} and x_{II} to denote the sequences of the moves made by Players I and II respectively in the game.

Definition 10.2. We say that σ is a *strategy* for Player I for a game $G(A)$ if σ is a function from $\omega^{<\omega}$ to ω, such that if x is the outcome of the game, then $x(2n) = \sigma(x \restriction 2n)$. We say that a strategy is a *winning strategy* if it guarantees victory, namely if $x(2n) = \sigma(x \restriction 2n)$ for all n, then $x \in A$. A strategy for Player II is defined similarly (here the victory is when $x \notin A$).

If A is a set such that $G(A)$ has winning strategy for one of the players, we say that A is determined.

Of course, at most one player can have a winning strategy. But is there always such a strategy?

Proposition 10.3. *If $A \subseteq \omega^\omega$ is countable, then Player* II *has a winning strategy.*

Proof. Let $A = \{a_n \mid n < \omega\}$, then on the $2n+1$-th move, Player II simply plays $a_n(2n+1)+1$, thus guaranteeing that if x is the outcome of the game, then $x \neq a_n$ for all $n < \omega$. □

Theorem 10.4. *There is a game without a winning strategy.*

Proof. Let $\{\sigma_\alpha, \tau_\alpha \mid \alpha < 2^{\aleph_0}\}$ be an enumeration of all the winning strategies such that σ_α is a winning strategy for Player I and τ_α is a winning strategy for Player II (not for the same game, of course).

Define by recursion a set $\{a_\alpha, b_\alpha \mid \alpha < 2^{\aleph_0}\}$. Let a_α be an outcome of a game where σ_α was used by Player I, but $a_\alpha \notin \{b_\beta \mid \beta < \alpha\}$. We can find such a_α, since the possible games where Player I played using σ_α has cardinality $2^{\aleph_0}$. Similarly, let b_α be an outcome obtained from a game where Player II played using τ_α and $b_\alpha \notin \{a_\beta \mid \beta < \alpha\}$.

Now we claim that $X = \{b_\alpha \mid \alpha < 2^{\aleph_0}\}$ is not determined. For every α, a_α is an outcome of a game where Player I used σ_α, and $a_\alpha \notin X$, therefore σ_α cannot be a winning strategy for I; and b_α is an outcome of a game where II used τ_α, but $b_\alpha \in X$, so τ_α is not a winning strategy for II. □

Nevertheless, if A is an open, closed or even Borel, in the product topology on ω^ω, then it is determined even if we assume choice .This means that the set A defined in the proof above is somewhat "pathological".

Definition 10.5. The Axiom of Determinacy (AD) states that every set is determined.

Corollary 10.6. AD *implies* $\neg\mathsf{AC}$.

But not all is lost, and we can still get some choice. The following is a very typical proof using AD.

Theorem 10.7. *Assume* $\mathsf{ZF} + \mathsf{AD}$. *Then* $\mathsf{AC}_{\aleph_0}(\omega^\omega)$ *holds.*

Proof. Let $\{X_n \mid n < \omega\}$ be a countable family of non-empty sets such that $X_n \subseteq \omega^\omega$. We define the game where I first chooses n, and then II has to construct (in the odd-indexed turns) an element of X_n. Namely, II wins if the outcome x is such that $x_{\mathrm{II}} \in X_{x(0)}$.

Clearly, I cannot possibly win, since once I played the first turn, II can choose a sequence from the relevant X_n, and just play it. By AD it has to be the case that II has a winning strategy, τ. Then we can define $f(X_n)$ to be x_{II}, where x is the outcome of τ and I playing n, and then only 0. $\square$

Proofs of this flavor are the staple of determinacy proofs. We can use such arguments to show that every subset of ω^ω is very "nice" from a topological and measure theoretic point of view.

Remark. One might wonder about the consistency of AD. Unlike with the case of AC, where ZFC is consistent if ZF is consistent, to prove that $\mathsf{ZF} + \mathsf{AD}$ is consistent we need to assume additional hypotheses which exceed what ZFC can prove by a lot. So for example, if we assume $\mathsf{ZF} + \mathsf{AD}$, then ω_1 is a strongly inaccessible cardinal in L, and in fact we can say much much more.

Exercise 10.1. AD implies that ω_1 is regular.

Exercise 10.2 (*). AD implies that there are no free ultrafilters on ω.

Exercise 10.3 (**). AD implies that the club filter on ω_1 is an ultrafilter.