The Open Dihypergraph Dichotomy for Definable Subsets of Generalized Baire Spaces

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joint work with Philipp Schlicht

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The κ-Baire space ${}^{\kappa}\kappa$ is the set of functions $f : \kappa \to \kappa$, with the bounded topology: basic open sets are of the form

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κ-Borel sets: close the family of open subsets under intersections and unions of size $\leq \kappa$ and complementation.

$\mathbf{\Sigma}^1_1(\kappa)$ sets: continuous images of κ-Borel sets;
equivalently: continuous images of closed sets.

The open graph dichotomy for subsets of ${}^\kappa\kappa$

Let κ be an infinite cardinal such that $\kappa^{<\kappa} = \kappa$. Let $X \subseteq {}^\kappa\kappa$. A graph G on X is an open graph if it is an open subset of $X \times X$.

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$\mathrm{OGD}^{2}_{\kappa}(X)$ implies the open graph dichotomy $\mathrm{OGD}_{\kappa}(X)$.

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Suppose X *is a separable metric space such that* $\mathrm{OGD}_{\omega}^{\omega}(X)$ *holds.*

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- *Several other applications ...*

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Then $x \in V[G_\alpha]$ for some $\alpha < \lambda$. Let $\dot{x}$ be a $\mathbb{P}_\alpha$-name for x.

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These arguments rely on the following lemma.

Lemma 1 (Solovay)

For all countable sequences y of ordinals in $V[G]$, $V[G]$ is a $\mathbb{P}_\lambda$-generic extension of $V[y]$.

- This lemma fails when $\kappa > \omega$ (Schlicht).

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By the next lemma, ι can be defined in such a way that $\dot{x}^{g_y} \in X$ for all $y \in {}^{\kappa}\delta$, and the (continuous) map

$$f : {}^{\kappa}\delta \to X;\ y \mapsto \dot{x}^{g_y}$$

is a homomorphism from $\mathbb{H}_\delta$ to H.

Sketch of the proof (the $\kappa = \omega$ case)

For any forcing $\mathbb{Q}$, any $q \in Q$ and any $\mathbb{Q}$-name σ, define

$$T_{\mathbb{Q}}^{\sigma,q} = \{t \in {}^{<\kappa}\kappa : (\exists r \leq q)\, r \Vdash_{\mathbb{Q}}^{V} t \subseteq \sigma\},$$

the tree of possible values for σ below q.

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Lemma 2

There exists $p \in \mathbb{P}_\alpha$ such that the following hold.

1. $p \Vdash^{V}_{\mathbb{P}_\alpha}$ "$\varphi_X(\dot{x}, a)$ holds in every further $\mathbb{P}_\lambda$-generic extension of $V[\dot{x}]$."

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1. $p \Vdash^{V}_{\mathbb{P}_\alpha}$ "$\varphi_X(\dot{x}, a)$ holds in every further $\mathbb{P}_\lambda$-generic extension of $V[\dot{x}]$."
2. For all $r \in \mathbb{P}_\alpha$ below p, there exists (in $V[G]$) a sequence

$$\langle t_i \in T^{\dot{x},r}_{\mathbb{P}_\alpha} : i < \delta \rangle$$

such that (in $V[G]$)

$$\prod_{i<\delta} N_{t_i} \cap X \subseteq R.$$

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Lemma 3

There exists $\gamma < \lambda$ and an $\mathrm{Add}(\kappa, 1)$-name $\sigma \in V[G_\gamma]$ such that the following hold:

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2. For all $r \in \mathrm{Add}(\kappa, 1)$, there exists a sequence

$$\bar{t}(r) = \langle t_i(r) \in T^{\sigma,r}_{\mathrm{Add}(\kappa,1)} : i < \delta \rangle \in V[G_\gamma]$$

such that in $V[G]$,

$$\prod_{i<\delta} N_{t_i(r)} \cap X \subseteq R.$$

Sketch of the proof (the $\kappa > \omega$ case)

Let $\mathbb{Q}_\sigma$ consist of those partial maps p from ${}^{<\kappa}\delta$ to ${}^{<\kappa}\kappa$ such that

1. $\mathrm{dom}(p)$ is a subtree of ${}^{<\kappa}\delta$ of size $< \kappa$.
2. For all $t, u \in \mathrm{dom}(p)$,
   $$t \subseteq u \text{ implies } p(t) \subseteq p(u), \text{ and } t \perp u \text{ implies } p(t) \perp p(u).$$
3. A technical requirement, involving the sequences $\bar{t}(r)$ from Lemma 3, holds for p.

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We let $p \leq_{\mathbb{Q}_\sigma} q$ if and only if $\mathrm{dom}(p) \supseteq \mathrm{dom}(q)$, and

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A $\mathbb{Q}_\sigma$-generic filter K adds a $\subseteq$ and $\perp$-preserving map

$$\iota_K : {}^{<\kappa}\delta \to {}^{<\kappa}\kappa;\ t \mapsto \bigcup\{p(t) : p \in K\}.$$

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Working in $V[G]$, let $g : {}^\kappa\delta \to {}^\kappa\kappa;\ y \mapsto \bigcup\{\iota_K(t) : t \subsetneq y\}$.

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Let $y \in {}^\kappa\delta$.

1. $g(y)$ is $\mathrm{Add}(\kappa, 1)$-generic over $V[G_\gamma]$.
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Let

$$f : {}^\kappa\delta \to X;\ y \mapsto \sigma^{g(y)}.$$

f is a continuous map and is a homomorphism from $\mathbb{H}_\delta$ to R. (Item 3 in the definition of $\mathbb{Q}_\sigma$ guarantees this).

Questions

- Suppose $\kappa > \omega$. Is it consistent that $\mathrm{OGD}^{\kappa}_{\kappa}(X)$ (i.e., for all box-open κ-dimensional dihypergraphs) holds for $\mathbf{\Sigma}^1_1(\kappa)$ subsets $X \subseteq {}^{\kappa}\kappa$?

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  Is OGA_{κ} consistent? If so, how does it influence the structure of the κ-Baire space?

Thank you!