DOWNWARD LÖWENHEIM-SKOLEM THEOREMS AND CHOICE PRINCIPLES

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ABSTRACT. We prove that the principle of Dependent Choice is equivalent to the existence of a countable elementary submodel of every model of a theory in a countable language. We attempt in extending this to infinite cardinals as well, with partial success.

1. INTRODUCTION

Let DC abbreviate the following statement, "Every tree T of height ω and without terminal nodes has an infinite branch". And let LS abbreviate the statement "Every infinite model $\mathfrak{M}$ of a countable language $\mathcal{L}$, has a countable elementary submodel".

In this note we prove the equivalence of the two as weak choice principles. After writing the initial draft of this paper (including the equivalence of these), it was revealed that in fact Christian Espíndola has proved the above, and extended these results. In his unpublished note (at time of writing) he points out that the equivalence was known, but not widely known, and appears in a book by G. Boolos. Espíndola also proves a stronger result which we discuss in the third section.

Before we proceed, a word about the notation. We will denote $\mathcal{L}$-structure by Gothic letters, and their universes by the corresponding Latin letters. To ease the reading we write $\vec{a} \subseteq A$ to mean that $\vec{a}$ is a finite tuple of the appropriate length, all whose elements are from A. It should be clear that all the proofs appearing here are in ZF.

To avoid confusion in the absence of choice, $\mathfrak{A}$ is an elementary submodel of $\mathfrak{M}$ if it is a substructure and for every $\vec{a} \subseteq A$ and $\varphi(\vec{x})$ we have $\mathfrak{M} \models \varphi(\vec{a}) \iff \mathfrak{A} \models \varphi(\vec{a})$. If $\mathfrak{M}$ is an $\mathcal{L}$-structure, A is a subset of M, and φ is the $\mathcal{L}$-formula $\exists y \psi(\vec{x}, y)$, we say that f is an **A-Skolem function for** φ when $\operatorname{dom} f = \{\vec{a} \subseteq A \mid \mathfrak{M} \models \exists y \psi(\vec{a}, y)\}$, its range is in M and for all $\vec{a} \in \operatorname{dom} f$ we have $\mathfrak{M} \models \psi(\vec{a}, f(\vec{a}))$.

Some uses of DC which we will use include the axiom of choice for countable families of sets, and the fact that countable unions of countable sets are countable. It is not hard to show that DC implies the first (by considering the tree defined by extension of choice functions from the first n sets), which in turn implies the latter (by the standard proof from ZFC).

Lemma 1. *Suppose that T is a countable tree that has height ω and no terminal nodes, then it has a branch.*

Proof. Enumerate T as $\{t_n \mid n \in \omega\}$, and without loss of generality t_0 is the root of T. we define a function by recursion, let $f(0)$ to be the root. Suppose that $f(n)$ was defined from the n-th level, then $f(n+1) = t_k$ such that k is the least index of an immediate successor of $f(n)$. The set $\operatorname{rng} f$ is an infinite branch by definition. □

The Downward Löwenheim-Skolem theorem often allows us to "generate" a submodel from a given set, whereas LS only assures the existence of some submodel. However the two are equivalent over ZF as the following lemma shows.

Lemma 2. LS *is equivalent to the statement "If $\mathfrak{M}$ is an infinite model of a countable language, and $B \subseteq M$ is countable, then there is an elementary submodel $\mathfrak{A}$ of $\mathfrak{M}$ such that $B \subseteq A$".*

Proof. It is clear that LS follows from the statement. Assume LS and let $\mathfrak{M}$ be an infinite model of a countable language $\mathcal{L}$, and $B \subseteq M$ a countable set. Augment $\mathcal{L}$ by adding constant symbols c_b for $b \in B$ and let $\mathfrak{M}^+$ be the model where we interpret c_b as b itself. From LS there is $\mathfrak{A}^+$ which is a countable elementary submodel, and since $c_b^{\mathfrak{M}} = b$ we have that $B \subseteq A$ as wanted, restricting back to $\mathcal{L}$ finishes the proof of the statement. □

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2. THE MAIN THEOREM

Theorem 3. DC *and* LS *are equivalent.*

Proof. Assume LS, and let T be a tree of height ω without terminal nodes. We can see T as a structure of the language $\mathcal{L} = \{<\}$ satisfying the axioms that T is a partial order, and T satisfies the axiom $\forall x \exists y (x < y)$, i.e. there are no terminal nodes. Let T' be a countable elementary submodel of T, then T' is a tree of height ω without terminal nodes. Moreover if $x \in T'$ then its level in T and in T' are the same, since if T' satisfies "There exists exactly n different elements below x", then so must T by elemetnarity. Therefore if $A \subseteq T'$ is a branch in T' it is a branch in T.

Assume DC, and let $\mathfrak{M}$ be an infinite model of a countable language $\mathcal{L}$, which we may assume contains only relation symbols. We define by induction a sequence of substructures of $\mathfrak{M}$, the collection of all these definitions is naturally ordered as a tree, if we succeed in our induction process then we prove that the tree is of height ω and has no terminal nodes. Therefore by DC there exists a branch, and we will show that this branch defines an elementary submodel.

Let $A_0 = \varnothing$, and let F_0 be a collection of A_0-Skolem functions. Since $\mathcal{L}$ is countable, we can use DC to choose such function for every φ. Suppose that A_k was defined and F_k was defined, let $A_{k+1} = A_k \cup \bigcup \{\operatorname{rng} f \mid f \in F_k\}$ and let F_{k+1} be a collection of A_{k+1}-Skolem functions extending F_k, that is if $f \in F_k$ an A-Skolem function for φ and $f' \in F_{k+1}$ is a Skolem function for φ, then for every $\vec{a} \subseteq A_n$ the functions agree: $f(\vec{a}) = f'(\vec{a})$. We can extend F_k in such manner because again we only make countably many choices (there are countably many new elements, so countably many new tuples to handle).

Let $A = \bigcup \{A_n \mid n \in \omega\}$, then A is countable, as a countable union of countable sets. Let $\mathfrak{A}$ be the substructure of $\mathfrak{M}$ whose universe is A and the interpretation of the relations are just their restriction to A. We will show that $\mathfrak{A}$ is an elementary submodel of $\mathfrak{M}$, by induction on the complexity of formulas φ.

If φ is atomic then there is some relation symbol R such that $\varphi(\vec{x}) = R(\vec{x})$, then by the definition of $\mathfrak{A}$ we have that $\mathfrak{M} \models \varphi(\vec{a}) \iff \mathfrak{A} \models \varphi(\vec{a})$, for all $\vec{a} \subseteq A$. For φ which is the negation or connection (conjunction, disjunction, material implication) of shorter formulas the statement follows from the definition of truth tables of negation and the various connectives.

Finally, if $\varphi(\vec{x}) = \exists y \psi(\vec{x}, y)$ and $\vec{a} \subseteq A$ then there is some n such that $\vec{a} \subseteq A_n$ and $f \in F_n$ which is an A_n-Skolem function for φ. If $\mathfrak{M} \models \varphi(\vec{a})$, then $f(\vec{a}) = b$ is such that $\mathfrak{M} \models \psi(\vec{a}, b)$ and $b \in A_{n+1} \subseteq A$. By the induction hypothesis $\mathfrak{A} \models \psi(\vec{a}, b)$ and therefore $\mathfrak{A} \models \varphi(\vec{a})$. If $\mathfrak{A} \models \varphi(\vec{a})$ then for some $b \in A$ we have $\mathfrak{A} \models \psi(\vec{a}, b)$, and so $\mathfrak{M} \models \psi(\vec{a}, b)$ and therefore $\mathfrak{M} \models \varphi(\vec{a})$ as wanted. $\square$

3. ...TO THE UNCOUNTABLE

The previous section shows that for the countable case there is a full equivalence between the known Löwenheim-Skolem theorem, and DC. It is natural to ask whether or not one can repeat the same proof on a larger scale. For an infinite $\aleph$ cardinal κ we define DC_κ as the principle stating "Every tree of height κ, where every branch of length $<\kappa$ can be extended, has a branch of length κ". Similarly $\mathsf{LS}(\kappa)$ is defined as "Every infinite model in a language of cardinality $\leq\kappa$ has an elementary submodel of cardinality $\leq\kappa$". We also introduce AC_κ as the abbreviation that every family of κ non-empty sets admits a choice function.

Let us quickly review some properties of DC_κ and AC_κ. Both are continuous, meaning if κ is singular and for every $\lambda < \kappa$, DC_λ holds then DC_κ holds (and similarly for AC_κ). Both principles also reflect down, DC_κ implies DC_λ for every $\lambda < \kappa$ (and similarly for AC_κ). Finally, it can be shown that DC_κ implies AC_κ and that this implication is strict, moreover if $\lambda < \kappa$ then AC_κ and DC_λ are completely independent - neither one implies the other. All these results can be found in Chapter 8 of [Jec73].

We can try and retrace the above proof, it is not very hard to repeat the proof that DC_κ readily implies $\mathsf{LS}(\kappa)$ holds. However when one tries to prove that $\mathsf{LS}(\kappa)$ implies DC_κ, one runs into a problem. The reason is that whereas the notion of a "short branch" in a tree of height ω is just a finite chain, which can be fully expressed using first-order logic as a schema, but when trying to talk about infinite branches, first-order logic is too weak, and the proof fails. Perhaps there is another way to do that without modifying our definitions for choice principles? No, we can't.

Theorem 4. *For every $\aleph$ cardinal κ,* $\mathsf{LS}(\kappa)$ *is equivalent to the conjunction of* DC *and* AC_κ.

Proof. To show that DC and AC_κ imply $\mathsf{LS}(\kappa)$, we can repeat the proof of Theorem 3. Note that the inductive definition of the A_n themselves only used AC_ω, and the only real use of DC in the proof was in the passing from the inductive definition to the existence of an actual sequence. In this case, observe that the inductive definition requires AC_κ to be applied where AC_ω was applied, but we only need a sequence of length ω, so DC suffices, and the proof carries perfectly.

In the other direction, note that the formulation of $\mathsf{LS}(\kappa)$ immediately implies LS and so DC holds. Given a family of non-empty sets $\{M_\alpha \mid \alpha < \kappa\}$, consider the language which has κ unary predicates R_α, let T be the theory $\{\exists x R_\alpha(x) \mid \alpha < \kappa\}$, and let $\mathfrak{M}$ be a structure for the language of T whose universe is $\bigcup\{M_\alpha \mid \alpha < \kappa\}$ and $R_\alpha^{\mathfrak{M}} = M_\alpha$, it is not hard to see that indeed $\mathfrak{M} \models T$. By $\mathsf{LS}(\kappa)$ we have that $\mathfrak{M}$ has an elementary submodel of size $\leq\kappa$, $\mathfrak{A}$. Therefore $\mathfrak{A} \models \exists x R_\alpha(x)$ for every α, and since it is a substructure of $\mathfrak{M}$ we have that $R_\alpha^{\mathfrak{M}} = A \cap M_\alpha$. Finally, A can be well-ordered, so we can simply pick an element from each $A \cap M_\alpha$, which ends up as a choice function for the original family as well. □

Corollary 5. *The statement "for every $\aleph$ cardinal κ, $\mathsf{LS}(\kappa)$" is equivalent to "for every $\aleph$ cardinal κ, AC_κ".*

Proof. The first statement obviously implies the second. The second statement implies DC (see [Jec73, Theorem 8.2]), and therefore the first statement as well. □

The question remains open whether or not we can find an equivalent for DC_κ, and while we're on the subject for $\mathsf{AC}_\mathfrak{p}$ for non-$\aleph$ cardinals. The daunting task would be to find a logic which has a downward Löwenheim-Skolem theorem to begin with, which can express the notion that every "short" branch can be continued to solve the former; and to find a way to circumvent the deep use of the well-orderability of κ that we did in the above proof for the latter.

If anything is likely, then it seems that DC_κ would require an infinitary logic, or some second-order quantifier. Possibly the combination of those two.

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References

[Jec73] Thomas J. Jech, *The axiom of choice*, North-Holland Publishing Co., Amsterdam, 1973, Studies in Logic and the Foundations of Mathematics, Vol. 75. MR 0396271 (53 #139)

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