THE AXIOM OF CHOICE AND SELF-DUALITY OF VECTOR SPACES

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Abstract. Let V be a vector space over $\mathbb{R}$. Denote by V^* the algebraic dual of V, and for a topological vector space denote by V' the continuous dual. We will show that the following implications are not provable without the axiom of choice: (1) $V \cong V^*$ implies that $\dim V < \infty$ (2) $V \cong V^{**}$ by a natural isomorphism if and only if $\dim V < \infty$; (3) If V is a Banach space, V' is reflexive if and only if V is reflexive; (4) If V is a reflexive Banach space, $W \subseteq V$ is a closed subspace, then W is also reflexive; (5) If V' is separable then V is separable.

1. Introduction

For a general vector space V denote its algebraic dual by V^*, the set of linear functions from V to the field, and for a topological vector space V denote by V' the subspace of V^* which exactly the continuous linear functionals.

To allow functional analysis behave nicely will make heavy use of the *Principle of Dependent Choice*, DC, which states that if S is a non-empty set, and R is a binary relation on S whose domain is S and for every $x \in S$ there is some $y \in S$ such that $x \mathrel{R} y$, then there is a sequence s_n for $n \in \omega$ such that $s_n \mathrel{R} s_{n+1}$ for all $n \in \omega$.

This choice principle implies the Baire Category theorem [Her06, Th. 4.106], and that the product of a countable collection of non-empty set is non-empty. Under this choice principle we have that in metric spaces compactness is equivalent to the assertion that every infinite set contains an accumulation point [Her06, Th. 3.27].

If X is a topological space and $A \subseteq X$ we say that A has the *Baire property* has the Baire property if there is an open set U such that $A \triangle U$ is a countable union of nowhere dense sets. Let the assertion "Every set of real numbers has the Baire property" be abbreviated as BP. Consider a model of $\mathsf{ZF}+\mathsf{DC}+\mathsf{BP}$, such as Solovay's model [Sol70] or Shelah's model [She84]. In models of $\mathsf{ZF}+\mathsf{DC}+\mathsf{BP}$ as these two we have that every ultrafilter on ω is principal, since free ultrafilters fail to have the Baire property.

We will show in this document the following things:

- It is consistent that V is a vector space which is not finitely generated, and $V \cong V^*$ (in contrast to ZFC where $\dim V < \dim V^*$ for infinitely dimensional spaces), Corollary 6.
- It is consistent that V is a vector space which is not finitely generated and V is naturally isomorphic to V^{**} (in contrast to ZFC where this holds if and only if $\dim V < \infty$), Theorem 5.
- It is consistent that $W \subseteq V$ is a closed subspace of a Banach space, and $V \cong V''$, but $W \ncong V''$ (in contrast to ZFC where a closed subspace of a reflexive Banach space is reflexive), Corollary 8.
- It is consistent that V is a Banach space such that V' is reflexive, but V is not (in contrast to ZFC where the reflexivity of V' implies the reflexivity of V), Corollary 8.
- It is consistent that V is a Banach space such that V' is separable, but V is not separable (in contrast to ZFC where the separability of V' implies separability of V), Corollary 9.

2. Some Preliminary Theorems

Let us quote some theorems and draw some basic conclusions to be used later.

Theorem 1. Let G, H be topological groups and $\varphi\colon G \to H$ a homomorphism. If for every $A \subseteq H$ that has the Baire property $\varphi^{-1}(A)$ has the Baire property, and H is separable then φ is continuous.

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The proof of this theorem appears in [Kec95, Th. 9.10]. Assuming $\mathsf{ZF}+\mathsf{DC}+\mathsf{BP}$ this theorem implies that if V, W are Banach space, W separable and $T\colon V \to W$ is linear then T is continuous. We remark that assuming that every set of real numbers is Lebesgue measurable can yield a seemingly stronger theorem which drops the requirement that W is separable, see [Gar74].

Let us assume from this point onwards that we work in a model whose theory includes $\mathsf{ZF}+\mathsf{DC}+\mathsf{BP}$.

Corollary 2. If V is a Banach space over a separable field K then $V^* = V'$. □

As remarked above, the Baire category theorem holds in $\mathsf{ZF}+\mathsf{DC}$ and therefore the open mapping theorem holds:

Theorem 3 (Open Mapping Theorem)**.** Let V, W be Banach spaces. If $T\colon E \to F$ is a linear operator and T is surjective then T is open.

We will consider the spaces of the form $\ell_p = \ell_p(\mathbb{N})$ over $\mathbb{R}$ for $1 \le p \le \infty$. Recall that ℓ_p is the space of all sequences $\langle a_n \mid n \in \mathbb{N}\rangle$ of real numbers such that $\sum_{n=1}^{\infty} |a_n|^p$ is finite. Even without the full axiom of choice these are still Banach spaces, and ℓ_2 is a Hilbert space with the inner product:

$$\langle a, b\rangle = \sum_{n=1}^{\infty} a_n b_n$$

Note that the closed unit ball of any ℓ_p is not compact since the standard Schauder basis is a closed and discrete set on the unit sphere without an accumulation point (and as remarked in the introduction under the assumption of DC this is equivalent to non-compactness in metric spaces).

Theorem 4 (Pitt)**.** If $1 < p < q$ then every continuous linear operator $T\colon \ell_p \to \ell_q$ is compact, namely $\{Tx \colon \|x\|_p \le 1\}$ is compact in ℓ_q.

The open mapping theorem appears in [Ped89, Th. 2.24], whereas a proof of Pitt's theorem can be found in [Del09]. We remark that Pitt's theorem proof does not use more choice than DC, since all the spaces in question are metrizable.

One direct corollary from these two theorems is that if $p < q$ for $p, q \in [1, \infty]$ then ℓ_p is not linearly isomorphic to ℓ_q. Otherwise there was a linear bijection $T\colon \ell_q \to \ell_p$, since ℓ_p is separable (this would be false if $p = \infty$ but $p < q \le \infty$) Theorem 1 ensures that T is continuous and by the open mapping theorem open. We have, if so, that it is a homeomorphism between the spaces as well. Pitt's theorem now tells us that T is a compact operator, in turn this means that T maps the closed unit ball of ℓ_q to a compact set in ℓ_p, and that T^{-1} maps a compact set to a non-compact set in contradiction to the fact it is a homeomorphism.

The above is in contrast with ZFC where all these spaces have Hamel basis of size $2^{\aleph_0}$ and therefore isomorphic as vector spaces (such linear isomorphism would have to be discontinuous, of course).

3. The Main Result

We finally arrive at the main results. One could have asked, what is all the hard work for? We already know that $\ell_2' = \ell_2$, that would have given the nice counterexample that we wanted and we can move on. However we shall exhibit a continuum of non-isomorphic examples for self-dual Banach spaces, at the cost of a slightly lengthier discussion for the interested reader.

Theorem 5. Let $p \in (1, \infty)$ and q such that $\frac{1}{p} + \frac{1}{q} = 1$, then $\ell_p^* = \ell_q$. Furthermore the natural map from ℓ_p to ℓ_p^{**} defined by $x \mapsto \operatorname{eval}_x$, where $\operatorname{eval}_x(\varphi) = \varphi(x)$, is the identity.

Proof. For $p > 1$ let $q = \frac{p}{p-1}$. The proof that $\ell_p' = \ell_q$ and vice versa is fully in ZF by using the Hölder inequality. Using Corollary 2 we have that $\ell_p^* = \ell_p' = \ell_q$, therefore $\ell_p^{**} = \ell_p$.

Suppose φ is a linear functional on ℓ_p, then there is $u \in \ell_q$ such that:

$$\varphi(x) = \sum_{n=1}^{\infty} x_n u_n$$

Similarly the evaluation function eval_x takes $u \in \ell_q$ and maps it to the above sum, we can now reconstruct x by applying eval_x to the standard Schauder basis and restore x_n for all n, and therefore the map is the identity as wanted. □

Corollary 6. Let $p \in (0, \infty)$ and q as above, and denote by $V = \ell_p \oplus \ell_q$. This is a Banach space, and therefore $V' = V^*$, and therefore $V^* = \ell_q^* \oplus \ell_p^* = \ell_p \oplus \ell_q = V$, and as before V is naturally isomorphic to V^{**}. □

Having cleared all the finite pairs of (p, q) we finish with a note on the case of $p = 1, q = \infty$.

Theorem 7. Recall that $\ell_1' = \ell_\infty$. If $\ell_1 \subsetneq \ell_\infty'$ then BP does not hold.

The proof appears as Theorem 29.38 in [Sch97]. From this we have that $\ell_1^{**} = \ell_1'' = \ell_\infty' = \ell_\infty^* = \ell_1$ We therefore have that in Solovay's model for every $p \in [1, \infty]$ we have that ℓ_p is a reflexive space both in the topological sense as well as in the algebraic sense.

Corollary 8. c_0 is a closed subspace of ℓ_∞, and $c_0' = \ell_1$. Therefore $c_0'' = \ell_\infty \neq c_0$. In particular we have an irreflexive closed subspace of a reflexive space, as well c_0' is reflexive, but c_0 is not.

Corollary 9. $\ell_\infty' = \ell_1$ is separable, however ℓ_∞ is not separable.

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