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Vector Spaces and Antichains of Cardinals in Models of Set Theory

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Abstract

Läuchli constructed a model of ZF in which there is a vector space which is not of finite dimension, but every proper subspace is of a finite dimension. In Läuchli's model the axiom of choice fails completely, there is a countable family from which we cannot choose representatives.

In this work we generalize Läuchli's original proof. In the proof presented here we show that we may choose any cardinal μ and construct a model of ZF in which there is a vector space such that every proper subspace has dimension less than μ, but the vector space itself is not spanned by any linearly independent subset. The construction uses a technique called symmetric extensions, which is used to create models in which the axiom of choice fails. In the first chapter we will review this technique, and weak versions of the axiom of choice. We show that in our construction we may preserve relatively large fragments of choice in the universe.

We also generalize a theorem by Monro which states that it is consistent without the axiom of choice that there are infinite sets which have no countably infinite subset, but can be mapped onto very large ordinals. Our proof uses the method of symmetric extensions, in contrast to Monro which took a different approach, and we show that for any two regular cardinals $\lambda \leq \kappa$ we may construct a model of ZF in which there is a set that can be mapped onto κ, and λ is the least ordinal which cannot be injected into this set.

In the third chapter we present a recent paper of Feldman, Orhon and Blass. In this paper the authors prove that if there is a finite bound on the size of antichains of cardinals then the axiom of choice holds. We review the original results and extend them to hold for a weaker notion of a quasi-ordering of the cardinals. We also answer one of the questions presented in the paper, and add questions of our own.

”וְחֶזְוֵי רֵאשִׁי יְבַהֲלֻנַּנִי“

סֵפֶר דָּנִיֵּאל, פֶּרֶק ז׳ פָּסוּק ט״ז

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[1] `http://math.stackexchange.com/users/7085`
[2] `http://math.stackexchange.com/q/28145`

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Introduction

In modern mathematics one usually works within the framework of ZFC. Assuming the axiom of choice every vector space has a basis, and therefore has many non-trivial linear functionals and linear endomorphisms. In fact, Blass proved that the assertion that every vector space has a basis implies the axiom of choice (see [Bla84]).

Before Blass' theorem it was already known that it is consistent that there are vector spaces without a basis. Läuchli constructed a model of ZF without the axiom of foundation, in which the axiom of choice fails and there is a vector space over a countable field which is not spanned by any finite subset, but every proper subspace has a finite dimension (see [Läu63]). The same proof can be carried in ZFA, the theory of ZF weakened to allow atoms (non-sets).

Several years later Jech and Sochor proved a transfer theorem which allowed the construction of Läuchli, when carried in ZFA to be transferred into a model of ZF (see [JS66a, JS66b], and [Jec73, Chapter 6]). Finally the consistency of ZF with the existence of a vector space which had no basis was proved. The transfer theorem, however, has a limited power in transferring unbounded statements. Namely, it can be used to show the existence of such a peculiar vector space, but it cannot be used to transfer a statement of the form "*Every set has property* φ".

In the years to come it seems that some writers have forgotten parts of Läuchli's model. They presented it as a model in which there exists a vector space which has no basis, or as a vector space which is not finitely generated and every proper subspace has a finite dimension (see [Jec73, Theorem 10.11]). These presentations would not mention the fact that the dual space is trivial, or that there are only scalar endomorphisms.

A main goal of this work is to present a generalization of Läuchli's model, which will establish that a weakened version of the axiom of choice (or rather Zorn's lemma) is not sufficient to prove the existence of non-scalar endomorphisms and non-zero linear functionals for every non-trivial vector space. We do so in the second chapter, where we present a forcing construction directly in ZF, rather than ZFA or ZF without foundation.

The first chapter will be dedicated to cover the techniques required for this proof, in particular **symmetric extensions**. Symmetric extensions are inner models to generic extensions obtained by forcing, these lie between the ground model and the extension and are models of ZF which usually do not satisfy the axiom of choice. The idea of symmetries goes back to Fraenkel in the early 1920's, who constructed models with atoms which contradicted the axiom of choice. The original method had a mistake which was later corrected by Fraenkel in 1937 and improved by Mostowski in the subsequent years. The final touch of the technique was given by Specker in 1956. We will not present this technique, but rather jump directly to Cohen's construction with forcing which draws from the Fraenkel-Mostowski-Specker method.

We will also review three common weakening versions of the axiom of choice, and we will define a technical construct called **2-permutations** which will play a role in defining the symmetric extension in the generalization of Läuchli's theorem.

The third chapter will review a recent paper by Feldman, Orhon and Blass (see [FOB08]), in which they prove that if there is a finite bound over antichains of cardinals then the axiom of choice holds. This is a generalization of a classical theorem by Hartogs which shows that if the cardinals are linearly ordered then the axiom of choice holds. We will present the original results from the paper, extend them and discuss open questions appearing in the paper.

1

Preliminaries

In this chapter we review the basics of symmetric extensions by forcing, as well as weak choice principles. We also introduce the notions of 2-permutations and affine ideals which will be used in chapter 2.

Some notational conventions are given first. If A is a set we will denote by $\mathcal{P}(A)$ its power set, and by $|A|$ **the cardinal number** of A, which is defined to be the least ordinal in bijection with A if such ordinal exists and otherwise it is the set

$$\{B \mid \exists f\colon A \to B \text{ a bijection} \wedge \operatorname{rank}(B) \text{ is minimal}\}.$$

If $|A|$ is an infinite ordinal we say that $|A|$ is an $\aleph$ cardinal, or a well-ordered cardinal, or we say that $|A| \in \mathbf{Ord}$ where $\mathbf{Ord}$ is the class of ordinals. If $|A| \in \omega$ we say that A is finite.

The cardinal numbers are partially ordered by $\leq$ where $|A| \leq |B|$ is to say that there is a subset B' of B such that $|A| = |B'|$, in particular this means that there exists a bijection between A and B', and an injection from A into B. We can consider another ordering, $\leq^*$ defined by surjections, namely $|A| \leq^* |B|$ if either $A = \varnothing$ or there is a surjective function from B onto A.

These two orders are equal under the axiom of choice, however without it they may differ. Whereas $\leq$ is always a partial order (anti-symmetry is guaranteed by the Cantor-Bernstein theorem which holds in ZF), the relation $\leq^*$ need not be anti-symmetric (see Proposition 3.13).

We say that A is finite if $|A| \in \omega$, and that A is infinite otherwise. For an infinite set A we associate two particular well-ordered cardinals:

- $\aleph(A)$, the first ordinal not injectable into A, known as the **Hartogs number** of A,

  $$\aleph(A) = \min\{\alpha \in \mathbf{Ord} \mid |\alpha| \nleq |A|\}.$$

  Equivalently we can define $\aleph(A)$ as $\sup\{\alpha \in \mathbf{Ord} \mid |\alpha| \leq |A|\}$,

- $\aleph^*(A)$, the least ordinal which A cannot be mapped onto, defined as

  $$\aleph^*(A) = \min\{\alpha \in \mathbf{Ord} \mid |\alpha| \nleq^* |A|\}.$$

  This is equivalent to $\aleph^*(A) = \sup\{\alpha \in \mathbf{Ord} \mid |\alpha| \leq^* |A|\}$.

If A is finite, we define $\aleph(A) = \aleph^*(A) = \omega$. It is always the case that $\aleph(A) \leq \aleph^*(A)$. If A is well-ordered then $\aleph(A) = \aleph^*(A) = |A|^+$. We will later see that the gap between the two can be made arbitrarily large in models without the axiom of choice. We also note that $|A| < \aleph(A)$ if and only if A can be well-ordered.

We will use κ, λ, μ to denote $\aleph$ cardinals. The Greek letters $\alpha, \beta, \gamma \ldots$ will always denote ordinals, and Gothic capital letters (e.g. $\mathfrak{M}, \mathfrak{N}$) will denote models of **ZF**.

If α is an ordinal and κ is an $\aleph$ cardinal we will write $\kappa^{+\alpha}$ to be the unique $\aleph$ cardinal λ such that α is the order type of $\{\mu \in [\kappa, \lambda) \mid \mu \text{ is an } \aleph\}$.

We assume that the reader is familiar with forcing, including the basic theorems and the definitions. If $\mathbb{P} = \langle P, \leq \rangle$ is a notion of forcing we will say that p is **stronger** than q if and only if $p \leq q$. We shall always assume that $\mathbb{P}$ has a maximum (weakest) element and it will be denoted by $1_\mathbb{P}$, or 1 where $\mathbb{P}$ is clear from context. For $A \subseteq P$ we say that A is **dense below** p if for every $q \leq p$ there is $r \in A$ such that $r \leq q$. We say that A is dense if it is dense below $1_\mathbb{P}$.

Recall that a $\mathbb{P}$-name in $\mathfrak{M}$ is a set which is a member of the class $\mathfrak{M}^\mathbb{P}$, defined by induction in $\mathfrak{M}$:

- $\mathfrak{M}^\mathbb{P}_0 = \varnothing$;
- $\mathfrak{M}^\mathbb{P}_{\alpha+1} = \mathcal{P}(P \times \mathfrak{M}^\mathbb{P}_\alpha)$;
- $\mathfrak{M}^\mathbb{P}_\delta = \bigcup_{\gamma<\delta} \mathfrak{M}^\mathbb{P}_\gamma$, If δ is a limit ordinal;
- $\mathfrak{M}^\mathbb{P} = \bigcup_{\alpha \in \mathbf{Ord}} \mathfrak{M}^\mathbb{P}_\alpha$.

We shall denote with $\dot{x}$ a $\mathbb{P}$-name, and we say a name $\dot{x}$ has $\mathbb{P}$-rank α if α is the least ordinal such that $\dot{x} \in \mathfrak{M}^\mathbb{P}_\alpha$. For $x \in \mathfrak{M}$ we shall denote by $\check{x}$ the canonical $\mathbb{P}$-name for x. For a set of $\mathbb{P}$-names, $\{\dot{x}_i \mid i \in I\}$ we define the canonical name of this set to be $\{\dot{x}_i \mid i \in I\}^\bullet = \{\langle 1_\mathbb{P}, \dot{x}_i \rangle \mid i \in I\}$, similarly $\langle \dot{x}, \dot{y} \rangle^\bullet$ is the canonical name $\{\langle 1_\mathbb{P}, \{\dot{x}\}^\bullet \rangle, \langle 1_\mathbb{P}, \{\dot{x}, \dot{y}\}^\bullet \rangle\}$.

Lastly, if G is a $\mathbb{P}$-generic filter over $\mathfrak{M}$, and $\dot{x}$ is a $\mathbb{P}$-name we will denote by $\dot{x}^G$ as the interpretation of the name $\dot{x}$ by the filter G, as an element of $\mathfrak{M}[G]$.

1.1 Symmetric Extensions by Forcing

Let $\mathfrak{M}$ be a transitive model of **ZFC**, in this section it will always play the role of the ground model. Let $\mathbb{P} = \langle P, \leq \rangle$ be a notion of forcing, it is a basic theorem that in every generic extension of $\mathfrak{M}$ the axiom of choice holds (see [Jec03, Theorem 14.24]). We will use automorphisms of $\mathbb{P}$ to define an intermediate model between $\mathfrak{M}$ and a generic extension $\mathfrak{M}[G]$, which will violate the axiom of choice. From here on, $\mathbb{P}$ will always denote an arbitrary notion of forcing.

Recall that an automorphism of $\mathbb{P}$ is a $\leq$-preserving bijection of P with itself. We denote the group of automorphisms of $\mathbb{P}$ as $\mathrm{Aut}(\mathbb{P})$. Given

$\pi \in \mathrm{Aut}(\mathbb{P})$ we extend it to $\mathfrak{M}^{\mathbb{P}}$ by induction:

$$\pi\dot{x} = \{\langle \pi p, \pi\dot{y}\rangle \mid \langle p, \dot{y}\rangle \in \dot{x}\}$$

Proposition 1.1. *For every $x \in \mathfrak{M}$ and every $\pi \in \mathrm{Aut}(\mathbb{P})$ we have $\pi\check{x} = \check{x}$.*

Proof. By induction on rank,

$$\pi\check{x} = \{\langle \pi 1, \pi\check{y}\rangle \mid y \in x\} = \{\langle 1, \check{y}\rangle \mid y \in x\} = \check{x}.$$ □

Proposition 1.2. *For every $\pi \in \mathrm{Aut}(\mathbb{P})$ and every $\dot{x} \in \mathfrak{M}^{\mathbb{P}}$, if $\dot{x} \in \mathfrak{M}^{\mathbb{P}}_{\alpha}$ then $\pi\dot{x} \in \mathfrak{M}^{\mathbb{P}}_{\alpha}$.*

Proof. By induction on the $\mathbb{P}$-rank of $\dot{x}$. □

Definition 1.1 (The Forcing Relation)**.** Let $\dot{a}, \dot{b} \in \mathfrak{M}^{\mathbb{P}}$ be two $\mathbb{P}$-names, and let p be a condition in $\mathbb{P}$.

- $\mathbf{p} \Vdash \dot{\mathbf{a}} = \dot{\mathbf{b}}$ if and only if
  For every $\langle s, \dot{x}\rangle \in \dot{a}$ the following set is dense below p:

  $$\left\{q \in P \;\middle|\; q \leq s \to \exists \langle r, \dot{y}\rangle \in \dot{b} : (q \leq r \wedge q \Vdash \dot{x} = \dot{y})\right\},$$

  and for every $\langle r, \dot{y}\rangle \in \dot{b}$ the following set is dense below p as well:

  $$\{q \in P \mid q \leq r \to \exists \langle s, \dot{x}\rangle \in \dot{a} : (q \leq s \wedge q \Vdash \dot{x} = \dot{y})\}.$$

- $\mathbf{p} \Vdash \dot{\mathbf{a}} \in \dot{\mathbf{b}}$ if and only if the following set is dense below p:

  $$\left\{q \in P \;\middle|\; \exists \langle r, \dot{y}\rangle \in \dot{b} : q \leq r \wedge q \Vdash \dot{a} = \dot{y}\right\}.$$

Of course the above hides an induction on the maximal $\mathbb{P}$-rank of the names $\dot{a}, \dot{b}$. We now extend this to general formulas by induction on the complexity of the formula. If $\varphi(u), \psi(u)$ are formulas in the language of forcing then

- $\mathbf{p} \Vdash \varphi \wedge \psi$ if and only if $p \Vdash \varphi$ and $p \Vdash \psi$.
- $\mathbf{p} \Vdash \neg\varphi$ if and only if there is no $q \leq p$ such that $q \Vdash \varphi$.
- $\mathbf{p} \Vdash \exists \mathbf{x}\varphi(\mathbf{u})$ if and only if the following set is dense below p:

  $$\left\{q \leq p \;\middle|\; \exists \dot{x} \in \mathfrak{M}^{\mathbb{P}} : q \Vdash \varphi(\dot{x})\right\}.$$

An interesting remark regarding the definition of $p \Vdash \exists x\varphi$, is that there is a very useful lemma (often called The Fullness Lemma, see [Jec03, Lemma 14.19]) asserting that $p \Vdash \exists x\varphi(u)$ if and only if there exists a $\mathbb{P}$-name $\dot{x}$ such that $p \Vdash \varphi(\dot{x})$. This lemma is in fact equivalent to AC (see [Mil11]).

Lemma 1.3 (The Symmetry Lemma). *Suppose $p \in \mathbb{P}$, $\pi \in \mathrm{Aut}(\mathbb{P})$, $\dot{x}$ a $\mathbb{P}$-name and $\varphi(x)$ is a formula. Then $p \Vdash \varphi(\dot{x})$ if and only if $\pi p \Vdash \varphi(\pi\dot{x})$.*

Proof. We prove this by induction on the complexity of φ and the $\mathbb{P}$-rank of $\dot{a}, \dot{b}$. Suppose that p is a condition, $\dot{a}, \dot{b}$ are names and $\pi \in \operatorname{Aut}(\mathbb{P})$

$$\begin{aligned}
&p \Vdash \dot{a} \in \dot{b} \\
&\iff \left\{q \in P \;\middle|\; \exists \langle r, \dot{y}\rangle \in \dot{b} : q \leq r \wedge q \Vdash \dot{a} = \dot{y}\right\} \\
&\qquad\qquad \text{is dense below } p \\
&\iff \left\{\pi q \in P \;\middle|\; \exists \langle r, \dot{y}\rangle \in \dot{b} : q \leq r \wedge q \Vdash \dot{a} = \dot{y}\right\} \\
&\qquad\qquad \text{is dense below } \pi p \\
&\iff \left\{\pi q \in P \;\middle|\; \exists \langle \pi r, \pi\dot{y}\rangle \in \pi\dot{b} : \pi q \leq \pi r \wedge \pi q \Vdash \pi\dot{a} = \pi\dot{y}\right\} \\
&\qquad\qquad \text{is dense below } \pi p \\
&\iff \left\{q \in P \;\middle|\; \exists \langle r, \dot{y}\rangle \in \pi\dot{b} : q \leq r \wedge q \Vdash \pi\dot{a} = \dot{y}\right\} \\
&\qquad\qquad \text{is dense below } \pi p \\
&\iff \pi p \Vdash \pi\dot{a} \in \pi\dot{b}.
\end{aligned}$$

For more complicated formulas, as well as for $p \Vdash \dot{a} = \dot{b}$, the argument is similar. □

Definition 1.2. Let $\mathscr{G}$ be a group. We say that a non-empty $\mathcal{F} \subseteq \mathcal{P}(\mathscr{G})$ is a **normal filter of subgroups** if the following holds:

- If $H \in \mathcal{F}$ then H is a non-trivial subgroup of $\mathscr{G}$.
- If H, K are subgroups of $\mathscr{G}$ and $H \in \mathcal{F}$ and $H \leq K$ then $K \in \mathcal{F}$.
- If $H, K \in \mathcal{F}$ then $H \cap K \in \mathcal{F}$.
- For every $\pi \in \mathscr{G}$ and every $H \in \mathcal{F}$, $\pi^{-1}H\pi \in \mathcal{F}$.

Suppose that $\mathscr{G}$ is a group of permutations of a collection A, and $B \subseteq A$ is a subcollection. We define two subgroups of $\mathscr{G}$: the **stabilizer of** B, $\operatorname{sym}_{\mathscr{G}}(B) = \{\pi \in \mathscr{G} \mid \pi''B = B\}$; and the **pointwise stabilizer of** B, $\operatorname{fix}_{\mathscr{G}}(B) = \{\pi \in \mathscr{G} \mid \forall b \in B : \pi(b) = b\}$. If $\mathscr{G}$ is clear from context we omit it and write $\operatorname{sym}(B)$ and $\operatorname{fix}(B)$.

If $\mathscr{G}$ acts on $\mathbb{P}$ we extend the above definitions to $\mathbb{P}$-names in a slightly different manner. For a name $\dot{x}$:

- $\operatorname{sym}_{\mathscr{G}}(\dot{x}) = \{\pi \in \mathscr{G} \mid \pi(\dot{x}) = \dot{x}\}$, and
- $\operatorname{fix}_{\mathscr{G}}(\dot{x}) = \{\pi \in \operatorname{sym}_{\mathscr{G}}(\dot{x}) \mid \forall \langle p, \dot{a}\rangle \in \dot{x} : \pi(\dot{a}) = \dot{a}\}$.

We will never use the first meanings when applying sym or fix to $\mathbb{P}$-names, only on rare occasions fix will be used for names, and never sym will be used for arbitrary sets.

Definition 1.3. Let $\mathscr{G} \in \mathfrak{M}$ be a group of permutations of $\mathbb{P}$ and $\mathcal{F}$ a normal filter of subgroups of $\mathscr{G}$. Let $\dot{x}$ be a $\mathbb{P}$-name, we say that $\dot{x}$ is **$\mathcal{F}$-symmetric** if $\operatorname{sym}_{\mathscr{G}}(\dot{x}) \in \mathcal{F}$.

We define by induction the class of **hereditarily $\mathcal{F}$-symmetric names**, denoted by $\mathsf{HS}_\mathcal{F}$:

$$\dot{x} \in \mathsf{HS}_\mathcal{F} \iff \{\dot{y} \mid \exists p : \langle p, \dot{y}\rangle \in \dot{x}\} \subseteq \mathsf{HS}_\mathcal{F} \wedge \dot{x} \text{ is } \mathcal{F}\text{-symmetric}.$$

As usual if $\mathscr{G}$ and $\mathcal{F}$ are clear from context we omit them completely.

Proposition 1.4. *If $\dot{x} \in \mathsf{HS}_\mathcal{F}$ and $\pi \in \mathscr{G}$ then $\pi\dot{x} \in \mathsf{HS}_\mathcal{F}$.*

Proof. We will prove this by induction on the $\mathbb{P}$-rank of $\dot{x}$, but first we will show that $\operatorname{sym}(\pi\dot{x}) = \pi^{-1}\operatorname{sym}(\dot{x})\pi$:

$$\begin{aligned} \sigma \in \operatorname{sym}(\pi\dot{x}) &\iff \sigma(\pi\dot{x}) = \pi\dot{x} \iff (\sigma\pi)\dot{x} = \pi\dot{x} \\ &\iff \pi^{-1}\sigma\pi\dot{x} = \dot{x} \iff \sigma \in \pi^{-1}\operatorname{sym}(\dot{x})\pi. \end{aligned}$$

Since $\mathcal{F}$ is a normal filter we have that if $\operatorname{sym}(\dot{x}) \in \mathcal{F}$ then $\operatorname{sym}(\pi\dot{x}) \in \mathcal{F}$ for every $\pi \in \mathscr{G}$. The inductive argument is as usual. Therefore the proposition holds. □

Theorem 1.5. *Let $\mathscr{G} \in \mathfrak{M}$ be a group of automorphisms of $\mathbb{P}$, and $\mathcal{F} \in \mathfrak{M}$ a normal filter of subgroups. Let G be a $\mathbb{P}$-generic filter over $\mathfrak{M}$, and $\mathsf{HS}_\mathcal{F}$ the class of hereditarily $\mathcal{F}$-symmetric names. We define the interpretation of $\mathsf{HS}_\mathcal{F}$ by G to be the class $N = (\mathsf{HS}_\mathcal{F})^G = \left\{\dot{x}^G \bigm| \dot{x} \in \mathsf{HS}_\mathcal{F}\right\}$ and $\mathfrak{N} = \langle N, \in\rangle$, then $\mathfrak{M} \subseteq \mathfrak{N} \subseteq \mathfrak{M}[G]$ and $\mathfrak{N}$ is a transitive model of ZF.*

Before the proof we must remark that the requirement that $\mathscr{G}, \mathcal{F}$ are in the ground model is somewhat essential, otherwise we might be able to code generic sets via automorphisms in the generic extension and the names from the ground model will not suffice to construct $\mathfrak{N}$.

Proof. We will omit $\mathscr{G}$ and $\mathcal{F}$ from the notation, as we only have one group of automorphisms and one filter. From Proposition 1.1 we have that for every $x \in \mathfrak{M}$, $\operatorname{sym}(\check{x}) = \mathscr{G} \in \mathcal{F}$ therefore $\check{\mathfrak{M}} \subseteq \mathsf{HS}$ and so $\mathfrak{M} \subseteq \mathfrak{N}$, it is also clear that $\mathfrak{N} \subseteq \mathfrak{M}[G]$.

The transitivity of $\mathfrak{N}$ is immediate: if $y \in x \in \mathfrak{N}$ then there is a name $\dot{y}$ and some $p \in G$ such that $\langle p, \dot{y}\rangle \in \dot{x}$, where $\dot{x} \in \mathsf{HS}$. Therefore by the definition of HS we have that $\dot{y} \in \mathsf{HS}$ as so $y \in \mathfrak{N}$ as wanted.

To show that $\mathfrak{N}$ is a model of ZF we will show that it is almost-universal and closed under Gödel operations (see [Jec03, Theorem 13.9]). Recall that $\mathfrak{N}$ is almost-universal if for every $x \in \mathfrak{M}[G]$ if $x \subseteq N$ then there is $Y \in N$ such that $x \subseteq Y$.

To see almost universality holds we will simply show that the set $Y_\alpha = \left\{\dot{x}^G \bigm| \dot{x} \in \mathsf{HS} \cap \mathfrak{M}^\mathbb{P}_\alpha\right\} \in \mathfrak{N}$ for all α, then if $x \subseteq \mathfrak{N}$ then $x \subseteq Y_\alpha$ for some α. Consider the name

$$\dot{Y}_\alpha = \left\{\dot{x} \bigm| \dot{x} \in \mathsf{HS} \cap \mathfrak{M}^\mathbb{P}_\alpha\right\}^\bullet.$$

Using Proposition 1.2 and Proposition 1.4 we have that indeed $\operatorname{sym}(\dot{Y}_\alpha) = \mathscr{G}$, and $\dot{Y}_\alpha \subseteq P \times \mathsf{HS}$. It is clear that $\dot{Y}_\alpha^G = Y_\alpha$, and therefore $\mathfrak{N}$ is almost universal.

We recall the Gödel operations:

$$
\begin{aligned}
&G_1(X,Y) = \{X,Y\}\\
&G_2(X,Y) = X \times Y\\
&G_3(X,Y) = \{\langle u,v\rangle \in X \times Y \mid u \in v\}\\
&G_4(X,Y) = X \setminus Y\\
&G_5(X,Y) = X \cap Y\\
&G_6(X) = \bigcup X\\
&G_7(X) = \mathrm{dom}(X)\\
&G_8(X) = \{\langle u,v\rangle \mid \langle v,u\rangle \in X\}\\
&G_9(X) = \{\langle u,v,w\rangle \mid \langle u,w,v\rangle \in X\}\\
&G_{10}(X) = \{\langle u,v,w\rangle \mid \langle v,w,u\rangle \in X\}
\end{aligned}
$$

We observe that if $\dot{x},\dot{y} \in \mathsf{HS}$ then $\{\dot{x},\dot{y}\}^\bullet \in \mathsf{HS}$, so $\mathfrak{N}$ is closed under $G_1(X,Y)$; in turn this implies that $\langle\dot{x},\dot{y}\rangle^\bullet \in \mathsf{HS}$ as well, so for G_2 we can see that the following name is hereditarily symmetric, whose stabilizer contains $\mathrm{sym}(\dot{x}) \cap \mathrm{sym}(\dot{y})$:

$$\{\langle 1, \langle\dot{u},\dot{v}\rangle^\bullet\rangle \mid \exists p,q \in \mathbb{P} : \langle p,\dot{u}\rangle \in \dot{x} \wedge \langle q,\dot{v}\rangle \in \dot{y}\}\,.$$

We define similar names for the operations $G_3, G_4, G_5, G_7, G_8, G_9, G_{10}$. We only need to verify that $\mathfrak{N}$ is closed under G_6, that is to say, $\mathfrak{N}$ satisfies the axiom of union.

Suppose that $\dot{x} \in \mathsf{HS}$, we define the name

$$\dot{Y} = \{\langle r,\dot{u}\rangle \mid \exists p,q \in \mathbb{P}\exists\dot{y} : \langle p,\dot{y}\rangle \in \dot{x} \wedge \langle q,\dot{u}\rangle \in \dot{y} \wedge r \leq p,q\}\,.$$

It is immediate that $\dot{Y} \subseteq P \times \mathsf{HS}$, and if $\pi \in \mathrm{sym}(\dot{x})$ then $\pi\dot{Y} = \dot{Y}$, so $\mathrm{sym}(\dot{x}) \leq \mathrm{sym}(\dot{Y})$, and so $\dot{Y} \in \mathsf{HS}$. To see that $\dot{Y}^G = \bigcup \dot{x}^G$, we see that $\dot{u}^G \in \dot{Y}^G$ if and only if for some $r \in G$ we have $\langle r,\dot{u}\rangle \in \dot{Y}$ if and only if there is some $p,q \in P$ and $\dot{y} \in \mathsf{HS}$ such that $r \leq p,q$, $\langle p,\dot{y}\rangle \in \dot{x}$ and $\langle q,\dot{u}\rangle \in \dot{y}$. Since G is a filter we have that $p,q \in G$ as well, therefore $\dot{u}^G \in \dot{y}^G \in \dot{x}^G$. □

Lemma 1.6. *Suppose that $\mathfrak{N}$ is a symmetric extension of $\mathfrak{M}$, and $A \in \mathfrak{N}$, such that $\dot{A} \in \mathsf{HS}$ is a name of A and $\mathrm{fix}(\dot{A}) \in \mathcal{F}$, then A can be well-ordered in $\mathfrak{N}$.*

Proof. In $\mathfrak{M}$ let $\langle \dot{a}_\alpha \mid \alpha < \kappa\rangle$ be an injective enumeration of all $\dot{a}$ such that $\langle p,\dot{a}\rangle \in \dot{A}$ for some $p \in \mathbb{P}$. We define the following name

$$\dot{f} = \{\langle\check{\alpha},\dot{a}_\alpha\rangle^\bullet \mid \alpha < \kappa\}^\bullet$$

First we claim that $\dot{f}$ is symmetric. Suppose that $\pi \in \mathrm{fix}(\dot{A})$ then $\pi\langle\check{\alpha},\dot{a}_\alpha\rangle^\bullet = \langle\check{\alpha},\dot{a}_\alpha\rangle^\bullet$, and therefore $\pi \in \mathrm{fix}(\dot{f})$. We also have that $\dot{f} \subseteq P \times \mathsf{HS}$, and therefore $\dot{f} \in \mathsf{HS}$ as well.

We see that $f = \dot{f}^G$ is a function whose domain is κ, and the range of f contains A, therefore A can be well-ordered in $\mathfrak{N}$ as wanted. □

1.2 An Example of a Symmetric Extension

Let $\mathfrak{M}$ be a transitive model of **ZFC**, and let $\lambda \leq \kappa$ be two regular cardinals. We will construct a symmetric extension of $\mathfrak{M}$ in which there exists a set X such that:

1. $\aleph(X) \leq \lambda$, and

2. $\kappa \leq \aleph^*(X)$.

We improve a result by Monro [Mon75] which assumed that $\mathfrak{M} \models V = L$ and $\lambda = \omega$.

Let $\mathbb{P} = \langle P, \leq \rangle$ be the following notion of forcing: $p \in P$ is a function from $\kappa \times \kappa$ into 2, such that $|\operatorname{dom}(p)| < \kappa$, and $p \leq q$ if and only if $q \subseteq p$. Note that by the regularity of κ this forcing is κ-closed, and therefore it collapses no cardinals below κ^+.

It is not hard to see that if G is $\mathbb{P}$-generic over $\mathfrak{M}$ then in $\mathfrak{M}[G]$ we adjoined κ new subsets of κ,

$$x_\alpha = \left\{\beta < \kappa \;\middle|\; \bigcup G(\alpha, \beta) = 1\right\}.$$

Let $\dot{x}_\alpha = \left\{\left\langle p, \check{\beta}\right\rangle \;\middle|\; p(\alpha, \beta) = 1\right\}$ be the canonical name of x_α. For the set $X = \{x_\alpha \mid \alpha < \kappa\}$ we give the canonical name $\dot{X} = \{\dot{x}_\alpha \mid \alpha < \kappa\}^\bullet$. We remark that $1 \Vdash \dot{x}_\alpha \neq \dot{x}_\beta$ for $\alpha \neq \beta$, by a simple genericity argument.

Let $S(\kappa)$ denote the group of permutations of κ. For $\pi \in S(\kappa)$ we define its action on $\mathbb{P}$ as follows:

$$\pi p = \{\langle \pi\alpha, \beta, \varepsilon\rangle \mid p(\alpha, \beta) = \varepsilon\}.$$

We check how π acts on the $\dot{x}_\alpha$:

$$\pi \dot{x}_\alpha = \left\{\left\langle \pi p, \check{\beta}\right\rangle \;\middle|\; p(\alpha, \beta) = 1\right\} = \left\{\left\langle p, \check{\beta}\right\rangle \;\middle|\; p(\pi\alpha, \beta) = 1\right\} = \dot{x}_{\pi\alpha}.$$

Therefore $\dot{x}_\alpha$ is fixed whenever $\pi \in \operatorname{fix}_{S(\kappa)}(\{\alpha\})$, and $\pi\dot{X} = \dot{X}$ for every $\pi \in S(\kappa)$.

Consider now the filter generated by the ideal $[\kappa]^{<\lambda}$, namely

$$\mathcal{F} = \left\{H \leq S(\kappa) \;\middle|\; \exists E \in [\kappa]^{<\lambda} : \operatorname{fix}(E) \leq H\right\}.$$

We observe that $[\kappa]^{<\lambda}$ is λ-closed and therefore $\mathcal{F}$ is λ-closed as well.

Let $\mathfrak{N} = (\mathsf{HS}_\mathcal{F})^G$. The above shows that $\dot{x}_\alpha$ and $\dot{X}$ all in $\mathsf{HS}_\mathcal{F}$, so x_α is in $\mathfrak{N}$ as well as X. We will show that in $\mathfrak{N}$ it is true that $\kappa \leq^* X$ and $\lambda \nleq X$, these two imply that $\mathfrak{N} \not\models \mathsf{AC}$, since in **ZFC** if $\lambda \leq \kappa \leq^* X$ then we have that $\lambda \leq X$.

In the next section we see prove that in fact $\lambda = \aleph(X)$ in $\mathfrak{N}$, it will be an immediate consequence of Lemma 1.9.

To see that $\lambda \nleq X$, suppose that $\dot{f}$ was a hereditarily symmetric name for an injective function from λ into X, and let $E \in [\kappa]^{<\lambda}$ such that whenever $\pi \in S(\kappa)$ and $\pi\restriction_E = \operatorname{id}_E$ we have $\pi\dot{f} = \dot{f}$.

Suppose that $p \Vdash \dot{f}\colon \check{\lambda} \to \dot{X}$ is injective, let $q \leq p$ be an extension such that for some $\alpha \notin E$ we have that $q \Vdash \dot{f}(\check{\gamma}) = \dot{x}_\alpha$, for some $\gamma < \lambda$. Take $\beta \notin E$ such that there is no $\tau < \kappa$ for which $\langle \beta, \tau\rangle \in \operatorname{dom}(q)$, we can find such β since $|\operatorname{dom}(q)| < \kappa$. Let π be the permutation such that $\pi(\alpha) = \beta$ and $\pi(\beta) = \alpha$, and $\pi(\tau) = \tau$ otherwise. It is clear that $\pi \in \operatorname{fix}(E)$ and therefore $\pi\dot{f} = \dot{f}$.

To see that πq is compatible with q, note that $\langle \delta, \tau\rangle \in \operatorname{dom}(q) \cap \operatorname{dom}(\pi q)$ then $\delta \notin \{\alpha, \beta\}$ and therefore $\pi q(\delta, \tau) = q(\delta, \tau)$. On the other hand $\pi q \Vdash \dot{f}(\check{\gamma}) = \dot{x}_\beta$. Take $r \leq q, \pi q$; then $r \Vdash \dot{x}_\alpha = \dot{f}(\check{\gamma}) = \dot{x}_\beta \wedge \dot{x}_\alpha \neq \dot{x}_\beta$, which is a contradiction.

On the other hand, we will construct in $\mathfrak{N}$ a function from X onto κ defined as follows:

$$f(x) = \min \{\beta < \kappa \mid \beta \notin x\}.$$

This function is definable in $\mathfrak{N}$, and it is well-defined since $x \neq \kappa$ for all $x \in X$. Genericity implies that f is surjective, since for every $\beta < \kappa$ if $p \in P$, letting $\alpha < \kappa$ be such that for all $\langle \gamma, \delta\rangle \in \operatorname{dom}(p)$ we have $\gamma \neq \alpha$, we can extend p to $q = p \cup \{\langle \alpha, \gamma, 1\rangle \mid \gamma < \beta\} \cup \{\langle \alpha, \beta, 0\rangle\}$.

We remark that if κ^+ was not collapsed (e.g. $\kappa^{<\kappa} = \kappa$) then it is clear that there is no function from X onto κ^+ in $\mathfrak{M}[G]$ and therefore there is no such function in $\mathfrak{N}$, and so $\aleph^*(X) = \kappa^+$.

1.3 Choice Principles

We say that a sentence in the language of set theory φ is a **choice principle** if it is provable from ZFC, but not from ZF[1]. There is a plethora of choice principles in different areas of mathematics, but we will focus our interest on three particular families of choice principles defined below.

Definition 1.4. Let κ be an $\aleph$ cardinal. We define the following axioms:

DC_κ For every binary relation R, if X is such set that for every $\alpha < \kappa$ and every α-sequence in X ($f\colon \alpha \to X$), there is some $x \in X$ such that fRx, then there is a κ-sequence f in X, such that for all $\alpha < \kappa$, $(f\restriction \alpha)Rf(\alpha)$.

AC_κ Every family of κ many non-empty set has a choice function.

W_κ Every infinite set is either well-ordered or has a subset of size κ.

It is clear that AC_κ is simply a restriction of AC to families of size κ, and W_κ is the restriction of the well-ordering principle to sets of size κ. However it is not clear where DC_κ is coming from. The best way to understand it is to think of it as a restricted version of Hausdorff's maximality principle, to maximal chains of length κ. However an equivalent formulation of DC_κ which may be a bit clearer is actually the restriction of Zorn's lemma to chains of length κ (see [HR98, Forms 87, 87(A)]):

[1]This is not a canonical definition, see http://mathoverflow.net/questions/104016 for a related discussion.

Theorem 1.7. DC_κ *is equivalent to the assertion that for every partially ordered set $\langle P, \leq\rangle$ in which every well-ordered chain is of order-type $< \kappa$ and has an upper bound, then P has a maximal element.*

See [Wol83] for the proof.

We abbreviate by $\mathsf{DC}_{<\kappa}$ the statement $(\forall\lambda < \kappa)\mathsf{DC}_\lambda$. Similarly for $\mathsf{W}_{<\kappa}$ and $\mathsf{AC}_{<\kappa}$. We will give the proof of the following theorem describing the implications between the above principles, as it appears in [Jec73, Theorem 8.1], the following theorem and proof are due to Lévy (see [Lév64]).

Theorem 1.8. *Let κ be an $\aleph$ cardinal, the following implications hold in* ZF*:*

(a) *If $\lambda < \kappa$ then DC_κ implies DC_λ, W_κ implies W_λ, and AC_κ implies AC_λ.*

(b) *DC_κ implies both W_κ and AC_κ.*

(c) *$(\forall\kappa)\mathsf{DC}_\kappa$ and $(\forall\kappa)\mathsf{W}_\kappa$ both imply* AC.

(d) *If κ singular then $\mathsf{DC}_{<\kappa}$ implies DC_κ, and $\mathsf{AC}_{<\kappa}$ implies AC_κ.*

(e) *If κ is a limit cardinal then $\mathsf{W}_{<\kappa}$ and $\mathsf{AC}_{\mathrm{cf}(\kappa)}$ imply W_κ, in particular $\mathsf{AC}_{\aleph_0}$ implies $\mathsf{W}_{\aleph_0}$.*

Proof. *(a)* It is clear that W_κ implies W_λ, and AC_κ implies AC_λ. To see that DC_κ implies DC_λ, suppose that R is a binary relation and X a non-empty set such that whenever $\alpha < \lambda$ and g is an α-sequence in X there is some $x \in X$ such that gRx. Fix $x_0 \in X$ and extend R to R' such that whenever g is an α-sequence, for $\lambda \leq \alpha < \kappa$, it holds that $gR'x_0$. We have that R' satisfies the assumptions needed for using DC_κ.

DC_κ asserts the existence of a κ-sequence, such that for all $\alpha < \kappa$, $(f\restriction\alpha)R'f(\alpha)$. Observe that if $\alpha < \lambda$ then $(f\restriction\alpha)Rf(\alpha)$, therefore $f\restriction_\lambda$ is a λ-sequence needed to prove DC_λ holds.

(b) Assume DC_κ holds, suppose that X is such that $|X| \not< \kappa$, let R be the relation defined on α-sequences in X for $\alpha < \kappa$, where sRx if and only if $x \notin \operatorname{rng}(s)$. Since $\kappa \not\leq |X|$ for every s we can add another point x. By DC_κ we have that there is a κ-sequence f such that $(f\restriction\alpha)Rf(\alpha)$, i.e. f is injective, and therefore $\kappa \leq |X|$.

The proof of AC_κ is similar: let $A = \{A_\alpha \mid \alpha < \kappa\}$ be a family of non-empty sets, $|A| = \kappa$. We define R to be such that whenever $\alpha < \kappa$ and f an α-sequence in $\bigcup A$ such that $f(\gamma) \in A_\gamma$ for all $\gamma < \alpha$; for $a \in \bigcup A$ we have that fRa if and only if $a \in A_\alpha$. Since A_α is non-empty we can always find such a. It is clear that f given by DC_κ is a choice function from the family A.

(c) By the above we have that $(\forall\kappa)\mathsf{DC}_\kappa$ implies that $(\forall\kappa)\mathsf{W}_\kappa$, and if the latter holds then every set can be compared with its Hartogs number, therefore the well-ordering principle holds.

(d) Suppose now that κ is singular, $\mathrm{cf}(\kappa) = \mu$. Let $\{\kappa_\xi \mid \xi < \mu\}$ be a cofinal and increasing sequence of cardinals below κ.

Suppose $\mathsf{DC}_{<\kappa}$ holds. Let R be a binary relation and X non-empty such that whenever $\alpha < \kappa$ and $f\colon \alpha \to X$ there is some $x \in X$ such that fRx. By $\mathsf{DC}_{<\kappa}$, for every $\lambda < \kappa$ we have $f\colon \lambda \to X$ witnessing DC_λ. Let T be the set of all functions from κ_ξ into X for $\xi < \mu$. For an α-sequence s in T, letting s_t the concatenation of all the terms in s, then s_t is also a γ-sequence in X for some $\gamma < \kappa$.

We define R' to be such that if t is a ξ-sequence in T for $\xi < \mu$, and $z \in T$ then $tR'z$ if and only if z is a κ_ξ-sequence in X and $t^\frown(z \restriction \eta)Rz(\eta)$ for all $\eta < \kappa_\xi$. Since $\mathsf{DC}_{<\kappa}$ holds we can always extend ξ-sequences. Using DC_μ we have t a μ-sequence in T such that s_t is a κ-sequence in X as wanted.

Now suppose that $\mathsf{AC}_{<\kappa}$ holds, consider the family $A = \{A_\alpha \mid \alpha < \kappa\}$ where $A_\alpha \neq \varnothing$ for all α. Let C_ξ be the collection of choice functions on the family $\{A_\alpha \mid \kappa_\xi \leq \alpha < \kappa_{\xi+1}\}$, by the assumption of $\mathsf{AC}_{\kappa_{\xi+1}}$ those are non-empty and by AC_μ we can choose a $f_\xi \in C_\xi$, the union $\bigcup\{f_\xi \mid \xi < \mu\}$ is a choice function from A itself.

(e) Lastly, if κ is a limit cardinal with cofinality μ, and both $\mathsf{W}_{<\kappa}$ and AC_μ hold let $\{\kappa_\xi \mid \xi < \mu\}$ be an increasing cofinal sequence of cardinals in κ. Suppose now that X is a set such that $|X| \not< \kappa$. By $\mathsf{W}_{<\kappa}$ for every κ_ξ there is a subset of X of size κ_ξ. For $\xi < \mu$ let C_ξ be the family of pairs (A, R), $A \subseteq X$, where $|A| = \kappa_\xi$ and R is a well-order of A of type κ_ξ, by the assumption this set is non-empty. Using AC_μ we can choose A_ξ and a well-ordering of it. Let $A = \bigcup_{\xi<\mu} A_\xi \subseteq X$, this is a well-ordered union of enumerated sets which is itself well-ordered and of size κ as wanted. □

Two immediate corollaries from the above are that if λ is the first $\aleph$ cardinal for which DC_λ (or AC_λ) fails then λ is regular, and that if $W_{<\lambda}$ holds and A is not well-orderable then $\aleph(A) \geq \lambda$.

We will now proceed to prove a certain link between the construction of symmetric extensions and $\mathsf{DC}_{<\kappa}$.

Lemma 1.9. *Let $\mathfrak{M}$ be a transitive model of* ZFC*, $\mathbb{P}$ a κ^+-closed notion of forcing in $\mathfrak{M}$. Let $\mathfrak{M}[G]$ be a generic extension by a $\mathbb{P}$-generic filter. Suppose that $\mathfrak{N} \subseteq \mathfrak{M}[G]$ is a symmetric extension of $\mathfrak{M}$ generated by $\mathbb{P}$ and a normal filter of groups $\mathcal{F}$. If $\mathcal{F}$ is κ^+-closed then the symmetric model satisfies* DC_κ.

Proof. Suppose that $X \in \mathfrak{N}$ is a non-empty set, and R a binary relation such that for every $Y \subseteq X$ such that $|Y| < \kappa$ we have some $x \in X$ such that YRx. As $\mathfrak{M}[G] \models \mathsf{AC}$, there exists (in $\mathfrak{M}[G]$) a function $f\colon \kappa \to X$ such that for all $\alpha < \kappa$ it holds that $\{f(\beta) \mid \beta < \alpha\}\, Rf(\alpha)$.

Let $\dot{f}_0$ be a name for the function f, and let p be a condition which forces that $\dot{f}_0$ is a function whose domain is κ, and its range is a subset of $\mathfrak{N}$. Let p_α be a sequence of conditions such that $p_\alpha \leq p_\beta$ for $\beta < \alpha$, and $p_\alpha \Vdash \dot{f}_0(\check{\alpha}) = \dot{x}_\alpha$ for some $\dot{x}_\alpha \in \mathsf{HS}$. By κ^+-closure of $\mathbb{P}$ we have that the sequence can be defined at limit ordinals and at κ, so we have $q = p_\kappa \le p_\alpha$ for all $\alpha < \kappa$. We have that $q \Vdash \dot{f}_0(\check{\alpha}) = \dot{x}_\alpha$ for all $\alpha < \kappa$.

Define the name $\dot{f} = \{\langle\check{\alpha}, \dot{x}_\alpha\rangle^\bullet \mid \alpha < \kappa\}^\bullet$, it is clear that $q \Vdash \dot{f} = \dot{f}_0$. From κ^+-closure of $\mathcal{F}$ we have that

$$\bigcap_{\alpha<\kappa} \operatorname{sym}(\dot{x}_\alpha) = H \in \mathcal{F}.$$

It follows that $\dot{f}$ is fixed *pointwise* by H, and every name appearing in $\dot{f}$ is in HS. Therefore $\dot{f}$ hereditarily symmetric, so it is indeed in $\mathfrak{N}$ as wanted. □

It is not hard to observe that we in fact proved more than the above lemma, in fact we proved the following claim:

Claim. *Suppose the conditions of Lemma 1.9 hold, and $A \in \mathfrak{M}[G]$ is such that $|A|^{\mathfrak{M}[G]} < \kappa^+$ and $A \subseteq \mathfrak{N}$, then $A \in \mathfrak{N}$.*

By the above lemma, in the model constructed in section 1.2 $\mathsf{DC}_{<\lambda}$ holds, and thus $\mathsf{W}_{<\lambda}$. In particular this means that $\lambda \le \aleph(X)$. Since we have shown that $\aleph(X) \le \lambda$, equality follows.

Theorem 1.8 tells us that indeed $(\forall\kappa)\mathsf{DC}_\kappa$ and $(\forall\kappa)\mathsf{W}_\kappa$ are equivalent to the axiom of choice, and therefore to each other. However the relationships between the three restrictions are not as trivial as they may appear at first glance. We quote several theorems describing some of the more surprising results in this context. For a full treatment of the subject see [Jec73, Chapter 8].

Theorem 1.10. *The following statements are true in* ZF*:*

(a) *$(\forall\kappa)\mathsf{AC}_\kappa$ implies $\mathsf{DC}_{\aleph_0}$.*

(b) *There is a model in which $(\forall\kappa)\mathsf{AC}_\kappa$ holds, but $\mathsf{DC}_{\aleph_1}$ and $\mathsf{W}_{\aleph_1}$ both fail.*

(c) *Let κ be a regular cardinal, then there is a model in which $\mathsf{AC}_{<\kappa}$ and $\mathsf{W}_{<\kappa}$ both hold, but $\mathsf{DC}_{\aleph_0}$ fails.*

(d) *Let κ be a regular, then there is a model of $\mathsf{ZF} + \mathsf{DC}_{<\lambda}$, but AC_λ fails on a family of pairs, and W_λ fails.*

(e) *Let κ be a singular with cofinality μ, then there is a model in which $\mathsf{DC}_{<\mu}$, $\mathsf{AC}_{<\mu}$ and $\mathsf{W}_{<\kappa}$ hold, but AC_μ and W_κ fail.*

We conclude this section with a question, which naturally rises from the proof of Lemma 1.9:

Question. Suppose that $\mathfrak{N}$ is a symmetric extension of $\mathfrak{M}$ by a κ^+-closed $\mathbb{P}$ and a filter $\mathcal{F}$. If $\mathfrak{N} \models \mathsf{DC}_\kappa$, does that mean that $\mathcal{F}$ is κ^+-closed?

1.4 2-Permutations and Affine Ideals

We will now define two related notions, 2-permutations (Definition 1.5) and affine-ideals (Definition 1.7) which will be used to generate symmetric models. The motivation is to somehow create a system of permutations of $A \times B$ where the permutations of A are "more important" and set the tone.

The main motivation is to later define a notion of forcing which has canonical names indexed by $A \times B$ where A is endowed with an additional structure. Using 2-permutations we will ensure the preservation of the structure from A to a generic set in the symmetric extension.

Definition 1.5. Let A, B be two non-empty sets, G_A and G_B groups of permutations of A and B respectively. A **2-permutation in** G_A **and** G_B is a pair $\langle \sigma, \pi \rangle$ such that $\sigma \in G_A$, and $\pi \colon A \to G_B$, we shall denote $\pi(a)$ by π_a. If G_A and G_B are clear from context then we will simply refer to it as a 2-permutation of A and B, or just 2-permutation.

If $\Sigma = \langle \sigma, \pi \rangle$ is a 2-permutation its action on $A \times B$ is a permutation of $A \times B$ defined as: $\Sigma(a, b) = \langle \sigma a, \pi_a b \rangle$.

Proposition 1.11. *Let A, B be two non-empty sets and G_A, G_B groups of permutations of A and B respectively. Let $\Sigma_1, \Sigma_2, \Sigma_3$ be 2-permutations in G_A and G_B, then:*

(a) *Σ_1 and Σ_2 act on $A \times B$ the same way if and only if $\Sigma_1 = \Sigma_2$.*

(b) *The composition $\Sigma_2 \circ \Sigma_1$ as permutations is a 2-permutation.*

(c) $(\Sigma_3 \circ \Sigma_2) \circ \Sigma_1 = \Sigma_3 \circ (\Sigma_2 \circ \Sigma_1)$.

(d) *The identity function of $A \times B$ is a 2-permutation.*

(e) *If Σ is a 2-permutation then Σ^{-1} is a 2-permutation.*

Proof. Let us denote $\Sigma_i = \langle \sigma^i, \pi^i \rangle$ for $i = 1, 2, 3$ and $\pi^i_a = \pi^i(a)$ for $a \in A$.

(a) It is clear that if $\Sigma_1 = \Sigma_2$ then they act the same way on $A \times B$. Suppose now that $\Sigma_1 \neq \Sigma_2$. If $\sigma^1 \neq \sigma^2$ then there is some $a \in A$ such that $\sigma^1(a) \neq \sigma^2(a)$. In this case for any pair $\langle a, b \rangle$ we have that $\Sigma_1(a, b) \neq \Sigma_2(a, b)$. The other case is that for some $a \in A$ we have $\pi^1_a \neq \pi^2_a$, so for some $b \in B$ those differ and we have that $\Sigma_1(a, b) \neq \Sigma_2(a, b)$ as wanted.

(b) We proceed to verify that the composition is a 2-permutation. Take a pair $\langle a, b \rangle \in A \times B$,

$$\begin{aligned} \Sigma_2 \circ \Sigma_1(a, b) &= \Sigma_2(\Sigma_1(a, b)) \\ &= \Sigma_2(\sigma^1(a), \pi^1_a(b)) \\ &= \left\langle \sigma^2 \sigma^1(a), \pi^2_{\sigma^1(a)} \pi^1_a(b) \right\rangle, \end{aligned}$$

therefore the 2-permutation $\langle \sigma, \pi \rangle$, $\sigma = \sigma^2\sigma^1$ and $\pi_a = \pi^2_{\sigma^1(a)}\pi^1_a$, have the same action on $A \times B$ as $\Sigma_2 \circ \Sigma_1$, so they are equal and the composition is well-defined.

(c) The composition is associative since the composition of permutations is associative. Since a composition of 2-permutations is a 2-permutation, and $\Sigma_3 \circ (\Sigma_2 \circ \Sigma_1)$ acts the same on $A \times B$ as $(\Sigma_3 \circ \Sigma_2) \circ \Sigma_1$ we have that they are indeed equal as 2-permutations.

(d) We define the 2-permutation $\mathsf{ID} = \left\langle \mathrm{id}_A, \pi^{id_B} \right\rangle$, where $\pi^{\mathrm{id}_B}_a = \mathrm{id}_B$ for all $a \in A$. If $\langle a, b \rangle \in A \times B$ is any pair then $\mathsf{ID}(a, b) = \langle a, b \rangle$ and therefore this is indeed $\mathrm{id}_{A \times B}$ as wanted.

(e) Lastly if $\Sigma = \langle \sigma, \pi \rangle$ is a 2-permutation, we define Σ^* to be the 2-permutation $\langle \sigma^{-1}, \rho \rangle$ such that $\rho_a = \pi^{-1}_{\sigma^{-1}(a)}$. For a pair $\langle a, b \rangle$ we have:

$$\begin{aligned}(\Sigma \circ \Sigma^*)(a, b) &= \Sigma\left(\sigma^{-1}(a), \pi^{-1}_{\sigma^{-1}(a)}(b)\right) \\ &= \left\langle \left(\sigma\sigma^{-1}\right)(a), \left(\pi_{\sigma^{-1}(a)}\pi_{\sigma^{-1}(a)}\right)(b) \right\rangle = \langle a, b \rangle .\end{aligned}$$

Therefore $\Sigma \circ \Sigma^*$ is the identity, and $\Sigma^* = \Sigma^{-1}$ as wanted. □

We therefore proved that given non-empty sets A, B and permutation groups G_A, G_B, the collection of all 2-permutations forms a group of permutations of $A \times B$, and we can now simply talk about groups of 2-permutations. We remark that if both sets have more than one element then not every permutation of $A \times B$ is a 2-permutation, although this is of no consequence in our case.

Since we would like to use 2-permutations for automorphisms of the forcing poset, we might as well define a matching notion of ideal of supports as a mean of generating a normal filter of subgroups.

Definition 1.6. Let A and B be non-empty sets, and let $\star$ be an element not in B. We define the **$\star$-product of A and B** to be $A \star B = A \times (B \cup \{\star\})$.

If $E \subseteq A \star B$ we shell denote by E^A the projection of E onto A, namely $\{a \in A \mid \exists b \in B \cup \{\star\} : \langle a, b \rangle \in E\}$, and for every $a \in A$ we shall denote by E_a the set $\{b \in B \mid \langle a, b \rangle \in E\}$ which is the section of E at a without, perhaps, the $\star$ element.

The reason we added $\star$ into the game is that later we will want to talk about E such that $E^A \neq \varnothing$ but for all $a \in A$, $E_a = \varnothing$, and this is impossible to achieve if we require $E \subseteq A \times B$.

We may assume as well that $\star$ is fixed throughout the entire discussion and does not appear in any set except $\star$-products, so it is meaningful to write $A \star \varnothing \subseteq A \star B$.

If $\Sigma = \langle \sigma, \pi \rangle$ is a 2-permutation of A and B then Σ acts on $A \star B$ the same way it does on $A \times B$ with the addition that $\Sigma(a, \star) = \langle \sigma a, \star \rangle$.

Definition 1.7. Let A, B be non-empty sets, I ideal on A containing all the singletons, and J ideal on B containing all the singletons.

$$I \star J = \left\{ E \subseteq A \star B \;\middle|\; E^A \in I \wedge \forall a \in A : E_a \in J \right\}$$

is called the **affine ideal on** $A \star B$ **generated by** I **and** J.

If $\mathscr{M}$ is a group of 2-permutations of A and B, we say that $I \star J$ **is normal in** $\mathscr{M}$ if for every $\Sigma \in \mathscr{M}$ and every $E \in I \star J$ we have $\Sigma'' E \in I \star J$.

We first observe that if $E \in I \star J$, and $a \notin E^A$ then automatically $E_a = \varnothing$, and therefore it is in J. It is not hard to see that $I \star J$ is an ideal over $A \star B$, and that if I and J are both λ-closed then $I \star J$ is λ-closed. The last thing remains to be proved before this can be applied to symmetric forcing is that normal ideals generate normal filters of subgroups.

Proposition 1.12. *Let A and B be non-empty sets, and let $\mathscr{M}$ be a group of 2-permutations of A and B, and $I \star J$ an affine ideal on $A \star B$ which is normal in $\mathscr{M}$. Then the set*

$$\mathcal{F} = \{H \leq \mathscr{M} \mid \exists E \in I \star J : \operatorname{fix}_{\mathscr{M}}(E) \leq H\}$$

is a normal filter of subgroups of $\mathscr{M}$. Furthermore, if I is λ-complete then $\mathcal{F}$ is λ-complete.

Proof. We will omit $\mathscr{M}$ from the notation in the following proof. First we see that if $H \in \mathcal{F}$ then there exists $E \in I \star J$ such that $\operatorname{fix}(E) \leq H$ and therefore if $H \leq K$ then $\operatorname{fix}(E) \leq K$ and so $K \in \mathcal{F}$; this also implies that $\mathscr{M} \in \mathcal{F}$ as well.

To show closure under conjugation it suffices to show that if $\Lambda \in \mathscr{M}$ then $\Lambda \operatorname{fix}(E)\Lambda^{-1} = \operatorname{fix}(\Lambda'' E)$, and from the normality of $I \star J$ we have that $\Lambda'' E \in I \star J$. Suppose now $H \in \mathcal{F}$, then for some $E \in I \star J$ we have $\operatorname{fix}(E) \leq H$ and therefore for every $\Lambda \in \mathscr{M}$, $\operatorname{fix}(\Lambda'' E) \leq \Lambda H \Lambda^{-1}$, and therefore $\Lambda H \Lambda^{-1} \in \mathcal{F}$.

Let $\Sigma \in \operatorname{fix}(E)$, and $\langle a, b\rangle \in \Lambda'' E$, then $\Lambda^{-1}(a, b) \in E$, so

$$\Lambda\Sigma\left(\Lambda^{-1}(a, b)\right) = \Lambda\Lambda^{-1}(a, b) = (a, b)$$

and $\Lambda\Sigma\Lambda^{-1} \in \operatorname{fix}(\Lambda'' E)$. The proof in the other direction is similar.

Lastly we will show that the completeness of $I \star J$ is the completeness of $\mathcal{F}$. Let μ be the completeness of $I \star J$, namely for every $\gamma < \mu$ and $\{E_\alpha \in I \star J \mid \alpha < \gamma\}$ we have that $E = \bigcup\{E_\alpha \in I \star J \mid \alpha < \gamma\} \in I \star J$.

Suppose that $H_\alpha \in \mathcal{F}$ for $\alpha < \gamma$ and let $\operatorname{fix}(E_\alpha) \leq H_\alpha$. Since $\operatorname{fix}(E) \leq H_\alpha$ for all α, it is also a subgroup of the intersection, and since $E \in I \star J$ we have the wanted intersection in $\mathcal{F}$. $\square$

2

Läuchli Spaces and Weak Choice

Läuchli proved in the early 1960's that it is consistent relative to **ZF** without the axiom of foundation that there exists a vector space which has no basis, but every proper subspace is of finite dimension. He also proved that the only endomorphisms of this vector space are scalar multiplications from the field (see [Läu63]). His proof was easily transferred to a proof in **ZFA**, but due to the methods of his construction there were two main limitations: it was restricted to models with atoms or with sets of the form $x = \{x\}$, and limited only to countable fields.

While the first limitation on atoms was remedied with the Jech-Sochor embedding theorem [Jec73, Chapter 6] and its various refinements (e.g. [Pin72]), the size of the field was not at all improved. In this chapter we present a new proof using a forcing argument based on Läuchli's original proof. We improve his results in two essential ways: the size of the field is no longer limited to a countable field, and for every cardinal μ we can construct a model which satisfies DC_μ (see Theorem 2.5).

2.1 Strange Vector Spaces

Before attempting to improve Läuchli's result, we first clearly state what is a vector space, and explore some of the implications of Läuchli's result. In our context a vector space is a quadruple $\langle V, F, +, \cdot \rangle$ such that the following axioms are true: $\langle V, + \rangle$ is an abelian group, $\langle F, +, \cdot \rangle$ is a field, and V is a module over F, namely we are allowed to multiply elements of F by elements of V. In this case we say that V is a vector space over F. We will abuse the language and notation and denote the vector space as V, intermittently remembering and forgetting about F, which will be called **the field** of V.

The endomorphisms of V are the homomorphisms from $\langle V, + \rangle$ to itself which respect scalar multiplication. We denote this ring as $\mathrm{End}(V)$, and its group of its invertible elements by $\mathrm{Aut}(V)$.

We say that a vector space V is a **Läuchli space** if $\mathrm{End}(V)$ is a field. In such case the action of an endomorphism on V is exactly a scalar multiplication, so we can think of V as a vector space over $\mathrm{End}(V)$. In **ZFC** we have that V is a Läuchli space if and only if $\mathrm{End}(V)$ is the field of V and $\dim V = 1$, because every vector space of dimension greater than 1 has a non-scalar endomorphism. In the case where $\mathrm{End}(V) \neq V$ we will say that V is a non-trivial Läuchli space.

We say that a vector space V over a field F is **indecomposable over** F if whenever $V = W_1 \oplus W_2$, either $W_1 = \{0\}$ or $W_1 = V$. It is not hard to see that in **ZFC**, V is indecomposable over F if and only if $V = F$. We will say that V is a **totally indecomposable** space over F if V is indecomposable over F and $V \neq F$.

Theorem 2.1. *If V is a non-trivial Läuchli space then V is totally indecomposable over* $\mathrm{End}(V)$.

Proof. Assume that V is not totally indecomposable over $\mathrm{End}(V)$, then there are two proper subspaces W_1, W_2 such that $V = W_1 \oplus W_2$. In this case every $x \in V$ has a unique decomposition $x_1 + x_2$ where $x_i \in W_i$. Let T be the linear operator $Tx = x_1$. Then $\ker T = W_2$ which is a non-trivial subspace, and since $W_1 \neq \{0\}$ as well we have that T is not zero nor is it invertible in $\mathrm{End}(V)$ which is to say that V is not Läuchli. □

Theorem 2.2. *If V is totally indecomposable over F then $V^* = \{0\}$.*

Proof. Suppose by contradiction that φ is a non-zero linear functional, and let $v \in V$ be such that $\varphi(v) = 1$. First we note that there is a non-zero $w \in V$ such that $\varphi(w) = 0$, else φ is injective which is impossible since $V \neq F$.

For every $x \in V$ we can write $x = (x - \varphi(x)v) + \varphi(x)v$, that is, we can decompose $V = \ker\varphi \oplus \mathrm{span}(\{v\})$, which is a contradiction to the fact that V is totally indecomposable. □

Theorem 2.3. *if V is totally indecomposable over F then V does not have a basis (as a vector space over F).*

Proof. If B is a basis, since $V \neq F$ we know that B contains at least two elements. Let $b \in B$ an arbitrary element, then $V = \mathrm{span}(B \setminus \{b\}) \oplus \mathrm{span}(\{b\})$.

Alternatively we can see that the function $\phi\colon B \to F$ such that $\phi(b) = 1$ and $\phi(x) = 0$ for $x \neq b$ can be extended to a non-trivial linear functional, in contradiction to the previous theorem. □

Whether or not every totally indecomposable space is Läuchli is an open question (see section 2.3). The other implications are not reversible in **ZF**, as the following example shows.

Example 2.4. *Suppose that V is a non-trivial Läuchli space, then $W = V \oplus V$ is not a Läuchli space, however $W^* = \{0\}$.*

To see that W is not a Läuchli space observe that the projection onto the first direct summand is a non-invertible non-zero endomorphism. Therefore $\mathrm{End}(W)$ *is not a field. Consider W as a vector space over* $\mathrm{End}(V)$ *then it is trivial to see that W is decomposable.*

On the other hand suppose that $\varphi\colon W \to \mathrm{End}(V)$ is a linear functional, then we can restrict it to each of the direct summands, this restriction is zero and therefore φ is zero as well.

2.2 Constructing a Läuchli Space

Läuchli proved that it is consistent with to ZF that there exists a non-trivial Läuchli space over a countable field, and that every proper subspace of it has a finite dimension. In such model $\mathsf{DC}_{\aleph_0}$ fails, and one may ask whether or not $\mathsf{DC}_{\aleph_0}$ could prove there are no non-trivial Läuchli spaces. This could be done, for example, by proving that $\mathsf{ZF} + \mathsf{DC}_{\aleph_0}$ proves that every non-zero vector space has a non-zero linear functional.

We show that in fact for any cardinal μ, it is consistent with $\mathsf{ZF} + \mathsf{DC}_\mu$ that a non-trivial Läuchli space exists.

Theorem 2.5. *Assuming* ZFC *is consistent, then for every field F and every cardinal μ we can construct a model of* $\mathsf{ZF} + \mathsf{DC}_\mu$ *in which there exists a non-trivial Läuchli space V over F.*

Of course that in the model we will construct AC will fail. We begin by deducing an immediate corollary from this and the theorems presented in the previous section:

Corollary 2.6. *For every cardinal μ,* $\mathsf{ZF} + \mathsf{DC}_\mu$ *cannot prove that for every non-zero vector space there is a non-zero functional.* □

Let $\mathfrak{M}$ be a transitive model of ZFC. Fix a field $\mathbb{F}$ in $\mathfrak{M}$, and let λ be a regular cardinal. We will construct a symmetric extension $\mathfrak{N}$ of $\mathfrak{M}$, in which $\mathsf{DC}_{<\lambda}$ holds, but DC_λ fails. In $\mathfrak{N}$ there is a Läuchli space $\mathscr{V}$ such that $\operatorname{End}(\mathscr{V}) = \mathbb{F}$, and every proper subspace of $\mathscr{V}$ has dimension $< \lambda$. Setting $\lambda > \mu$ will prove the theorem, since $\mathsf{DC}_{<\lambda}$ implies that DC_μ holds as well, as we wish to show.

The idea of the proof is to add generic subsets to a large enough cardinal, and to index these subsets using a vector space of a suitable dimension which will be the mould for $\mathscr{V}$. We will then use the automorphisms of the vector space to generate 2-permutations of the forcing, and this way ensure the preservation of the structure defined by the indexing when passing down to the symmetric extension. We wish to exclude "large enough" subspaces, so we will use an affine ideal whose main component is that of small-enough subsets. The resulting generic vector space has the desired properties, and will be in the symmetric model due to the choice of automorphisms.

Let κ be a regular cardinal such that $\kappa > |\mathbb{F}|$ and $\kappa \geq \lambda$. Fix an arbitrary $V \in \mathfrak{M}$ a vector space over $\mathbb{F}$ such that $\dim V = \kappa$. The cardinality of V itself is κ, since we have a basis $B \subseteq V$ of cardinality κ and every $v \in V$ is a unique finite sum of elements from B with non-zero coefficients from $\mathbb{F}$. Therefore V is equinumerous with the set $[B \times (\mathbb{F} \setminus \{0\})]^{<\omega}$, whose cardinality is $[\kappa]^{<\omega} = \kappa$.

Let $\mathbb{P} = \langle P, \leq \rangle$ be the following notion of forcing: $p \in P$ is a partial function from $(V \times \kappa) \times \kappa$ into $\{0, 1\}$, such that $|\operatorname{dom} p| < \kappa$. P is ordered by reverse inclusion, that is $p \leq q$ if and only if $q \subseteq p$.

Proposition 2.7. $\mathbb{P}$ *does not add new subsets to any ordinal $\alpha < \kappa$, and* $\mathbb{P}$ *preserves cardinalities and cofinalities below κ^+.*

Proof. κ is regular and therefore $\mathbb{P}$ is κ-closed, because the union of an increasing sequence of $<\kappa$ conditions is itself a condition in $\mathbb{P}$. Therefore no new bounded subsets of κ are added, and in particular if $\alpha < \kappa$ and $\dot{f}$ is a name for a function whose domain is α then $1 \Vdash \exists g \in \mathfrak{M} : \dot{f} = g$, and so κ and smaller cardinals are preserved. $\square$

We remark that $|P| = \kappa^{<\kappa}$, and if it happens that $|P| = \kappa$ then no cardinals are collapsed at all.

If G is a $\mathbb{P}$-generic filter over $\mathfrak{M}$ then $\bigcup G = f_G$ is a function from $(V \times \kappa) \times \kappa$ into 2.

We give canonical names for the following:

- The generic subset of κ, $a_{v,\beta} = \{\gamma \mid f_G(v, \beta, \gamma) = 1\}$, has the name $\dot{a}_{v,\beta} = \{\langle p, \check{\gamma}\rangle \mid p(v, \beta, \gamma) = 1\}$;
- the set $x_v = \{a_{v,\beta} \mid \beta < \kappa\}$ we has the name $\dot{x}_v = \{\dot{a}_{v,\beta} \mid \beta < \kappa\}^\bullet$;
- and $X = \{x_v \mid v \in V\}$ has the name $\dot{X} = \{\dot{x}_v \mid v \in V\}^\bullet$.

We remark that for distinct $\langle v_1, \beta_1\rangle, \langle v_2, \beta_2\rangle \in V \times \kappa$, $a_{v_1,\beta_1} \neq a_{v_2,\beta_2}$, as well as for two distinct $v, u \in V$ we have $x_v \neq x_u$.

There exists a natural vector space over $\mathbb{F}$ structure on X, defined by the bijection $v \mapsto x_v$. We give names to the addition and scalar multiplication defined this way:

- Addition has the name $\dot{\mathbf{add}} = \{\langle \dot{x}_v, \dot{x}_u, \dot{x}_w\rangle^\bullet \mid V \models v + u = w\}^\bullet$.
- For every $\mathbf{a} \in \mathbb{F}$ we define $\dot{\mathbf{a}} = \{\langle \dot{x}_v, x_w\rangle^\bullet \mid V \models \mathbf{a} \cdot v = w\}^\bullet$.

Let $\mathscr{G} \in \mathfrak{M}$ be the group of 2-permutations of V and κ in $\mathrm{Aut}(V)$ and $S(\kappa)$, where $S(\kappa)$ is the group of all permutations of κ. Given Σ a 2-permutation in $\mathscr{G}$, it acts on $\mathbb{P}$ by operating on the first two coordinates, namely $\Sigma p = \{\langle \Sigma(v, \beta), \gamma, \varepsilon\rangle \mid \langle\langle v, \beta\rangle, \gamma, \varepsilon\rangle \in p\}$.

Let $I \in \mathfrak{M}$ be the affine ideal $[V]^{<\lambda} \star [\kappa]^{<\kappa}$. Recall that λ is a regular cardinal for which we want to have that $\mathsf{DC}_{<\lambda}$ holds, but DC_λ fails. I is normal in $\mathscr{G}$ since permutations preserve cardinality. Let $\mathcal{F}$ be the normal filter of subgroups generated by I in $\mathscr{G}$, and let $\mathfrak{N} \subseteq \mathfrak{M}[G]$ be $(\mathsf{HS}_\mathcal{F})^G$. We will show that $\mathfrak{N}$ is a symmetric model with the promised properties.

If $A \in \mathfrak{N}$ and $\dot{A}$ is a hereditarily symmetric name for A, we say that $E \in I$ is **a support of** $\dot{A}$ if $\mathrm{fix}_\mathscr{G}(E) \leq \mathrm{sym}_\mathscr{G}(\dot{A})$. Since $\mathscr{G}$ is the only group of permutations we will deal with, we will omit it from the notation and simply write $\mathrm{fix}(E), \mathrm{sym}(\dot{A})$, etc.

Proposition 2.8. *I is λ-complete.*

Proof. Suppose we are given a set $\{E_\alpha \in I \mid \alpha < \gamma\}$ for some $\gamma < \lambda$. We wish to show that $E = \bigcup_{\alpha<\gamma} E_\alpha \in I$. Namely that $E^V \in [V]^{<\lambda}$ and $E_v \in [\kappa]^{<\kappa}$ for all $v \in V$.

The first set is a projection of a union, therefore it is the union of projections E^V_α, which is in $[V]^{<\lambda}$. For $v \in V$ we observe that E_v is again the union of $(E_\alpha)_v$ which is a union of $<\kappa$ sets of size $<\kappa$ and so $E_v \in [\kappa]^{<\kappa}$ for all $v \in V$. $\square$

We therefore have the following corollary:

Corollary 2.9. $\mathfrak{N} \models \mathsf{DC}_{<\lambda}$

Proof. By Proposition 2.8 and Proposition 1.12 we have that $\mathcal{F}$ is λ-closed, and thus by the fact $\mathbb{P}$ is κ-closed (therefore also λ-closed) and Lemma 1.9 $\mathfrak{N} \models \mathsf{DC}_{<\lambda}$, furthermore since $\mathbb{P}$ does not collapse any cardinal up to κ this is truly $\mathsf{DC}_{<\lambda}$, as wanted. $\square$

Proposition 2.10. *If $\Sigma = \langle T, \pi\rangle \in \mathscr{G}$ then $\Sigma\dot{a}_{v,\beta} = \dot{a}_{\Sigma(v,\beta)}$, and $\Sigma\dot{x}_v = \dot{x}_{Tv}$.*

Proof. Let $\Sigma \in \mathscr{G}$ as above, then:

$$\begin{aligned}\Sigma\dot{a}_{v,\beta} &= \{\langle \Sigma p, \check{\gamma}\rangle \mid p(v,\beta,\gamma) = 1\}\\ &= \{\langle p, \check{\gamma}\rangle \mid p(\Sigma(v,\beta),\gamma) = 1\} = \dot{a}_{\Sigma(v,\beta)},\end{aligned}$$

therefore we have:

$$\Sigma\dot{x}_v = \left\{\dot{a}_{\Sigma(v,\beta)} \;\middle|\; \beta < \kappa\right\}^\bullet = \{\dot{a}_{Tv,\beta} \mid \beta < \kappa\}^\bullet = \dot{x}_{Tv}. \qquad \square$$

Corollary 2.11. *For every $v \in V$ and $\beta \in \kappa$ the following names are hereditarily symmetric: $\dot{a}_{v,\beta}$, with a support $\{\langle v, \beta\rangle\}$; $\dot{x}_v$ with a support $\{v\} \star \varnothing$; and $\dot{X}$ as well as the names of the vector space operations are hereditarily symmetric with empty support.*

Proof. From Proposition 2.10 above we have that $\dot{a}_{v,\beta}$ is fixed whenever $\Sigma(v,\beta) = \langle v,\beta\rangle$ and thus hereditarily symmetric (all the names inside are canonical names for elements of $\mathfrak{M}$), similar argument holds for $\dot{x}_v$ (all the names are $\dot{a}_{v,\beta}$).

By this we have that $\dot{X}$, $\dot{\mathsf{a}}$ and $\dot{\mathsf{add}}$ are also hereditarily symmetric, since for every $\Sigma \in \mathscr{G}$, if $\Sigma = \langle T, \pi\rangle$ then:

$$\Sigma\dot{\mathsf{add}} = \{\langle \dot{x}_{Tv}, \dot{x}_{Tu}, \dot{x}_{Tw}\rangle^\bullet \mid V \models v + u = w\}^\bullet.$$

However since T is linear we have that $V \models v + u = w \iff V \models Tv + Tu = Tw$, so we are done. $\square$

We denote by $\dot{\mathscr{V}}$ the name of the vector space generated by the above operations, and if $\mathscr{V} = \dot{\mathscr{V}}^G$ then we have that $\mathscr{V} \in \mathfrak{N}$. For $W \subseteq V$, $W \in \mathfrak{M}$, let $\dot{\mathscr{W}} = \{\dot{x}_w \mid w \in W\}^\bullet$ be the natural name of W. Clearly if W is a subspace then $\dot{\mathscr{W}}^G = \mathscr{W}$ is a subspace of $\mathscr{V}$. Furthermore if $\dim W < \lambda$, letting B be a basis for W, then $\operatorname{fix}(B \star \varnothing)$ is a support for $\dot{\mathscr{W}}$, since every linear T fixing B pointwise must fix W pointwise.

Lemma 2.12. *Let $T \in \operatorname{Aut}(V)$ and $E \in I$ be such that $T\restriction_{E^V} = \operatorname{id}_{E^V}$, then for every $q \in \mathbb{P}$ there is $\Sigma \in \operatorname{fix}(E)$ such that $\Sigma = \langle T, \pi\rangle$ and Σq is compatible with q.*

Proof. Let W be the $\operatorname{span}(E^V)$. Since T is linear we have that $T\restriction_W = \operatorname{id}_W$. Fix $q \in \mathbb{P}$, we next define π_v for $v \in V$:

- If $v \in W$ then $\pi_v = \operatorname{id}_\kappa$.

- Otherwise, let $\varepsilon = \sup\{\beta + 1 \mid \exists z \exists \gamma : \langle z, \beta, \gamma\rangle \in \operatorname{dom} q\}$ and define:

$$\pi_v(\beta) = \begin{cases} \varepsilon + \beta & \beta < \varepsilon \\ \beta' & \beta = \varepsilon + \beta', \beta' < \varepsilon \\ \beta & \beta \geq \varepsilon + \varepsilon \end{cases}$$

To see that $\Sigma \in \operatorname{fix}(E)$ we note that if $\langle a, b\rangle \in E$ then $a \in E^V$ and therefore $\Sigma(a, b) = \langle a, b\rangle$.

Let Σ be $\langle T, \pi\rangle$, and suppose that $\langle w, \beta, \gamma\rangle \in \operatorname{dom}(q) \cap \operatorname{dom}(\Sigma q)$, then $\beta < \varepsilon$ and so $w \in W$, and therefore $\Sigma(w, \beta) = \langle w, \beta\rangle$. By the definition of how Σ acts on $\mathbb{P}$ we have that $q(w, \beta, \gamma) = \Sigma q(w, \beta, \gamma)$, therefore q and Σq are compatible as wanted. $\square$

Corollary 2.13. *Suppose that W is a subspace of V and $\dim W < \lambda$, $u, v \in V \setminus W$ and $q \in \mathbb{P}$. Then there exists $\Sigma \in \mathscr{G}$ such that $\Sigma \dot{x}_v = \dot{x}_u$ and Σq is compatible with q.*

Proof. It is enough to exhibit a linear operator T such that $T \restriction_W$ is the identity function and $Tv = u$, since by Proposition 2.10 every Σ such that $\Sigma = \langle T, \pi\rangle$ will have the property $\Sigma \dot{x}_v = \dot{x}_u$, and by Lemma 2.12 there is a Σ with Σq compatible with q.

Let B be a basis of W, then $|B| < \lambda$, so $B \star \varnothing \in I$. Extend B to a basis B_1 such that $v \in B_1$ and a basis B_2 such that $u \in B_2$. Let τ be any bijection of B_1 onto B_2 such that $\tau \restriction_B = \operatorname{id}_B$ and $\tau v = u$. Define T as the unique extension of τ to a linear automorphism, then T is as needed. $\square$

Theorem 2.14. *Suppose that $A \in \mathfrak{N}$ and $A \subseteq \mathscr{V}$, then there exists a subspace $\mathscr{W}$ of $\mathscr{V}$, such that $\dim \mathscr{W} < \lambda$ and either $A \subseteq \mathscr{W}$ or $\mathscr{V} \setminus A \subseteq \mathscr{W}$.*

Proof. Let $\dot{A}$ be a symmetric name for A and E a support of $\dot{A}$. Let W be the span of E^V and $\dot{\mathscr{W}}$ the natural name of W. We will show that if $p \Vdash \dot{A} \nsubseteq \dot{\mathscr{W}}$ then $p \Vdash \dot{\mathscr{V}} \setminus \dot{A} \subseteq \dot{\mathscr{W}}$, the other case where $p \Vdash \dot{\mathscr{V}} \setminus \dot{A} \nsubseteq \dot{\mathscr{W}}$ follows by changing the roles of A and $\mathscr{V} \setminus A$.

If $p \Vdash \dot{A} \subseteq \dot{\mathscr{W}}$ then we are done, similarly if $p \Vdash \dot{\mathscr{V}} \setminus \dot{A} \subseteq \dot{\mathscr{W}}$. Assume towards contradiction that $p \Vdash \dot{A} \nsubseteq \dot{\mathscr{W}}$ and $p \nVdash \dot{\mathscr{V}} \setminus \dot{A} \subseteq \dot{\mathscr{W}}$. Let $q \leq p$ such that $q \Vdash \dot{\mathscr{V}} \setminus \dot{A} \nsubseteq \dot{\mathscr{W}}$. We can therefore find $u, v \in V \setminus W$ such that an extension of q forces $\dot{x}_u \in \dot{A}$ and $\dot{x}_v \notin \dot{A}$. Without loss of generality, q already forces this.

Let Σ be a 2-permutation as guaranteed by Corollary 2.13 which fixes E, and such that $\Sigma \dot{x}_u = \dot{x}_v$.

Since $q \Vdash \dot{x}_u \in \dot{A}$ we have that $\Sigma q \Vdash \dot{x}_v \in \dot{A}$, by Lemma 2.12 these are compatible conditions which yields a contradiction as they force contradictory statements. $\square$

We deduce several corollaries on the structure of subspaces of $\mathscr{V}$. First we remark that it holds in **ZF** that if U is a vector space over a field F, and F is well-orderable, and moreover U has a well-orderable basis B, then U itself is well-orderable. This follows from the bijection between U and finite subsets of $B \times (F \setminus \{0\})$. We further remark that if U is a well-orderable vector space then the **ZFC** theorems about U are true (it has a basis; every subspace has a basis and a direct complement; etc.).

Proposition 2.15. *If $A \subseteq \mathscr{V}$ is such that $\operatorname{span}(A) = \mathscr{V}$, then $|A|^{\mathfrak{M}[G]} = \kappa$.*

Proof. By Proposition 2.7, $\mathbb{P}$ does not collapse cardinals below κ^+, therefore in $\mathfrak{M}[G]$ the dimension of V is κ, and so the dimension of $\mathscr{V}$ in $\mathfrak{M}[G]$ is κ as well. Therefore if $A \in \mathfrak{M}[G]$ is a spanning set then $|A|^{\mathfrak{M}[G]} = \kappa$. □

Corollary 2.16. *Let $\mathscr{W} \in \mathfrak{N}$ be a subspace of $\mathscr{V}$, then $\mathscr{W} = \mathscr{V}$ or $\dim \mathscr{W} < \lambda$.*

Proof. Let $\mathscr{W}$ be a proper subspace of $\mathscr{V}$. Pick $v \in \mathscr{V} \setminus \mathscr{W}$ and define $A = v + \mathscr{W} = \{v + w \mid w \in \mathscr{W}\}$. Since $A \cap \mathscr{W} = \varnothing$, by Theorem 2.14 we have that there is a subspace $\mathscr{W}_0 \subseteq \mathscr{V}$ such that $\dim \mathscr{W}_0 < \lambda$ and $A \subseteq \mathscr{W}_0$ or $\mathscr{W} \subseteq \mathscr{W}_0$.

If $\mathscr{W} \subseteq \mathscr{W}_0$ then by the remark above about well-ordered spaces we have that $\dim \mathscr{W} \leq \dim \mathscr{W}_0 < \lambda$. If $A \subseteq \mathscr{W}_0$ then by the fact that $v \in \mathscr{W}_0$ and that it is a subspace we have that $-v \in \mathscr{W}_0$. Therefore $-v + A = \mathscr{W} \subseteq \mathscr{W}_0$ and the conclusion holds. □

Corollary 2.17. *$\mathscr{V}$ has no basis in $\mathfrak{N}$.*

Proof. Suppose that $B \in \mathfrak{N}$ is a linearly independent subset of $\mathscr{V}$, pick some $b \in B$, then $B' = B \setminus \{b\}$ is also linearly independent. Therefore $\operatorname{span} B'$ is a proper subspace of $\operatorname{span} B$, and by Corollary 2.16 the former has dimension smaller than λ, and so $\operatorname{span} B = \operatorname{span} B' \oplus \operatorname{span}\{b\}$ has dimension smaller than λ. By Proposition 2.15 we have that $\operatorname{span}(B) \neq \mathscr{V}$. □

We can finally prove that indeed $\mathbf{DC}_\lambda$ fails in $\mathfrak{N}$.

Corollary 2.18. $\mathfrak{N} \models \neg\mathbf{DC}_\lambda$

Proof. Define R to be the following binary relation: $\langle f, x\rangle \in R$ if and only if $f \colon \alpha \to \mathscr{V}$ for some $\alpha < \lambda$ and $\operatorname{rng} f$ is a linearly independent subset of $\mathscr{V}$, and $x \notin \operatorname{span}(\operatorname{rng} f)$. Namely, $f R x$ if and only if $\operatorname{rng} f \cup \{x\}$ is linearly independent.

We have that every α-sequence in $\mathscr{V}$ can be extended, but no $f \colon \lambda \to \mathscr{V}$ can have a linearly independent range, since there is no subspace of $\mathscr{V}$ whose dimension is λ. □

Theorem 2.19. *Let $S \in \mathfrak{N}$ be a linear operator from $\mathscr{V}$ to itself, then there is $\mathrm{a} \in \mathbb{F}$ such that $Sx = \mathrm{a}x$ for all $x \in \mathscr{V}$.*

Proof. Let $\dot{S}$ be a symmetric name for S and E a support for it, as before denote by W the span of E^V, and $\mathscr{W} = \dot{\mathscr{W}}^G$.

We will first prove several helpful claims.

Claim. *If $x_v \in \mathscr{W}$ then $Sx_v \in \mathscr{W}$.*

Proof of the Claim. Suppose $p \Vdash \dot{x}_v \in \dot{\mathscr{W}}$, and by contradiction for some $q \leq p$, $q \Vdash \dot{S}\dot{x}_v = \dot{x}_u \notin \dot{\mathscr{W}}$. Let Σ be a 2-permutation which fixes W, and therefore E, and $\Sigma\dot{x}_u \neq \dot{x}_u$. Since Σq is guaranteed to be compatible with q we have that $\Sigma q \Vdash \dot{S}\dot{x}_v \neq \dot{x}_u$, which is a contradiction. Therefore $p \Vdash \dot{S}\dot{x}_v \in \dot{\mathscr{W}}$. □

Claim. *If $x_v \notin \mathscr{W}$ and $Sx_v \in \mathscr{W}$ then $Sx_v = x_0$.*

Proof of the Claim. Let p be a condition such that $p \Vdash \dot{S}\dot{x}_v = \dot{x}_w \in \dot{\mathscr{W}}$. If there exists $u \neq v$, $u \notin W$ such that $p \Vdash \dot{S}\dot{x}_u = \dot{x}_w$, let Σ be a 2-permutation fixing E such that $\Sigma\dot{x}_v = \dot{x}_{v+u}$. We have that $\Sigma p \Vdash \dot{S}\dot{x}_{v+u} = \dot{x}_w$ as well, and Σp and p are compatible. Taking q stronger than both these conditions, we have that $q \Vdash \dot{S}\dot{x}_v = \dot{S}\dot{x}_{v+u}$. Recall that by the definition of $\dot{\mathbf{add}}$ we have that $1 \Vdash \dot{x}_{v+u} = \dot{x}_v + \dot{x}_u$, so we actually have $q \Vdash \dot{S}\dot{x}_v = \dot{S}\dot{x}_v + \dot{S}\dot{x}_u$, and therefore $q \Vdash \dot{S}\dot{x}_v = \dot{S}\dot{x}_u = \dot{x}_0$.

If no such u exists, then either for some $q \leq p$ there exists $u \neq v$ and $u \notin W$ such that $q \Vdash \dot{S}\dot{x}_u = \dot{x}_v$, and we are back to the previous case. Else there is $q \leq p$ and $u \notin W$ such that $q \Vdash \dot{S}\dot{x}_u \neq \dot{S}\dot{x}_v$. By Corollary 2.13 we have Σ fixing E which switches between $\dot{x}_u$ and $\dot{x}_v$, and Σq is compatible with q. This is a contradiction, since $\Sigma q \Vdash \dot{S}\dot{x}_u = \dot{S}\Sigma\dot{x}_v = \dot{x}_w$. We therefore have that the set $\left\{q \leq p \;\middle|\; q \Vdash \dot{S}\dot{x}_v = \dot{x}_0\right\}$ is dense below p, as wanted. □

Suppose now that p is a condition forcing that $\dot{S}$ is not the zero map. We will show that there is $q \leq p$ and $\mathbf{a} \in \mathbb{F}$ such that $q \Vdash \dot{S} = \mathbf{a}$, namely that the set of conditions which force that $\dot{S}$ is a scalar multiplication is dense, by genericity this would mean that $S = \mathbf{a}$ as wanted.

If p already forces $\dot{S} = \dot{\mathbf{a}}$ we are done. Otherwise there are $u, v \in V$ and $q \leq p$ such that $q \Vdash \forall \mathbf{a} \in \mathbb{F} : \dot{S}(\dot{x}_u - \dot{x}_v) \neq \dot{\mathbf{a}}(\dot{x}_u - \dot{x}_v)$. Since q forces $\dot{S}$ is not zero, it forces that $\ker(S)$ is a proper subspace. Without loss of generality, E is a support for $\ker(S)$ as well. By the two claims above, if $v \notin W$ and $q \Vdash \dot{S}\dot{x}_v \in \dot{\mathscr{W}}$ then $q \Vdash \dot{x}_v \in \ker\dot{S}$, in which case $v \in W$. Therefore for $v \notin W$ we have $q \Vdash \dot{S}\dot{x}_v \notin \dot{\mathscr{W}}$.

For every $v \in V \setminus W$ let $E(v) = E \cup (\{v\} \star \varnothing)$ and $W_v = W \oplus \langle v \rangle = \operatorname{span}(E(v))^V$, then by the claims $q \Vdash \dot{S}''\dot{\mathscr{W}} \subseteq \dot{\mathscr{W}}$, in particular we have that $q \Vdash \dot{S}\dot{x}_v \in \dot{\mathscr{W}}_v$, so we must have that $q \Vdash \dot{S}\dot{x}_v = \dot{\mathbf{a}}_v\dot{x}_v$ for some non-zero scalar $\mathbf{a}_v \in \mathbb{F}$.

Claim. *Suppose that $v, u \notin W$ and $u \notin W_v$ (so $v \notin W_u$ as well), then $\mathbf{a}_v = \mathbf{a}_u$.*

Proof of the Claim. Let u, v linearly independent over W, namely $\mathbf{c}u + \mathbf{d}v \in W$ if and only if $\mathbf{c} = \mathbf{d} = \mathbf{0}$. We have that $v + u \notin W$, therefore $\mathbf{a}_v, \mathbf{a}_u, \mathbf{a}_{v+u}$ are all non-zero.

$$\begin{aligned} q \Vdash \dot{x}_0 = \dot{S}\dot{x}_0 = \dot{S}(\dot{x}_v + \dot{x}_u - \dot{x}_{v+u}) &= \dot{\mathbf{a}}_v\dot{x}_v + \dot{\mathbf{a}}_u\dot{x}_u - \dot{\mathbf{a}}_{v+u}(\dot{x}_v + \dot{x}_u) \\ &= (\dot{\mathbf{a}}_v - \dot{\mathbf{a}}_{v+u})\dot{x}_v + (\dot{\mathbf{a}}_u - \dot{\mathbf{a}}_{v+u})\dot{x}_u \end{aligned}$$

By linear independence $\mathbf{a}_u = \mathbf{a}_v = \mathbf{a}_{v+u}$ as wanted. □

Fix now some $v \notin W$ and we have that for all $u \notin W_v$ we have to have $a_u = a_v$. Therefore

$$q \Vdash \forall x(x \notin \dot{\mathscr{W}}_v \rightarrow \dot{S}x = \dot{a}_v x)$$

Because q also forces that $\dot{S}$ is a linear operator and that the span of $\mathscr{V} \setminus \mathscr{W}_v$ is the entire space, we have to have that for all $v \in V$ $q \Vdash \dot{S}\dot{x}_v = \dot{\mathbf{a}}\dot{x}_v$, as wanted. $\square$

We have proved that if $S \in \mathrm{End}(\mathscr{V}) \cap \mathfrak{N}$ then $Sx = \mathbf{a}x$. This means that $\mathrm{End}(\mathscr{V})$ is a field, but $\dim \mathscr{V} \neq 1$. That is $\mathscr{V}$ is a non-trivial Läuchli space, and by Theorem 2.1 it is also indecomposable, and by Theorem 2.2 it has no non-zero linear functionals.

The reader could see that the fact $\mathscr{V}$ is totally indecomposable over $\mathbb{F}$ could have been proved just as easily as an immediate consequence of Corollary 2.16 and Proposition 2.15, and Corollary 2.17 could have been easily deduced now from the above conclusions of the theorem.

2.3 Consequences and Questions

Proof of Theorem 2.5. In $\mathfrak{M}$ let μ be an $\aleph$ cardinal, and fix $\mathbb{F}$ to be any field in $\mathfrak{M}$, and take $\lambda > \mu$. The above construction guarantees that the symmetric model satisfies DC_μ, but a Läuchli space exists. $\square$

We conclude with several questions we find interesting and have no answer to:

Question. Does $\mathsf{ZF} + \neg\mathsf{AC}$ prove the existence of a non-trivial Läuchli space?

Question. Is it consistent with ZF that there is a totally indecomposable vector space that is not a Läuchli space?

Theorem 2.2 shows that totally indecomposable spaces (and in particular non-trivial Läuchli spaces) have trivial duals. We ask the following questions about the choice strength of the existence of functionals and non-scalar automorphisms.

Question. Does the assertion "If V is a non-zero vector space, then V has a non-zero linear functional" imply AC in ZF?

Question. Is it consistent with ZF and $\neg\mathsf{AC}$ and that for every non-zero vector space there is a non-scalar endomorphism?

Question. Is it consistent with ZF that every non-zero vector space has a non-scalar endomorphism, but there is a non-zero vector space that has a trivial dual?

3

On Antichains of Cardinals

In this chapter we present and extend the work of Feldman, Orhon, and Blass in [FOB08]. All proofs in this chapter are in **ZF**. The theorem was independently proved by Tarski in 1964 the proof was announced in [Tar64], and appears in [RR85, pp.22-23][1].

Recall that if A is a set then the Hartogs number of A, denoted by $\aleph(A)$, is the least infinite ordinal α such that $\alpha \nleq |A|$. If A is well-ordered then $|A| < \aleph(A)$, otherwise we simply have $\aleph(A) \nleq |A|$.

We will slightly abuse the notation and if $\mathfrak{a} = |A|$ we will write $\aleph(\mathfrak{a})$ instead of $\aleph(A)$. If $\mathfrak{a}, \mathfrak{b}$ are cardinals of A and B respectively we will denote $\mathfrak{a} + \mathfrak{b} = |A \cup B|$, assuming we take disjoint copies of these sets; $\mathfrak{a} \cdot \mathfrak{b} = |A \times B|$; $\mathfrak{a}^{\mathfrak{b}} = \left|A^B\right|$. We will also use $2^{\mathfrak{a}} = |\mathcal{P}(A)|$. We will make free use of the basic properties of cardinal arithmetic which hold in **ZF**.

We say that a set is **Dedekind-finite** if it has no countably infinite subset, equivalently this is to state that its Hartogs number is $\aleph_0$, or that every proper subset has a strictly smaller cardinality. The assertion that there exists an infinite Dedekind-finite set is equivalent to $\neg\mathsf{W}_{\aleph_0}$. If a set is not Dedekind-finite we say that it is **Dedekind-infinite**. A Dedekind-finite (Dedekind-infinite) cardinal is the cardinal number of a Dedekind-finite (Dedekind-infinite) set.

Truss published an extensive paper [Tru74] reviewing various forms of Dedekind-finite sets. In the construction of a symmetric extension in section 1.2 assuming $\lambda = \omega$ (as Monro originally did) results in an infinite Dedekind-finite set.

Lemma 3.1. *Suppose that A is an infinite Dedekind-finite set, then $\mathcal{P}(\mathcal{P}(A))$ is Dedekind-infinite.*

Proof. The following map is an injection from ω into $\mathcal{P}(\mathcal{P}(A))$:

$$n \mapsto \{B \subseteq A \mid |B| = n\}$$

This is a well-defined function, and moreover it is injective by the following argument. Since A is infinite it has subsets of every finite size, so no n is mapped to the empty set and clearly every $m \neq n$ are mapped to different sets. $\square$

[1]**Form T3**(n)

Proposition 3.2. *If $\mathfrak{a}$ is a Dedekind-infinite cardinal and $\aleph(\mathfrak{a}) > \lambda$ then $\mathfrak{a} + \lambda = \mathfrak{a}$.*

Proof. We may assume that λ is infinite. Let A be a set such that $|A| = \mathfrak{a}$. Since $\lambda < \aleph(\mathfrak{a})$, there is $B \subseteq A$ such that $|B| = \lambda$. Since $\lambda + \lambda = \lambda$, given an arbitrary set C such that $|C| = \lambda$ and $C \cap A = \varnothing$ we have a bijection $f\colon B \cup C \to B$. The function $F = f \cup \mathrm{id}_{A \setminus B}$ is a bijection from $A \cup C$ to A and therefore $\mathfrak{a} + \lambda = \mathfrak{a}$ as wanted. □

Proposition 3.3. *If $\mathfrak{a}$ an is infinite cardinal, then for all $0 < n < \omega$, $\aleph(\mathfrak{a}) = \aleph(n\mathfrak{a})$.*

Proof. Let A be a set such that $\mathfrak{a} = |A|$. Assume towards a contradiction that $\aleph(\mathfrak{a}) < \aleph(n\mathfrak{a})$. Then there is an injection $f\colon \aleph(\mathfrak{a}) \to A \times n$.

For $i < n$ let $X_i = \{\alpha < \aleph(\mathfrak{a}) \mid f(\alpha) \in A \times \{i\}\}$. By the pigeonhole principle there is at least one i such that $|X_i| = \aleph(\mathfrak{a})$, and therefore an injective function from $\aleph(\mathfrak{a})$ into A which is a contradiction. □

Theorem 3.4 (Hartogs). *If for every A and B, $|A| \leq |B|$ or $|B| \leq |A|$, then every set can be well-ordered, i.e. the axiom of choice holds.*

Proof. Let A be an arbitrary set. Since $\aleph(A) \nleq |A|$ we have that $|A| < \aleph(A)$. In particular we have an injection from A into an ordinal, therefore A can be well-ordered. □

3.1 Generalizing Hartogs Theorem

Hartogs theorem shows that if the cardinals are linearly ordered then the axiom of choice holds. We know that a partial order is linear if and only if every antichain is a singleton. The following is a natural generalization of Hartogs theorem, given in [FOB08].

Definition 3.1. The k-Trichotomy Principle, $\mathsf{Tri}(k)$:

If $\mathfrak{a}_1, \ldots, \mathfrak{a}_k$ are distinct cardinals, then there are $i \neq j$ such that $\mathfrak{a}_i \leq \mathfrak{a}_j$.

Namely every antichain of cardinals has less then k elements.

We observe that for $k = 1$ this is vacuously true, and from here on end we will always assume that $k > 1$ when asserting $\mathsf{Tri}(k)$. We can restate Hartogs theorem as "$\mathsf{Tri}(2)$ implies the axiom of choice", so the natural generalization of Hartogs theorem would be:

Theorem 3.5 (Tarski; Feldman-Orhon). *For every $k < \omega$, $\mathsf{Tri}(k)$ implies the axiom of choice.*

For the proof of this theorem we need two preliminary claims. Feldman and Orhon approach these claims somewhat differently. In their paper they first prove that under the assumption of $\mathsf{Tri}(k)$ there are no infinite Dedekind-finite sets, and that every cardinal $\mathfrak{a}$ has some finite n such that $n\mathfrak{a} + n\mathfrak{a} = n\mathfrak{a}$. We replace these two proofs with the following lemma, and give a slightly more general argument to Specker's theorem.

Lemma 3.6. *For every cardinal $\mathfrak{a}$ there exists $\mathfrak{a}'$ such that $\mathfrak{a} \leq \mathfrak{a}' = \mathfrak{a}' + \mathfrak{a}'$.*

Proof. Let A be a set such that $|A| = \mathfrak{a}$. Denote $A' = A \times \omega$ and let $\mathfrak{a}' = |A'|$. Clearly $\mathfrak{a} \leq \mathfrak{a}' = \mathfrak{a} \cdot \aleph_0$. Using distributivity of multiplication over addition we have

$$\mathfrak{a}' + \mathfrak{a}' = \mathfrak{a} \cdot \aleph_0 + \mathfrak{a} \cdot \aleph_0 = \mathfrak{a}(\aleph_0 + \aleph_0) = \mathfrak{a} \cdot \aleph_0 = \mathfrak{a}'. \qquad \square$$

Theorem 3.7 (Specker). *Suppose that $\mathfrak{a}$ is an infinite cardinal such that $\mathfrak{a} + \mathfrak{a} = \mathfrak{a}$, and $\mathfrak{b}$ is such that $\mathfrak{a} + \mathfrak{b} = 2^{\mathfrak{a}}$, then $\mathfrak{b} = 2^{\mathfrak{a}}$.*

Proof. From the assumption $\mathfrak{a} + \mathfrak{a} = \mathfrak{a}$ we have that $2^{\mathfrak{a}} \cdot 2^{\mathfrak{a}} = 2^{\mathfrak{a}+\mathfrak{a}} = 2^{\mathfrak{a}} = \mathfrak{a} + \mathfrak{b}$. Let A, B be disjoint sets of cardinality $\mathfrak{a}, \mathfrak{b}$ respectively. We therefore have a bijection $f \colon \mathcal{P}(A) \times \mathcal{P}(A) \to A \cup B$.

There is some $X \in \mathcal{P}(A)$ such that $f''(\{X\} \times \mathcal{P}(A)) \cap A = \varnothing$. Otherwise composing $f^{-1} \restriction_A$ with the projection onto the left coordinate of $\mathcal{P}(A) \times \mathcal{P}(A)$ gives a surjection from A onto $\mathcal{P}(A)$ in contradiction to Cantor's theorem.

Fix X as above, and define $g \colon \mathcal{P}(A) \to B$ defined by $g(Y) = f(X, Y)$, this is an injective function. Therefore $\mathfrak{b} \leq 2^{\mathfrak{a}} \leq \mathfrak{b}$ as wanted. $\square$

Finally we can prove Theorem 3.5.

Proof of Theorem 3.5. Suppose that $\mathsf{Tri}(k)$ holds for some $k \in \omega$. Let A be an infinite set, we will show that A can be well-ordered. We may assume that A is Dedekind-infinite, if not we can take $\mathcal{P}(\mathcal{P}(A))$ instead and if it can be well-ordered then A can be well-ordered as well. We shall denote by $\mathfrak{a}$ the cardinal of A.

We may assume that $\mathfrak{a} + \mathfrak{a} = \mathfrak{a}$, otherwise by Lemma 3.6 there is some $\mathfrak{a}' = |A'|$ such that $\mathfrak{a}' + \mathfrak{a}' = \mathfrak{a}'$, and $\mathfrak{a} \leq \mathfrak{a}'$. By showing that A' can be well-ordered we will show that A itself can be well-ordered as well. We may also assume that $\mathcal{P}^n(A)$ does not contain ordinals for all $n < k$.

For $i < k$ we define the following cardinals: let $\mathfrak{b}_0 = \mathfrak{a}$, $\mathfrak{b}_{i+1} = 2^{\mathfrak{b}_i}$, and let $\kappa_i = \aleph(\mathfrak{b}_{k-i})^{+k-i}$. We observe that if $n < m$ then $\kappa_m < \kappa_n$. Finally, we define

$$\mathfrak{p}_i = \mathfrak{b}_i + \kappa_i = |\mathcal{P}^i(A) \cup \kappa_i|$$

From $\mathsf{Tri}(k)$ we know that there are two comparable cardinals in this family, $\mathfrak{p}_m, \mathfrak{p}_n$ where $n < m$. Similarly as in the proof of Lemma 3.6 we can deduce that $\mathfrak{p}_m \leq \mathfrak{p}_n$, for otherwise we can find an injective function from $\kappa_n > \aleph(\mathfrak{b}_m)$ into $\mathfrak{b}_m$.

We therefore have an injective $f \colon \mathcal{P}^m(A) \cup \kappa_m \to \mathcal{P}^n(A) \cup \kappa_n$. Let $|(f''\mathcal{P}^m(A)) \cap \mathcal{P}^n(A)| = \mathfrak{m}_0 \leq \mathfrak{b}_n < \mathfrak{b}_m$ and $|(f''\mathcal{P}^m(A)) \cap \kappa_n| = \mathfrak{m}_1$, we have that $\mathfrak{m}_0 + \mathfrak{m}_1 = \mathfrak{b}_m$.

By the definition of $\mathfrak{b}_m$, $\mathfrak{b}_m \geq 2^{\mathfrak{b}_n}$, and by Specker's theorem we therefore have that $\mathfrak{m}_1 = \mathfrak{b}_m$ and thus $\mathcal{P}^m(A)$ can be well-ordered and in particular A can be well-ordered, as wanted. $\square$

It follows that in $\mathsf{ZF} + \neg\mathsf{AC}$ for every k there exists a set of k cardinals mutually incomparable. The proof above gives us an antichain of cardinals in the case where $\mathfrak{a}$ we began with was not well-orderable.

However the proof relies on the fact that $\mathfrak{a} + \mathfrak{a} = \mathfrak{a}$. In case this is not true we can turn to the proof of Lemma 3.6 and see that it constructs the wanted antichains, while the assumption in the lemma was that $\mathfrak{a}$ is Dedekind-infinite the construction yields an antichain if $\mathfrak{a}$ is Dedekind-finite.

Blass presents a direct construction in the paper. Given a set A which cannot be well-ordered he constructs an antichain of any finite length. Instead of presenting the construction directly, we will give it a slight modification and prove something stronger.

3.2 Generalizing the Generalization

Instead of talking about injections in this section we will talk about surjections. Recall that $|A| \leq^* |B|$ if A is empty or if B can be mapped onto A. This relation is reflexive and transitive but it does not need to be anti-symmetric. For example if there exists an infinite Dedekind-finite cardinal $\mathfrak{b}$ then there exists an infinite Dedekind-finite cardinal $\mathfrak{a}$ such that

$$\mathfrak{a} < \mathfrak{a} + \mathfrak{a} \leq^* \mathfrak{a}.$$

The above claim along with its proof are given in Proposition 3.13.

Definition 3.2. $\mathsf{Tri}^*(k)$:

If $\mathfrak{a}_1, \dots, \mathfrak{a}_k$ are distinct cardinals, then there are $i \neq j$ such that $\mathfrak{a}_i \leq^* \mathfrak{a}_j$.

Of course for $k = 1$ this is vacuously true, and we will always assume that $k > 1$. It is easy to see that $\mathsf{Tri}(k)$ implies $\mathsf{Tri}^*(k)$ because whenever $\mathfrak{a} \leq \mathfrak{b}$ we have that $\mathfrak{a} \leq^* \mathfrak{b}$. We would like to prove an analogue of Theorem 3.5 for $\mathsf{Tri}^*(k)$. We begin with an analogue of Hartogs theorem, namely a statement equivalent to $\mathsf{Tri}^*(2)$. Recall the $\leq^*$ analogue of $\aleph(A)$ is $\aleph^*(A)$ which is the least ordinal that A cannot be mapped onto, and as before we write $\aleph^*(\mathfrak{a})$ for $\aleph^*(A)$ when $\mathfrak{a} = |A|$.

Theorem 3.8 (Lindenbaum[2]). *If for every two sets A and B $|A| \leq^* |B|$ or $|B| \leq^* |A|$, then every set can be well-ordered.*

Proof. Let A be an infinite set, take $B = \aleph^*(A)$ then from the assumption either $|A| \leq^* |B|$ or $|B| \leq^* |A|$. However the definition of $\aleph^*(A)$ was the least ordinal β such that $\beta \nleq^* |A|$. Therefore $|A| \leq^* |B|$.

Let $f\colon \aleph^*(A) \to A$ be a surjective function, then $g\colon A \to \aleph^*(A)$ defined as $g(a) = \min\{\alpha < \aleph^*(A) \mid f(\alpha) = a\}$ is well-defined and injective, therefore A can be well-ordered. □

We shall now proceed to generalize Lindenbaum's theorem in a similar manner to the generalization of Hartogs' theorem. The approach is based on Blass' proof, in contrast to the approach in the proof of Theorem 3.5 which was based on the original argument given by Feldman and Orhon.

[2]The result was announced by Lindenbaum in a joint paper with Tarski in 1924, but Sierpiński was the first to publish a proof only in 1948 [Moo82, p. 216].

Let us denote by $\mathcal{Q}(A) = A \times \mathcal{P}(A)$, and by induction let $\mathcal{Q}^{k+1}(A) = \mathcal{Q}(\mathcal{Q}^k(A))$. Again we abuse the notation and if $\mathfrak{a}$ is the cardinal of A we will interchange $\mathcal{Q}(\mathfrak{a})$ and $\mathcal{Q}(A)$ freely.

Lemma 3.9. *Let $\mathfrak{a}$ be a cardinal such that for some $\aleph$ cardinal κ we have*

$$\mathcal{Q}(\mathfrak{a}) \leq^* \mathfrak{a} + \kappa$$

Then $\mathfrak{a}$ is a well-orderable cardinal.

Proof. Let A be a set such that $|A| = \mathfrak{a}$. Without loss of generality $A \cap \kappa = \varnothing$. From the assumption there exists a surjective $f\colon A \cup \kappa \to \mathcal{Q}(A)$. Without loss of generality $f{\restriction}_\kappa$ is injective and $f''A \cap f''\kappa = \varnothing$, if this is not the case we can remove ordinals from κ and rearrange it to a smaller ordinal.

There is some $B \in \mathcal{P}(A)$ such that $A \times \{B\} \subseteq f''\kappa$. Otherwise composing f with the projection from $\mathcal{Q}(A)$ to $\mathcal{P}(A)$ is a surjection from A onto its power set, in contradiction to Cantor's theorem.

We define $g\colon A \to \kappa$ by $g(a) = f^{-1}(a, B)$. This is an injective function since $f{\restriction}_\kappa$ is injective, and $A \times \{B\} \subseteq f''\kappa$, therefore A can be well-ordered and $\mathfrak{a} \leq \kappa$ is a well-orderable cardinal. □

Blass original argument was assuming $\mathfrak{a} + \kappa \geq \mathcal{Q}(\mathfrak{a})$, but his argument is a corollary of the above lemma. This is because $\mathfrak{a} + \kappa \geq \mathcal{Q}(\mathfrak{a})$ implies $\mathfrak{a} + \kappa \geq^* \mathcal{Q}(\mathfrak{a})$. We can actually show now that $2^{\mathfrak{a}}$ can be well-ordered: we have that $\mathfrak{a} < \kappa$, therefore $\mathfrak{a} + \kappa = \kappa$ and so $\mathcal{Q}(\mathfrak{a}) \leq^* \kappa$, and therefore can be well-ordered. We point that out because it is a natural question with an easy answer, but in fact this is merely a piece of trivia that has no use in the rest of our proof.

Theorem 3.10. *For every $k < \omega$, $\mathsf{Tri}^*(k)$ implies the axiom of choice.*

Proof. Suppose that the axiom of choice fails, and let A be an infinite set that cannot be well-ordered whose cardinal is $\mathfrak{a}$. For $k \in \omega$ we define a collection of cardinals. Let $\kappa = \aleph^*(\mathcal{Q}^k(\mathfrak{a}))$, we define the following:

$$\mathfrak{p}_i = \mathcal{Q}^i(\mathfrak{a}) + \kappa^{+k-i}$$

If $\{\mathfrak{p}_i \mid i < k\}$ was not an antichain then there were $i, j < k$ such that $\mathfrak{p}_i \leq^* \mathfrak{p}_j$.

It would be impossible for $i < j$, because in such case we would have a surjection $f\colon \mathcal{Q}^j(A) \cup \kappa^{+k-j} \to \mathcal{Q}^i(A) \cup \kappa^{+k-i}$. Since $\kappa^{+k-j} < \kappa^{+k-i}$ we have that $f''\kappa^{+k-j} \cap \kappa^{+k-i}$ has cardinality strictly smaller than κ^{k-i}. Therefore $f''\mathcal{Q}^j(A) \cap \kappa^{+k-i}$ is of cardinality κ^{+k-i} which is impossible because it means that

$$\kappa^{+k-i} < \aleph^*(\mathcal{Q}^j(A)) \leq \aleph^*(\mathcal{Q}^k(A)) = \kappa.$$

However we cannot have $j < i$ either, as in this case we have that $\mathcal{Q}^i(\mathfrak{a}) \leq^* \mathcal{Q}^j(\mathfrak{a}) + \kappa^{+k-j}$. Since $\mathcal{Q}(\mathcal{Q}^j(\mathfrak{a})) \leq \mathcal{Q}^i(\mathfrak{a})$ by Lemma 3.9 $\mathcal{Q}^i(\mathfrak{a})$ is well-orderable, and therefore $\mathfrak{a}$ is well-orderable, but we assumed that A cannot be well-ordered.

Therefore $\{\mathfrak{p}_i \mid i < k\}$ is a $\leq^*$ antichain of length k, and $\mathsf{Tri}^*(k)$. □

3.3 Discussion and Open Questions

We have seen that both in the case of $\leq$ and $\leq^*$ if there is a finite bound on the size of antichains then the axiom of choice holds. Feldman, Orhon and Blass ask in the paper the following questions:

Question. Does $\mathsf{ZF} + \mathsf{Tri}(\omega)$ prove AC? Where $\mathsf{Tri}(\omega)$ states that every antichain is finite.

Question. Does $\mathsf{ZF} + \mathsf{Tri}(\infty)$ prove AC? Where $\mathsf{Tri}(\infty)$ states that every antichain is Dedekind-finite.

The authors of [FOB08] believe that the answer to the first question is negative. Note that all the antichains that we have created used decreasing sequences of ordinals, to construct an infinite antichain would require either a decreasing sequence of cardinals, or a whole new understanding in what structure is provably true from ZF on the order of cardinals. In his notice Tarski asks whether or not the theorem can be extended to infinite, and in particular countable, sets of cardinals.

We will now prove the equivalence between $\mathsf{Tri}(\infty)$ and $\mathsf{Tri}(\omega)$. If every antichain is finite, in particular it is Dedekind-finite, and so $\mathsf{Tri}(\omega)$ implies $\mathsf{Tri}(\infty)$, so we need to show now that the reverse implication holds.

We shall see that if there exists an infinite Dedekind-finite set then there is a countably infinite (and so Dedekind-infinite) antichain of cardinals. Therefore $\mathsf{Tri}(\infty)$ implies that Dedekind-finite sets are finite, in particular antichains of cardinals. So $\mathsf{Tri}(\infty)$ implies that every antichain of cardinals is finite, which is to say it implies $\mathsf{Tri}(\omega)$.

Lemma 3.11. *If there exists an infinite Dedekind-finite set, then there exists a countably infinite antichain of cardinals.*

To prove this lemma we will first prove the following lemma by Tarski:

Lemma 3.12 (Tarski). *Assume that there exists an infinite Dedekind-finite set A. Then there exists an order preserving embedding of $\mathbb{R}$ with its natural order into the class of Dedekind-finite cardinals.*

Proof. We observe that the set $S = \{f \in A^{<\omega} \mid f \text{ injective}\}$ is a Dedekind-finite set as well, otherwise it has a countably infinite subset $\{f_n \mid n \in \omega\}$. However $\bigcup_{n<\omega} \operatorname{rng} f_n$ is an enumerated union of finite sets and it is infinite, and therefore countably infinite. Since A itself is Dedekind-finite this is a contradiction.

Let $\{X_r \subseteq \omega \mid r \in \mathbb{R}\}$ be a chain of order type $\mathbb{R}$ (e.g. indices of a fixed enumeration of the rational numbers under Dedekind cuts). For $r \in \mathbb{R}$ define $A_r = \{f \in S \mid \operatorname{dom} f \in X_r\}$. Since A_r is a subset of a Dedekind-finite set, it is a Dedekind-finite set as well and if $r < s$ then $A_r \subsetneq A_s$.

Letting $\mathfrak{A} = \{\mathfrak{a}_r = |A_r| \mid r \in \mathbb{R}\}$ finishes the proof. Since by the fact $A_s \subsetneq A_r$ for $s < r$ we have $\mathfrak{a}_s < \mathfrak{a}_r$, and it is clear that every two cardinals in this set are comparable. $\square$

Proof of Lemma 3.11. We will now use $\mathfrak{A}$ to define a countably infinite antichain of cardinals. First we observe that if $r, s \in \mathbb{R}$ and $\mathfrak{b}$ is any cardinal such that $\mathfrak{a}_r + \mathfrak{b} = \mathfrak{a}_s$ then $\mathfrak{b}$ is not well-ordered. To see this, note that $\mathfrak{b}$ cannot be an infinite $\aleph$ number, otherwise $\mathfrak{a}_s$ would not be Dedekind-finite. On the other hand if $\mathfrak{b}$ was a finite set, we could not have had infinitely many cardinals $\mathfrak{a}_r < \mathfrak{p} < \mathfrak{a}_s$.

Consider now the family of cardinals $\{\mathfrak{p}_n = \mathfrak{a}_{-n} + \aleph_n \mid n \in \omega\}$. If there were $n, m \in \omega$ such that $\mathfrak{p}_n \leq \mathfrak{p}_m$ then we would have $f\colon A_{-n} \cup \omega_n \to A_{-m} \cup \omega_m$ injective. However the usual argument shows that if $m < n$ then $\aleph_n \leq \mathfrak{a}_{-m}$, and if $n < m$ then we have that $\mathfrak{a}_m = \mathfrak{a}_n + \alpha$ for some ordinal α. Both contradictory to our assumptions. $\square$

This brings us to a natural question. What does the assertion "there is no decreasing sequence of cardinals" imply in terms of choice principles? It turns out that the answer in unknown. It clearly implies there are no infinite Dedekind-finite sets, but does it imply much more? There is very little known about this principle, as checking in [HR98][3] shows.

It should be pointed that merely a decreasing sequence would not suffice, we would need that the difference between infinitely many of its members is not well-ordered. It seems like a reasonable conjecture that this property is true for decreasing sequences of cardinals by the fact that an infinite family of those would have the same Hartogs number and therefore the differences between them cannot be well-ordered.

We saw that $\mathsf{Tri}(\omega)$ implies that there are no family of cardinals whose order is linear and dense, and in particular that there are no infinite Dedekind-finite sets.

Let us take a moment to prove an earlier remark, and to show that $\leq^*$ may behave very strangely in the absence of choice.

Proposition 3.13. *If there exists an infinite Dedekind-finite cardinal, then there exists a Dedekind-finite cardinal $\mathfrak{a}$ such that $\mathfrak{a} < \mathfrak{a} + \mathfrak{a} \leq^* \mathfrak{a}$.*

Proof. Let X be an infinite Dedekind-finite set. We have that $X \times \{0, 1\}$ is also Dedekind-finite by Proposition 3.3: $\aleph(X) = \aleph(X \times 2) = \omega$.
Let $A = \{f \in (X \times 2)^{<\omega} \mid f \text{ is injective}\}$, as in the proof of Lemma 3.12 A is Dedekind-finite. We will show that A can be mapped onto $A \times 2$. And so $|A| = \mathfrak{a}$ proves the proposition.

We will now define a surjective function $g\colon A \to A \times \{0, 1\}$. For $f \in A$ denote by $k_f = \max \operatorname{dom} f$, we define g as follows:

$$g(f) = \begin{cases} \langle f\restriction_{k_f}, 0\rangle & f(k_f) \in X \times \{0\} \\ \langle f\restriction_{k_f}, 1\rangle & \text{otherwise.} \end{cases}$$

To see that g is surjective, if $\langle f, 0\rangle \in A \times 2$ then there is some $y \in X \times \{0\}$ such that $y \notin \operatorname{rng} f$, and $g(f \cup \{\langle k_f + 1, y\rangle\}) = \langle f, 0\rangle$. Similarly for pairs of the form $\langle f, 1\rangle$. Since A and $A \times 2$ are both Dedekind-finite we also have that $|A| < |A \times 2|$ as wanted. $\square$

[3] This statement is **Form 7** in the book.

This motivates the following questions. Note that each question implies the subsequent questions.

Question. Does $\mathsf{Tri}^*(\omega)$ imply the axiom of choice?

Question. Does $\mathsf{Tri}^*(\omega)$ imply $\mathsf{Tri}(\omega)$?

As with the finite case, we see quite easily that $\mathsf{Tri}(\omega)$ implies $\mathsf{Tri}^*(\omega)$. The converse is true for finite k because both $\mathsf{Tri}(k)$ and $\mathsf{Tri}^*(k)$ imply the axiom of choice, however is it still true in the case of $k = \omega$?

Question. Does $\mathsf{Tri}^*(\omega)$ imply $\mathsf{W}_{\aleph_0}$, i.e. that there are no infinite Dedekind-finite sets?

The Dedekind-finite sets constructed in Proposition 3.13 and in Tarski's lemma might have peculiar properties in terms of $\leq^*$. For example it might be possible that even when $r < s$ we have $\mathfrak{a}_s \leq^* \mathfrak{a}_r$. This would necessitate a different technique than the one employed in the proof of Lemma 3.11.

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