THE FIVE WH'S OF SET THEORY (AND A BIT MORE)

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Abstract. What is set theory, why should you study it and how should you study it? Where and when should you study it? We tackle these questions from a contemporary point of view.

1. What is set theory and why study it?

Sets are mathematical objects which are themselves collections of mathematical objects. Set theory is a branch of mathematics which studies the properties of universes of set theory. In addition to that, set theory provides a common basic language to mathematics. This means that we can formalize mathematics like the real numbers or differentiation of real-valued functions and treat all these objects as sets. So the properties of the universe of sets are the properties of the universe of mathematics. Therefore the study of set theory is the study of properties of the mathematical universe.

1.1. **What are sets?** The idea that sets are collections of mathematical objects, which are themselves mathematical objects, is just an informal intuition as to how we treat sets. The reality is that sets are primitive to mathematics, they cannot be defined in "simpler terms". To understand that sets are truly fundamental to mathematics, let us entertain ourselves with a quick thought experiment often used to describe the origin of mathematics. We go back into prehistory of the human race. Long before the Greek, or the Babylonians. Long before them, we like to imagine, men counted mammoths, or rocks, or tribe members, and at some point someone realized that quantity is abstract. So the quantity obtained by adjoining two apples to another apple is the same as the number of three children. From there, we like to think, arithmetic developed, and as society became more complex the need for better abstractions grew and fractions, zero, irrational numbers and many other concepts became commonplace.

But this story is missing a very fine point. In order to count how many apples you have, first you need to identify that there is a collection of apples to count. In order to count the members of your tribe, first you need to identify that there is a collection "of people in your tribe". We see, if so, that the notion of a collection is as fundamental than the idea of counting. And in mathematics we can argue the same. We can use sets, their (formal) properties and the membership relation, and construct the rest of mathematics.

So what *are* sets? How do we know what properties they have? Well, we postulate a list of axioms which seem self-evident from our understanding of collections in real life, and finite collections we can "test by hand". In modern times the list of axioms that has been accepted by set theorists as the canonical list of axioms is the theory of Zermelo and Fraenkel with the Axiom of Choice, and it is often denoted by ZFC.

1.2. **Why study set theory?** While the axioms of ZFC tell us more or less everything we want to know about the finite sets, there is a lot that we can neither prove nor disprove from these axioms about infinite sets. The prime example is the fact that ZFC proves that the set of real

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numbers is strictly larger than the set of rational numbers, but is there a set whose cardinality lies strictly between them? Cantor hypothesized that the answer is negative, but Gödel and Cohen proved that we can neither prove nor disprove that. Such questions can be seen as unnatural, or even irrelevant to mathematics. Sets like these are not of particular interest to the general mathematician. Maybe we can just ignore them?

It turns out the answer is negative in two aspects. The first is that questions like the problem of intermediate cardinality between that of the rational and that of the real numbers end up affecting questions in operator theory, commutative algebra and other areas in mathematics. Other assertions which cannot be proved from ZFC end up having effects on the structure of the sets of real numbers, and even if that is not directly affecting the end result of what analysts do, the truth value of these assertions tells us if certain examples can be "simple" or not. For example, how complex is a description of a non-measurable set? Some set theoretic axioms imply that there are relatively simple description of non-measurable sets, whereas other axioms tell us that only quite complicated sets can be non-measurable.

The other aspect which makes it hard to ignore "less-natural sets" is that modern mathematical research strives to a certain aesthetic in generality and formulation of statements. This is one of the reasons that the axiom of choice has been accepted into the mathematical canon. Instead of formulating things like "Every commutative ring with a unit whose underlying set can be well-ordered has a maximal ideal" (which will still not capture the full generality of the theorem), the axiom of choice allows us to simply say "Every commutative ring with a unit has a maximal ideal". Similarly to that, some theorems in algebra or topology end up depending on the existence (or lack thereof) of certain infinite sets. The most striking example is Whitehead's Problem which can be easily proved for finitely generated abelian groups, and induction can push this to countably generated abelian groups. But Shelah showed that the general statement is neither provable nor refutable from the axioms of ZFC.

So why study set theory? First and foremost because it is fun and interesting. But even if you are not interested in set theory for its own sake, understanding how infinite sets behave, and what structure the universe of sets *can* have, means that you are better capable understanding how ordinary objects in mathematics behave when you move from the realm of countable objects. If modern mathematics deals with abstract infinite objects, understanding the set theoretic universe gives you a slightly better understanding how the mathematical universe can behave, and by extension what we can prove about it.

2. When, where and how to study set theory?

Let me begin this section by addressing the second question in its title. How to study set theory. Over the course of our lives we develop intuition that allows us to say whether or not something is true or false. This intuition is usually developed from our physical experience, and the way we reason internally to understand the things we see. Mathematics in general, and set theory in particular do not deal directly with reality, but rather with well-defined abstract objects. In set theory the objects of interest are arbitrary sets which represent nothing physical. They might be infinite, they might be finite but very large. And these sets will have properties that at first defy our expectations and often confuse.

How do you overcome this? Slowly. The first step is to accept that your intuition is most likely going to fail you at first. The second step is to work very carefully with the definitions. Make sure that you understand the assumptions in every statement, and what you are trying to prove. If you work slowly by unwinding the definition and reconstructing them slowly, with time you develop a new intuition, an intuition that matches the definitions and theorems that you used over and over again. At this point it is often a good idea to look back, both to congratulate yourself on the progress that you made as well to reinspect what you felt uneasy with at first. You will often find that after developing sufficient intuition going back to the beginning to review a certain definition, theorem or proof will result in an enlightenment. This is true for every field in mathematics, not just set theory.

2.1. **Elementary set theory.** So when should you study set theory? You will study some set theoretic basics wherever you go in modern mathematics. Unions, intersections, equivalence relations and other basic tools appear in almost every introductory book to calculus and algebra. These are the very basic and naive tools of set theory. But these things are considered set theoretic as much as differentiating polynomials is considered calculus.

Good introductions to set theory will include in addition to the basic operations we can perform on sets, the following topics:

(1) The basic paradoxes of naive set theory. Namely, not every collection we can talk about forms a set.
(2) Formalization of ordered pairs, equivalence and order relations, as well as functions using sets.
(3) The notion of equipotence and cardinality.
(4) The properties of finite and countable sets.
(5) The axiom of choice and its important equivalents.
(6) Well-orders and von Neumann ordinals and transfinite induction and recursion.
(7) The $\aleph$ numbers and their basic properties.

Since these topics become very abstract very quickly, some mathematical maturity is required to properly be able to tackle this list. While first year students are certainly capable to do this, it is recommended to have some basic experience with mathematical reasoning and various mathematical constructions before that. Therefore set theory is a second and third year topic in many universities (and rightfully so for the majority of students).

This is as far as elementary set theory goes. The introductory part which will give you the basic toolkit to deal with infinite sets in modern mathematics. And while many people might argue that you don't really need a lot of these topics to do analysis or algebra, this might be a good place to remind you that you don't need a lot of modern mathematics to build bridges or paint a portrait. We study these things for their inherent beauty, and set theory as a foundation for mathematics is a pinnacle of postmodernist minimalism: with just a few axioms and a binary relation for membership, we can build the whole world from one empty set.

The following is a list, which is far from complete, of introductory books to set theory which cover the above material.

- Enderton, **Elements of set theory.** *Academic Press*, 1977. ISBN: 0122384407
- Hrbacek and Jech, **Introduction to set theory.** *Marcel Dekker, Inc.*, 1999. ISBN: 0824779150
- Goldrei, **Classic Set Theory: For Guided Independent Study.** *Chapman and Hall/CRC Press*, 1996. ISBN: 0412606100
- Just and Weese, **Discovering Modern Set Theory. I.** *American Mathematical Society*, 1996. ISBN: 0821802666
- Moschovakis, **Notes on set theory.** Second edition. *Springer*, 2006. ISBN: 0387287221
- Lévy, **Basic Set Theory.** *Springer-Verlag*, 1979. ISBN: 3540084177 (Reprinted by *Dover Publications, Inc.*, 2002. ISBN: 0486420795)

The last two books already include material beyond the scope of your usual introductory course to set theory. This brings us to the question, how to continue from the introduction into the actual world of set theory?

2.2. **Advanced of set theory.** Advanced set theory deals with set theory for its own sake. The focus shifts from the basics of set theory and the interactions of set theory with the mathematical universe (although it is not devoid of these interactions), to the study of models of set theory and the modern toolkit of a set theorist. These are usually advanced undergraduate and graduate level courses, and they require additional background, specifically in logic and basic model theory. Topics usually covered in first and second courses in advanced set theory include some of the following topics.

(1) The axioms of ZFC.
(2) Well-foundedness, ordinals, the axiom of foundation (also regularity) and its consistency through the von Neumann hierarchy.
(3) Cardinals, clubs and stationary sets, cofinality.
(4) Gödel's constructible universe and the consistency of the Axiom of Choice and the continuum hypothesis.
(5) Infinitary combinatorics: consistency and consequences of $\Diamond$ and $\Box$ principles; Suslin and Aronszajn trees.
(6) Martin's Axiom and its consequences.
(7) Basic descriptive set theory.
(8) Large cardinals: inaccessible cardinals, weakly compact cardinals, measurable cardinals and elementary embeddings.
(9) Forcing and the consistency of failure of the continuum hypothesis.
(10) Iterated forcing and the consistency of Martin's Axiom.
(11) Relative constructibility and the consistency of the negation of the Axiom of Choice with ZF.
(12) Basic PCF theory.

To a set theorist's eye the above list might seem a bit jumbled. It includes various subjects which might be the topic of very advanced courses and not at all a first, second or even third graduate level course in set theory.

But these are all possible topics that can be covered in advance courses and certainly the basic toolbox from which one can start and understand modern research in set theory. To complement the list of topics, we include a short and very incomplete list of books that should help accompany the young set theorist in their first steps into the field.

- Jech, **Set Theory.** Third Millennium Edition. *Springer-Verlag*, 2002. ISBN: 3540440852, 3642078990
- Kunen, **Set Theory.** *College Publications*, 2011. ISBN: 1848900503
- Schindler, **Set Theory.** *Springer*, 2014. ISBN: 3319067249
- Halbeisen, **Combinatorial Set Theory.** *Springer*, 2012. ISBN: 1447121724
- Just and Weese, **Discovering Modern Set Theory. II.** *American Mathematical Society*, 1997. ISBN: 0821805282
- Kanamori, **The Higher Infinite.** Second Edition. *Springer*, 2003. ISBN: 3540003843
- Foreman and Kanamori (eds.), **Handbook of Set Theory**. *Springer*, 2010. ISBN: 1402048432
- Holz, Steffens and Weitz, **Introduction to Cardinal Arithmetic.** Second Edition. *Birkhäuser Verlag*, 1999. ISBN: 3764361247

It is imperative to reiterate that the above list is by no means complete, and there are plenty of other books. Some may put more emphasis on one topic and others might put more emphasis on other topics. There are many specialized books written on one topic, or with one goal in mind. But as far as basic references to go to, the above list should cover most of the needs of a budding set theorist.

3. What else?

The majority of the above texts work in the context of ZFC. But this is not the only set theory out there. There are flavors of set theory which are very different from ZFC and its close relatives, and we would be remiss if this text would not mention at least some of them.

3.1. **Relatives of Zermelo-Fraenkel set theory.** Some collections which we can describe, but we can also prove that they cannot form a set. In ZFC the collection of all sets is not a set, but we can still talk about it. Such collection is called a class. It is possible to extend ZFC and include classes as objects of the language, such set theories include *von Neumann-Gödel-Bernays* (abbreviated NBG) and *Kelley-Morse* (abbreviated as KM). Another popular extension, especially in the context of set theory without the axiom of choice, is *Zermelo-Fraenkel with Urelements* (abbreviated ZFU), which is a set theory in which there are non-set objects called urelements, or atoms. These atoms can often be dispensed at the cost of removing the axiom of foundation, there are several axioms called *Anti-Foundation Axioms* which replace the axiom of foundation by a strong form of negation of the axiom. One example of such axiom is Aczel's Anti-Foundation Axiom. More details can be found in the following book:

Aczel, **Non-Well-Founded Sets.** *Stanford University, Center for the Study of Language and Information*, 1988. ISBN: 0937073229

3.2. **Quine's New Foundations.** The axioms of ZFC were born out of the remnants of the naive properties of sets formulated in the late 19th century. Quine suggested a different approach than the one taken by Zermelo, Fraenkel and their collaborators. Where ZFC limits the way we can define new sets by bounding them in preexisting sets, Quine's idea was to pose limitations on which properties can define sets to begin with. This theory is called *New Foundations* (abbreviated NF), and similar to ZFU it too has a variant with Urelements abbreviated NFU. One interesting feature of NF is that the set of all sets exists in this theory. It was also shown to be inconsistent with the axiom of choice, although NFU is consistent with the axiom. We mention this book by Forster:

Forster, **Set theory with a universal set.** Second Edition. *Oxford University Press*, 1995. ISBN: 0198514778

4. Final words

This document is by no means complete and it meant to be replaced by better alternatives that will surely be written in the future. Our perception of the mathematical world is ever-changing, and the emphasis on one topic today might be phased out tomorrow. There is more, a lot more, to say on the philosophical aspects of set theory. From interactions with logic and model theory, to justifying axioms, to various philosophical approaches to set theory and the phenomenon of independence. Not to mention the interaction of set theory with other foundational approaches to mathematics like type theory and category theory. We urge the philosophically inclined reader to seek out answers to these questions on their own over the course of their studies, as well many discussions over tea in your department and beer in your local pub.

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