Intermediate Models of Prikry Type Forcings

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A general basic problem is to classify the complete subalgebras of a given complete boolean algebra B. Alternatively, one can study the intermediate models of a V-generic extensions $V[G]$ of B i.e. transitive ZFC models $V \subsetneq M \subseteq V[G]$ definable in $V[G]$.

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2. *(D.Maharam) Any intermediate model of a Random real generic extension is a Random real generic extension.*

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Theorem (Gitik, Kanovei, Koepke, 2010 [6])

Let U be a normal measure over κ and $G \subseteq \mathbb{P}(U)$ be a V-generic set producing the Prikry sequence $C_G := \{C_G(n) \mid n < \omega\}$. Then for every $A \in V[G]$ there is $C \subseteq C_G$, such that $V[A] = V[C]$.

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Every such model is of the form $M = V[A]$ for some set $A \in V[G]$. By the theorem, $M = V[C]$ for some subsequence C of the Prikry sequence. By the Mathias criteria[10], C is itself a Prikry sequence for U.

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In this talk we will examine the intermediate models of **the tree Prikry** and the **Magidor-Radin** forcings.

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Let $\langle S, \leq_S \rangle$ be any poset, denote by $[S]^{<\omega}$ the tree of finite $\leq_S$-increasing sequences ordered by end-extension. Let $\vec{U} = \langle U_a \mid a \in [S]^{<\omega} \rangle$ be a tree of $|S|$-complete uniform ultrafilters over S.

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We will mostly be interested in ultrafilters over κ itself. It turns out (not surprisingly) that the structure of the intermediate models of the tree Prikry forcing depends on the combinatorical properties of the measures in $\vec{U}$.

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Assume GCH and let κ be a measurable cardinal. There is a cofinality preserving forcing extension $V \subseteq N$ and an ultrafilter $W \in N$ such that forcing with $P_T(W)$[a] over N adds a κ-Cohen real.

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Assume that $\vec{U} = \langle U_\alpha \mid \alpha < \kappa \rangle$ is a sequence of distinct normal measures. Then for every V-generic filter $G \subseteq P_T(\vec{U})^a$, there is no proper intermediate model $V \subsetneq M \subsetneq V[G]$.

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Let us sketch the main ideas of the proof:

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Proof.

The model N is obtained by forcing the Easton support iteration $\langle P_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa, \beta < \kappa \rangle$:

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$$f = \cup_{n_0 \leq n < \omega} f_{\kappa_n} \restriction [\kappa_{n-1}, \kappa_n) \in N[C_G]$$

is N-generic for $Add(\kappa, 1)$. □

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2. *For every $p \in \mathbb{P}$ there is a κ-complete ultrafilter $U_p \supseteq \mathcal{D}_p(\mathbb{P})$. Where $\mathcal{D}_p(\mathbb{P})$ is the filter of dense open subsets of $\mathbb{P}$ above p.*

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Definition (κ-compact Cardinal)

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Let Q be the forcing shooting a club through the singulars below κ[a]. Assume that there is a κ-complete ultrafilter extending the filter $D(Q)$ of dense open subset of Q. Then either there is an inner model for $\exists\lambda, o(\lambda) = \lambda^{++}$, or $o^{\mathcal{K}}(\kappa) \geq \kappa + \kappa$.

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Theorem

Assume GCH and let κ be a measurable cardinal.
Then there is a cofinality preserving forcing extension in which for any $\mathbb{Q} \in \mathcal{N}_\kappa$, there is a κ-complete ultrafilter $\mathcal{U}$ extending $\mathcal{D}_p(\mathbb{Q})$ for every $p \in \mathbb{Q}$.

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Assume that $o^{\vec{U}}(\kappa) = \omega$, thus $\mathrm{otp}(C_G) = \omega^\omega$. Consider the intermediate extension $V[\{C_G(\omega^n) \mid n < \omega\}]$ it is a diagonal Prikry generic extension for the sequence of measures $\langle U(\kappa, n) \mid n < \omega\rangle$.

Intermediate models of Magidor Extensions

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Let κ *be a cardinal such that* $o^{\vec{U}}(\kappa) < \delta_0 := \min(\alpha \mid o^{\vec{U}}(\alpha) = 1)$*. Let* $G \subseteq \mathbb{M}[\vec{U}]$ *be a* V*-generic set producing the Magidor club* C_G*. Then for every* $A \in V[G]$ *there is* $C \subseteq C_G$*, such that* $V[A] = V[C]$*.*

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As we have seen in the examples, it is not clear which are the forcings such that the models $V[C]$ are generic for. In our paper, we defined in the ground model a class of "Magidor-Type" forcing notions, denoted by $\mathbb{M}_I[\vec{U}]$, which is basically a Magidor forcing adding elements prescribed by the set I, where I is the set of indices of C inside C_G.

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Theorem

In the settings of the last theorem, let $V \subseteq M \subseteq V[G]$ be an intermediate ZFC model definable $V[G]$, $M = V[G']$ where $G' \subseteq \mathbb{M}_I[\vec{U}]$ is a generic filter for some $I \in V$.

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Example

Assume that $o^{\vec{U}}(\kappa) = \kappa$, and let $C_G = \{C_G(\alpha) \mid \alpha < \kappa\}$, and let $\kappa^* \in C_G$ is such that for any $\beta \in C_G \setminus \kappa^*$, $o^{\vec{U}}(\beta) < \beta$. In $V[G]$, define $\alpha_0 = \kappa^*$, and $\alpha_{n+1} = C_G(\alpha_n)$. Then $\{\alpha_n \mid n < \omega\}$ is a cofinal ω-sequence in κ. Also, it satisfy the Mathias criteria [1] for the tree Prikry forcing with respect to the measures on κ, $\langle U(\kappa, \alpha) \mid \alpha < \kappa\rangle$.

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The proof of the lemma used the strong Prikry property.

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Suppose that $p \in \mathbb{M}[\vec{U}]$ and $D \subseteq \mathbb{M}[\vec{U}]$ is a dense open subset. Then there is $p^* \geq^* p$ and a $\vec{U}$-fat tree T, such that for every maximal branch $\vec{b} \in T$, $p^{*\frown}\vec{b} \in D$.

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Proof of lemma.

The strong Prikry property for $\mathbb{M}[\vec{U}]$

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A tree $T \subseteq [\kappa]^{<\omega}$ is called a $\vec{U}$-fat tree, if $ht(T) < \omega$ and for every $t \in T$, either t is a maximal element of the tree, or $succ_T(t) := \{\alpha < \kappa \mid t^\frown\alpha \in T\} \in U(\beta, i)$ for some $\beta \leq \kappa$ and $i < o^{\vec{U}}(\beta)$.

Proposition (The strong Prikry Property)

Suppose that $p \in \mathbb{M}[\vec{U}]$ and $D \subseteq \mathbb{M}[\vec{U}]$ is a dense open subset. Then there is $p^* \geq^* p$ and a $\vec{U}$-fat tree T, such that for every maximal branch $\vec{b} \in T$, $p^{*\frown}\vec{b} \in D$.

Proof of lemma.

Assume for example that $A = \{a_n \mid n < \omega\}$ and let $\langle \underset{\sim}{a}_n \mid n < \omega\rangle$ be a sequence of $\mathbb{M}[\vec{U}]$-names for A.

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Proof of lemma.

Assume for example that $A = \{a_n \mid n < \omega\}$ and let $\langle \underset{\sim}{a}_n \mid n < \omega \rangle$ be a sequence of $\mathbb{M}[\vec{U}]$-names for A. Let $p \in \mathbb{M}[\vec{U}]$, for each n apply the Strong Prikry property to find $p \leq^* p_n$ and a $\vec{U}$-fat tree T_n such that for every $\vec{\beta} \in mb(T_n)$, there is γ $p_n^\frown\vec{\beta} \Vdash \underset{\sim}{a}_n = \gamma$. Denote by $f_n(\vec{\beta}) = \gamma$.

Sketch of the Proof

Using combinatorical properties of $\vec{U}$-fat trees, we can extend p_n to some p_n^* and collapse some of the levels of T_n to T_n^* such that the restriction of f_n to T_n^*, will be $1-1$.

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Say that $A \subseteq On$ stabilizes, if there is $\beta < \kappa$ such that $\forall \alpha < \sup(A). A \cap \alpha \in V[C_G \cap \beta]$.

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Using combinatorical properties of $\vec{U}$-fat trees, we can extend p_n to some p^*_n and collapse some of the levels of T_n to T^*_n such that the restriction of f_n to T^*_n, will be $1-1$. By $\leq^*$-closure, find a single p_ω such that $p_n \leq^* p_\omega$ for every $n < \omega$ and by density find such $p^* \in G$. There will always be a branch D_n such that $p^{*\frown} D_n \in G$. Since $(\underset{\sim}{a}_n)_G = a_n$ it follows that $f_n(D_n) = a_n$, define $C = \cup_{n<\omega} D_n$. In $V[C]$ we can construct $A = \{f_n(D_n) \mid n < \omega\}$ and in $V[A]$ we can calculate each D_n, $D_n = f_n^{-1}(a_n)$, then $C = \cup_{n<\omega} f_n^{-1}(a_n)$. Thus $V[A] = V[C]$. □

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An example for A that stabilizes if for example the canonical ω-sequence $\langle \alpha_n \mid n < \omega \rangle$ we defined in previous examples. Any bounded initial segment of it is final and therefore belongs to the ground model. So we can take $\beta = 0$ for example and $\forall \delta < \kappa . \{\alpha_n \mid n < \omega\} \cap \delta \in V = V[C_G \cap 0]$.

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If $A \subseteq \kappa$ does not stabilize, then $\theta_A := cf^{V[A]}(\kappa) < \kappa$.

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Consider the set $X_A = \{\nu < \kappa \mid cf^{V[A]}(\nu) < cf^V(\nu) = \nu\}$. Note that $Cl(X_A) \subseteq Lim(C_G)$, since only the points in $Lim(C_G)$ change cofinality in $V[G]$ (thus in $V[A]$).

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Sketch of the Proof- A which does not stabilize

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how do we construct C?

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2. For all $\delta \in Lim(D)$ and $Y \in \cap_{i<o^{\vec{U}}(\delta)} U(\delta, i)$, there is $\xi < \delta$ such that $D \cap (\xi, \delta) \subseteq Y$.

Sketch of the Proof- A which does not stabilize

how do we construct C? Fix a cofinal sequence $\langle \alpha_i \mid i < \theta_A \rangle \in V[A]$. For $i < \theta_A$, apply induction to find $C_i \subseteq C_G$ such that $V[C_i] = V[A \cap \alpha_i]$. The sequence $\langle C_i \mid i < \theta_A \rangle$ entails the information needed to construct A. It is tempting to define $C = \cup_{i<\theta_A} C_i$. However there are two problems here:

1. Although each $C_i \in V[A \cap \alpha_i] \subseteq V[A]$, the entire sequence $\langle C_i \mid i < \theta_A \rangle$ is not necessarily in $V[A]$.
2. When taking the union we might loss information i.e it is possible that some $C_i \notin V[C]$.

Definition (Mathias set)

A set $D \in V[A]$ is called a *Mathias set*, if

1. $Lim(D) \subseteq Cl(X_A)$.
2. For all $\delta \in Lim(D)$ and $Y \in \cap_{i<o^{\vec{U}}(\delta)} U(\delta, i)$, there is $\xi < \delta$ such that $D \cap (\xi, \delta) \subseteq Y$.

The main property of a Mathias set is:

Sketch of the Proof- A which does not stabilize

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To overcome the first problem we pick in $V[A]$ a sequence of Mathias sets $\langle D_i \mid i < \theta_A \rangle$ such that $V[A \cap \alpha_i] = V[D_i]$. Note that a new problem was created, $D := \cup_{i<\theta_A} D_i$ can accumulate infinite noise and seize to be Mathias. Fortunately, $|D \setminus C_G| \leq \theta_A < \kappa$, and by the Hausdorff property, there is Y which filters this noise. The second problem is a bit more technical to fix.

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For sets A such that $\sup(A) \geq \kappa^+$, we continue by induction on $sup(A)$. The idea at the induction step it to first find a single $C \subseteq C_G$, such that for every $\alpha < \sup(A)$, $A \cap \alpha \in V[C]$.

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Let $W \models ZFC$ and $\mathbb{P} \in W$ a forcing notion. Let $T \subseteq \mathbb{P}$ be any W-generic filter and λ a regular cardinal in $W[T]$. Assume $\mathbb{P}$ is λ-c.c. in $W[T]$. Then in $W[T]$ there are no fresh subsets of λ with respect to W.

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$\mathbb{M}[\vec{U}]/C$ is $\kappa^+ -$ c.c. in $V[G]$.

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The third condition might fail when intersecting large sets.

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We pick names for $\underset{\sim}{p}_i$ and find

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Note that we can also stabilize the part of the large set of q_i below α_n which guarantees that the intersection of r_i and r_j satisfy $(1), (2)$, as for (3), we can shrink even more $A_i \cap A_j$ to a set X so that (3) holds with respect to B_i, B_j.

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Intermediate models in the case $\delta_0 < o^{\vec{U}}(\kappa) < \kappa$

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Theorem ([2])

Assume that for every $\alpha <\leq \kappa$, $o^{\vec{U}}(\alpha) < \alpha$. Then for every V-generic filter $G \subseteq \mathbb{M}[\vec{U}]$ and every transitive ZFC intermediate model $V \subseteq M \subseteq V[G]$, there is a closed subset $C_{fin} \subseteq C_G$ such that:

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2. *There is a finite iteration $\mathbb{Q} := \mathbb{M}_{f_1}[\vec{U}] * \mathbb{M}_{\mathcal{L}_2}[\vec{U}] ... * \mathbb{M}_{\mathcal{L}_n}[\vec{U}]$, and $H^* \subseteq \mathbb{Q}$, V-generic H^* filter such that $V[H^*] = V[C_{fin}] = M$.*

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In the first example, the model is in fact a two steps iteration, the first parts adds a Prikry sequence to $C_G(\omega)$, so $f_1 : \omega \to \kappa$, $f_1(n) = 0$. The second part is of the form $\mathbb{M}_{\underset{\sim}{f}}[\vec{U}]$, where $\underset{\sim}{f}$ is a name for the function $f : \omega \to \delta_0$ defined by $f(n) = C_G(n)$.

References I

- Tom Benhamou, *Prikry Forcing and Tree Prikry Forcing of Various Filters*, Arch. Math. Logic **58** (2019), 787–817.
- Tom Benhamou and Moti Gitik, *Intermediate Models of Magidor-Radin Foring-Part I*, arXiv:2009.12775 (2020), submitted to Israely Journal of Mathematics.
- ———, *Sets in Prikry and Magidor Generic Extensions*, Annals of Pure and Applied Logic **172** (2021), no. 4, 102926.
- Moti Gitik, *Prikry-Type Forcings*, pp. 1351–1447, Springer Netherlands, Dordrecht, 2010.
- Moti Gitik, *On κ-Compact Cardinals*, preprint (to appear).
- Moti Gitik, Vladimir Kanovei, and Peter Koepke, *Intermediate Models of Prikry Generic Extensions*, Pre Print (2010).

References II

Akihiro Kanamori, *The Higher Infinite*, Springer, 1994.

Peter Koepke, Karen Rasch, and Philipp Schlicht, *Minimal Prikry-Type Forcing for Singularizing a Measurable Cardinal*, J. Symb. Logic **78** (2013), 85–100.

Menachem Magidor, *Changing the cofinality of cardinals*, Fundamenta Mathematicae **99** (1978), 61–71.

A. R. D. Mathias, *On Sequences Generic in the Sense of Prikry*, Journal of Australian Mathematical Society **15** (1973), 409–414.

Finish line

Thank you for your attention!