Paradoxes of perfectly small sets (abridged)

Freiling's argument

- Under CH, there is a subset S of the unit square with each horizontal slice of S and each vertical slice of $[0,1]^2 \setminus S$ countable
- Under MA, there is a subset S of the unit square with each horizontal slice of S and each vertical slice of $[0,1]^2 \setminus S$ measure 0 (in fact strong measure 0)
- A point in the unit square is chosen "at random." Depending on which coordinate you are told first, it will appear almost certain that that point is in S, or almost certain that it is not.
- So MA is false, right?

Perfectly small sets

- A set of reals X is called *perfectly small* if there is a perfect set P such that the translates $X + p$ for $p \in P$ are pairwise disjoint.
- These have a strong intuitive claim to the property that if a real is chosen randomly, it will almost certainly not be in X (indeed, that seems as unlikely as a random real being in some fixed singleton)
- Yet from a modest amount of choice, it can be shown that the class of perfectly small sets has terrible closure properties

Continuum Splitting Principle

- The Continuum Splitting Principle (CSP) asserts that every surjective function with domain R splits.
- Equivalently, every partition on the reals has a choice function
- Derives DCR (in fact, $DC_{\omega_1}(R)$) and is thus sufficient for the development of analysis and the theory of Lebesgue measure.
- Implies the Hausdorff, Banach-Tarski, and von Neumann paradoxes
- Not known to justify more pathological constructions like a Hamel basis for R and the three-spray theorem

Theorem 1. *(ZF + CSP)*

a *Every total extension μ of Lebesgue measure concentrates on a perfectly small set.*

b *There is a countable cofinality order $\prec_1$ and an uncountable cofinality ordering $\prec_2$ on $\mathbb{R}$ with every proper inital segment of each ordering perfectly small.*

c *$\mathbb{R}$ is an increasing union of perfectly small sets.*

d *There is $X \subset [0,1]^2$ with each horizontal slice of X and each vertical slice of $[0,1]^2 \setminus X$ perfectly small.*

- We'll focus on (b) today for lack of time
- (a) is a similar but more involved argument
- (c) and (d) are almost immediate corollaries of (b)

Let A be a perfect set of algebraically independent reals, e.g.

$$\left\{\sum_{n=0}^{\infty}\frac{2^{2^{\lfloor nr\rfloor}}}{2^{2^{n^2}}}\right\}_{r\in[1,2]}.$$

Fix an injective sequence $\{a_m\}_{m<\omega}\subset A$ and a perfect $P\subset A\setminus\{a_m : m<\omega\}$. Let $F=\mathbb{Q}(A)$ and $f:\mathbb{R}\to\mathbb{R}/F$ the projection function. CSP provides a right inverse $g:\mathbb{R}/F\to\mathbb{R}$. Define $h:\mathbb{R}\to F$ by $r\mapsto r-g(f(r))$, and let $F_n=\mathbb{Q}(A\setminus\{a_m : m\geq n\})$. Notice that the translates of F_n by $p\in a_nP$ are pairwise disjoint subsets of F, and $F=\bigcup_{n<\omega}F_n$.

Let $\mathbb{R}$ inherit the lexicographical ordering of $\omega\times\mathbb{R}$ via $r\mapsto(n,r)$, n least such that $h(r)\in F_n$. This is as desired.