Blowing up the power of cardinals which are singular in the ground models with collapses

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Definition

Let $\kappa < \kappa' < \lambda$. Extenders E on (κ, λ) and F on (κ', λ) are coherent *if $j_F(E) \upharpoonright \lambda = E$ where j_F is an embedding derived from F.*

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From the definition above, we have that E is Mitchell below F in the sense that $E \in \mathrm{Ult}(V, F)$.

Main theorem

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Given an increasing sequence of cardinals $\langle \kappa_\alpha : \alpha < \eta \rangle$ *where* $\eta < \kappa_0$ *is limit. Let* $\lambda = (\sup_{\alpha<\eta} \kappa_\alpha)^{++}$.

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1. *If* $j_\alpha : V \to M_\alpha = \mathrm{Ult}(V, E_\alpha)$*, we have* $\mathrm{crit}(j_\alpha) = \kappa_\alpha$*,* $j_\alpha(\kappa_\alpha) \geq \lambda$*,* ${}^{\kappa_\alpha}M_\alpha \subseteq M_\alpha$ *and* M_α *computes cardinals correctly up to and including* λ*.*

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2. *There is a function $s_\alpha : \kappa_\alpha \to \kappa_\alpha$ such that $j_\alpha(s_\alpha)(\kappa_\alpha) = \sup_{\beta<\eta} \kappa_\beta$.*

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Then there is a λ*-c.c. forcing extension such that in the generic extension, for limit* $\beta < \eta$, $2^{\aleph_\beta} > \aleph_{\beta+1}$ *and* $2^{\aleph_\eta} = \aleph_{\eta+2}$.

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The assumptions can be prepared with a sufficiently large Woodin cardinal.

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$\mathrm{mc}_\alpha(d_\alpha) = (j_\alpha \restriction d_\alpha)^{-1} = \{(j_\alpha(\gamma), \gamma) : \gamma \in d_\alpha\}$.

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$A \in E_\alpha(d_\alpha)$ iff $\mathrm{mc}_\alpha \in j_\alpha(A)$.

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4. *For* $\beta \in \mathrm{dom}(\mu) \cap \kappa_\alpha$, $\mu(\beta) = \beta$.

Lemma

$\mathrm{OB}_\alpha(d_\alpha) \in E_\alpha(d_\alpha)$.

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   $\text{dom}(\text{mc}_\alpha) \cap j_\alpha(\kappa_\alpha) = j_\alpha[d_\alpha] \cap j_\alpha(\kappa_\alpha) = d_\alpha \cap \kappa_\alpha$, and $\text{crit}(j_\alpha) = \kappa_\alpha$.

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- For $\alpha < \eta$, let $\overline{\kappa}_\alpha = \sup_{\beta < \alpha} \kappa_\beta$, and $\overline{\kappa}_0 = \max\{\omega, \eta\} < \kappa_0$.
- In particular, if $\alpha = \beta + 1$, $\overline{\kappa}_\alpha = \kappa_\beta$. If α is limit, $\overline{\kappa}_\alpha < \kappa_\alpha$.

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4. $\langle d_\alpha : \alpha < \eta \rangle$ is $\subseteq$-increasing.
5. For each $\mu \in A_\alpha$ with $\nu = \mu(\kappa_\alpha)$, $H^0_\alpha(\mu) \in \mathrm{Col}(\overline{\kappa}^+_\alpha, < \nu)$, $H^1_\alpha(\mu) \in \mathrm{Col}(\nu, s_\alpha(\nu)^+)$, and $H^2_\alpha(\nu) \in \mathrm{Col}(s_\alpha(\nu)^{+3}, < \kappa_\alpha)$.

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3. For $l = 0, 1$, $K^l_\alpha(\mu) \leq H^l_\alpha(\mu \restriction \mathrm{dom}(f_\alpha))$, and $K^2_\alpha(\mu(\kappa_\alpha)) \leq H^2_\alpha(\mu(\kappa_\alpha))$.

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One-step extension (example): p is pure and $\mu \in A_2$.

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One-step extension (example): p is pure and $\mu \in A_2$. One-step extension of p by μ, with $\nu = \mu(\kappa_2)$ is a condition q, written by $p + \mu$, such that:

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Forcing extensions

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Forcing extensions

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Forcing extensions

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- In particular, a few cardinals in the interval $(\kappa_1, \kappa_2]$ are preserved.

- Recursively define $p + \langle \mu_0, \ldots, \mu_{k-1} \rangle = (p + \mu_0) + \langle \mu_1, \ldots, \mu_{k-1} \rangle$. $q \leq p$ if $q \leq^* p + \langle \mu_0, \ldots, \mu_{k-1} \rangle$ for some objects $\mu_0, \ldots, \mu_{k-1}$.

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- In general, $p \leq^* q + \mu$ does not imply that $p = p' + \tau$ for some $p' \leq^* q$ and $\tau \supseteq \mu$.

Let's assume for a moment that η is limit.

Warm up

Lemma

Let $p = \langle p_\alpha : \alpha < \eta \rangle$ be a pure condition and $\alpha < \eta$. Let $d = \mathrm{dom}(f^p_\alpha)$. Set $q = j_\alpha(p) + \mathrm{mc}_\alpha(d)$. Then $q \restriction \alpha = p \restriction \alpha$.

The context in the lemma is reasonable, since $A^p_\alpha \in E_\alpha(d)$, so $\mathrm{mc}_\alpha(d) \in j_\alpha(A_\alpha)$.

Proof.

Say $p_\alpha = \langle f_\alpha, A_\alpha, \vec{H}_\alpha \rangle$. Then $j_\alpha(p) = \langle j_\alpha(p_\beta) : \beta < \eta \rangle$.

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For $\beta < \alpha$, $p_\beta = \langle f_\beta, A_\beta, \vec{H}_\beta \rangle$ where f_β is a partial function from λ to κ_β whose domain has size κ_β.

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Let $p = \langle p_\alpha : \alpha < \eta \rangle$ *be a pure condition and* $\alpha < \eta$*. Let* $d = \mathrm{dom}(f_\alpha^p)$*. Set* $q = j_\alpha(p) + \mathrm{mc}_\alpha(d)$*. Then* $q \restriction \alpha = p \restriction \alpha$*.*

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For $\beta < \alpha$, $p_\beta = \langle f_\beta, A_\beta, \vec{H}_\beta \rangle$ where f_β is a partial function from λ to κ_β whose domain has size κ_β. Thus $j_\alpha(f_\beta) = j_\alpha[f_\beta]$. Since $\mathrm{dom}(f_\beta) \subseteq d$, we have $\mathrm{dom}(j_\alpha(f_\beta) \circ \mathrm{mc}_\alpha^{-1}) = \{\gamma \in d : j_\alpha(\gamma) \in j_\alpha[d_\beta]\} = d_\beta$.
For $\gamma \in d_\beta$, $j_\alpha(f_\beta) \circ \mathrm{mc}_\alpha^{-1}(\gamma) = j_\alpha(f_\beta)(j_\alpha(\gamma)) = j_\alpha(f_\beta(\gamma)) = f_\beta(\gamma)$. The rests are similar.

□

The Prikry property

Theorem

For each forcing statement ϕ and a condition p there is $q \leq^* p$ such that either $q \Vdash \phi$ or $q \Vdash \neg\phi$.

Some conclusions

Assume η is limit. Recall $\lambda = \overline{\kappa}_\eta^{++}$.

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Assume η is limit.

Lemma

Let $\gamma \in [\overline{\kappa}_\eta, \lambda)$ and $\alpha < \eta$. Let D be the collection of $p \leq q$ such that

1. $\alpha \in \mathrm{supp}(p)$.
2. *If we enumerate* $\mathrm{supp}(p) \setminus \alpha$ *in decreasing order as* $\alpha_0 > \cdots > \alpha_{k-1} = \alpha$, *then* $\gamma \in \mathrm{dom}(f^p_{\alpha_0})$, *the sequence of ordinals defined inductively by setting* $\gamma_0 = \gamma$ *and* $\gamma_{i+1} = f^p_{\alpha_i}(\gamma_i)$ *for as long as* $\gamma_i \in \mathrm{dom}(f^p_{\alpha_i})$ *reaches a stage where* γ_{k-1} *is defined, and* $\gamma_{k-1} \in \mathrm{dom}(f^p_\alpha)$.
3. *For* $\beta \in (\alpha_{i+1}, \alpha_i)$, *for every* $\mu \in A^p_\beta$, $\gamma_{i+1} \in \mathrm{dom}(\mu)$.

Then D is open dense below q.

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For each such α and γ, find $p \in G$ satisfying the lemma. Define $t_\gamma(\alpha) = f^p_\alpha(\gamma_{k-1})$.

Lemma

$t_\gamma(\alpha)$ is well-defined.

Proof (Sketch).

For $p, q \in G$ as in the previous lemma, we show that p and q define $t_\gamma(\alpha)$ the same way. Let $r \in G$, $r \leq p, q$. Enough to show that p and r agree on $t_\gamma(\alpha)$ (Similar for q and r).

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$$\begin{aligned} & f^r_{\alpha_{k-1}} \circ \ldots f^r_{\alpha_n} \circ f^r_\beta \circ f^r_{\alpha_{n-1}} \circ \ldots f^r_{\alpha_0}(\gamma) \\ = {} & f^p_{\alpha_{k-1}} \circ \ldots f^r_{\alpha_n} \circ f^r_\beta \circ f^p_{\alpha_{n-1}} \circ \ldots f^p_{\alpha_0}(\gamma) \\ = {} & f^p_{\alpha_{k-1}} \circ \ldots f^p_{\alpha_n} \circ \mu^{-1} \circ \mu \circ f^p_{\alpha_{n-1}} \circ \ldots f^p_{\alpha_0}(\gamma) \\ = {} & f^p_{\alpha_{k-1}} \circ \ldots f^p_{\alpha_n} \circ f^q_{\alpha_{n-1}} \circ \ldots f^p_{\alpha_0}(\gamma). \end{aligned}$$

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In $V[G]$, $\langle t_\gamma : \gamma \in [\overline{\kappa}_\eta, \lambda)\rangle$ *is* $\leq^*$*-increasing.*

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Let $\gamma < \gamma'$ and $p \in \mathbb{P}$. By extending p, assume $\gamma, \gamma' \in \mathrm{dom}(f^p_{\alpha_0})$ where $\alpha_0 > \max(\mathrm{supp}(p))$. Assume that for $\alpha \geq \alpha_0$ domain of each object in A^p_α contains γ and γ'. Then $p \Vdash \forall\alpha \geq \alpha_0(\dot{t}_\gamma(\alpha) < \dot{t}_{\gamma'}(\alpha))$. The key point is every object is order-preserving.

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Each t_γ is a function in $\prod_{\alpha<\eta} \kappa_\alpha$. As a result, $\aleph_{\eta+2} = \lambda = 2^{\overline{\kappa}_\eta} = 2^{\aleph_\eta}$ in $V[G]$.