

Blowing up the power of cardinals which are singular in the ground models with collapses

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Definition

Let $\kappa < \kappa' < \lambda$. Extenders E on (κ, λ) and F on (κ', λ) are coherent if $j_F(E) \upharpoonright \lambda = E$ where j_F is an embedding derived from F .

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From the definition above, we have that E is Mitchell below F in the sense that $E \in \text{Ult}(V, F)$.

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- 1 If $j_\alpha : V \rightarrow M_\alpha = \text{Ult}(V, E_\alpha)$, we have $\text{crit}(j_\alpha) = \kappa_\alpha$, $j_\alpha(\kappa_\alpha) \geq \lambda$, $\kappa_\alpha M_\alpha \subseteq M_\alpha$ and M_α computes cardinals correctly up to and including λ .

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The assumptions can be prepared with a sufficiently large Woodin cardinal.

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$A \in E_\alpha(d_\alpha)$ iff $mc_\alpha \in j_\alpha(A)$.

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- 4 For $\beta \in \text{dom}(\mu) \cap \kappa_\alpha$, $\mu(\beta) = \beta$.

Lemma

$OB_\alpha(d_\alpha) \in E_\alpha(d_\alpha)$.

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Recall $mc_\alpha = \{(j_\alpha(\gamma), \gamma) : \gamma \in d_\alpha\}$. Also $(j_\alpha(\kappa_\alpha), \kappa_\alpha) \in mc_\alpha$ because $\kappa_\alpha \in d_\alpha$.

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- In particular, if $\alpha = \beta + 1$, $\bar{\kappa}_\alpha = \kappa_\beta$. If α is limit, $\bar{\kappa}_\alpha < \kappa_\alpha$.

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- 4 $\langle d_\alpha : \alpha < \eta \rangle$ is \subseteq -increasing.
- 5 For each $\mu \in A_\alpha$ with $\nu = \mu(\kappa_\alpha)$, $H_\alpha^0(\mu) \in \text{Col}(\bar{\kappa}_\alpha^+, < \nu)$, $H_\alpha^1(\mu) \in \text{Col}(\nu, s_\alpha(\nu)^+)$, and $H_\alpha^2(\nu) \in \text{Col}(s_\alpha(\nu)^{+3}, < \kappa_\alpha)$.

Forcing extensions

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Direct extension: $q \leq^* p$ if for all α we have

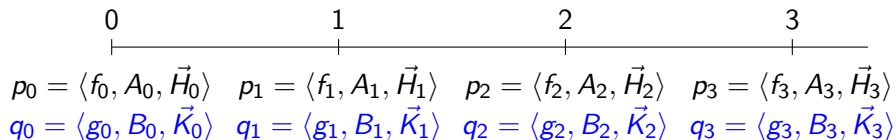
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- 3 For $l = 0, 1$, $K_\alpha^l(\mu) \leq H_\alpha^l(\mu \upharpoonright \text{dom}(f_\alpha))$, and $K_\alpha^2(\mu(\kappa_\alpha)) \leq H_\alpha^2(\mu(\kappa_\alpha))$.

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One-step extension (example): p is pure and $\mu \in A_2$. One-step extension of p by μ , with $\nu = \mu(\kappa_2)$ is a condition q , written by $p + \mu$, such that:

- 1 $q_\alpha = p_\alpha$ for $\alpha > 2$.

Forcing extensions

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One-step extension (example): p is pure and $\mu \in A_2$. One-step extension of p by μ , with $\nu = \mu(\kappa_2)$ is a condition q , written by $p + \mu$, such that:

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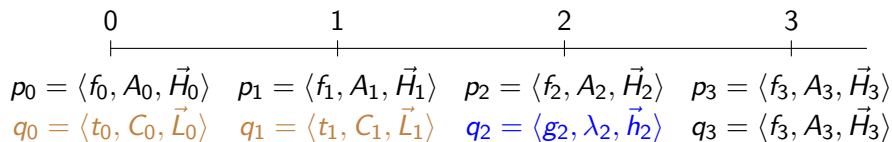
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- In particular, a few cardinals in the interval $(\kappa_1, \kappa_2]$ are preserved.

- Recursively define $p + \langle \mu_0, \dots, \mu_{k-1} \rangle = (p + \mu_0) + \langle \mu_1, \dots, \mu_{k-1} \rangle$.
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- In general, $p \leq^* q + \mu$ does not imply that $p = p' + \tau$ for some $p' \leq^* q$ and $\tau \supseteq \mu$.

Let's assume for a moment that η is limit.

Warm up

Lemma

Let $p = \langle p_\alpha : \alpha < \eta \rangle$ be a pure condition and $\alpha < \eta$. Let $d = \text{dom}(f_\alpha^p)$. Set $q = j_\alpha(p) + \text{mc}_\alpha(d)$. Then $q \upharpoonright \alpha = p \upharpoonright \alpha$.

The context in the lemma is reasonable, since $A_\alpha^p \in E_\alpha(d)$, so $\text{mc}_\alpha(d) \in j_\alpha(A_\alpha)$.

Proof.

Say $p_\alpha = \langle f_\alpha, A_\alpha, \vec{H}_\alpha \rangle$. Then $j_\alpha(p) = \langle j_\alpha(p_\beta) : \beta < \eta \rangle$.

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For $\gamma \in d_\beta$, $j_\alpha(f_\beta) \circ \text{mc}_\alpha^{-1}(\gamma) = j_\alpha(f_\beta)(j_\alpha(\gamma)) = j_\alpha(f_\beta(\gamma)) = f_\beta(\gamma)$. The rests are similar.



The Prikry property

Theorem

For each forcing statement ϕ and a condition p there is $q \leq^ p$ such that either $q \Vdash \phi$ or $q \Vdash \neg\phi$.*

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Assume η is limit. Recall $\lambda = \overline{\kappa_\eta}^{++}$.

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scale

Assume η is limit.

Lemma

Let $\gamma \in [\bar{\kappa}_\eta, \lambda)$ and $\alpha < \eta$. Let D be the collection of $p \leq q$ such that

- 1 $\alpha \in \text{supp}(p)$.
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- 3 For $\beta \in (\alpha_{i+1}, \alpha_i)$, for every $\mu \in A_\beta^p$, $\gamma_{i+1} \in \text{dom}(\mu)$.

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For each such α and γ , find $p \in G$ satisfying the lemma. Define

$$t_\gamma(\alpha) = f_\alpha^p(\gamma_{k-1}).$$

scale

Lemma

$t_\gamma(\alpha)$ is well-defined.

Proof (Sketch).

For $p, q \in G$ as in the previous lemma, we show that p and q define $t_\gamma(\alpha)$ the same way. Let $r \in G$, $r \leq p, q$. Enough to show that p and r agree on $t_\gamma(\alpha)$ (Similar for q and r).

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$$\begin{aligned} & f_{\alpha_{k-1}}^r \circ \dots \circ f_{\alpha_n}^r \circ f_\beta^r \circ f_{\alpha_{n-1}}^r \circ \dots \circ f_{\alpha_0}^r(\gamma) \\ &= f_{\alpha_{k-1}}^p \circ \dots \circ f_{\alpha_n}^r \circ f_\beta^r \circ f_{\alpha_{n-1}}^p \circ \dots \circ f_{\alpha_0}^p(\gamma) \\ &= f_{\alpha_{k-1}}^p \circ \dots \circ f_{\alpha_n}^p \circ \mu^{-1} \circ \mu \circ f_{\alpha_{n-1}}^p \circ \dots \circ f_{\alpha_0}^p(\gamma) \\ &= f_{\alpha_{k-1}}^p \circ \dots \circ f_{\alpha_n}^p \circ f_{\alpha_{n-1}}^q \circ \dots \circ f_{\alpha_0}^p(\gamma). \end{aligned}$$

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In $V[G]$, $\langle t_\gamma : \gamma \in [\bar{\kappa}_\eta, \lambda) \rangle$ is \leq^* -increasing.

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Let $\gamma < \gamma'$ and $p \in \mathbb{P}$. By extending p , assume $\gamma, \gamma' \in \text{dom}(f_{\alpha_0}^p)$ where $\alpha_0 > \max(\text{supp}(p))$. Assume that for $\alpha \geq \alpha_0$ domain of each object in A_α^p contains γ and γ' . Then $p \Vdash \forall \alpha \geq \alpha_0 (t_\gamma(\alpha) < t_{\gamma'}(\alpha))$. The key point is every object is order-preserving.

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Each t_γ is a function in $\prod_{\alpha < \eta} \kappa_\alpha$. As a result, $\aleph_{\eta+2} = \lambda = 2^{\bar{\kappa}_\eta} = 2^{\aleph_\eta}$ in $V[G]$.