Pseudo-Prikry Sequences

Chris Lambie-Hanson

Department of Mathematics and Applied Mathematics
Virginia Commonwealth University

Prikry Forcing Online

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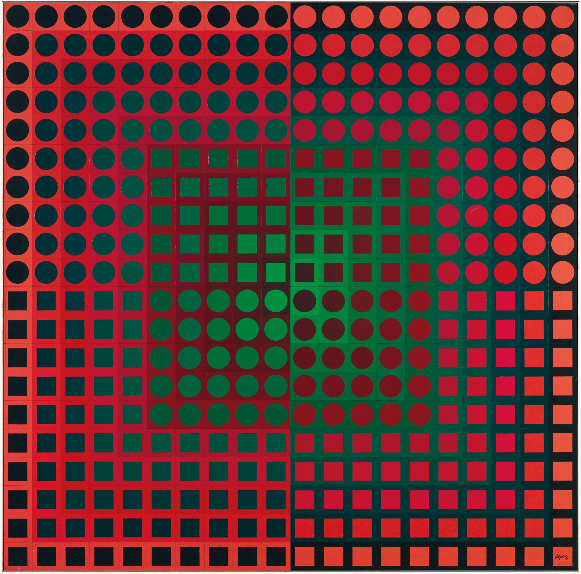
I. A brief history



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We call such objects *pseudo-Prikry sequences*.

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So our pseudo-Prikry sequences will be sequences in W which appropriately diagonalize certain club filters as defined in V.

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Then, in W, *there is a sequence* $\langle \gamma_i \mid i < \theta \rangle$ *of ordinals such that, for all* $\alpha < \kappa^+$ *and all sufficiently large* $i < \theta$, $\gamma_i \in C_\alpha$.

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A similar theorem is proven by Gitik [3].

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Then, in W, there is an increasing sequence $\langle x_i \mid i < \omega \rangle$ of elements of $(\mathcal{P}_\kappa(\kappa^{+n}))^V$ such that, for all $\alpha < \kappa^{+n+1}$ and all sufficiently large $i < \omega$, $x_i \in D_\alpha$.

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This can be seen as a "pseudo-" version of a generic sequence for supercompact Prikry forcing using a normal measure on $\mathcal{P}_\kappa(\kappa^{+n})$.

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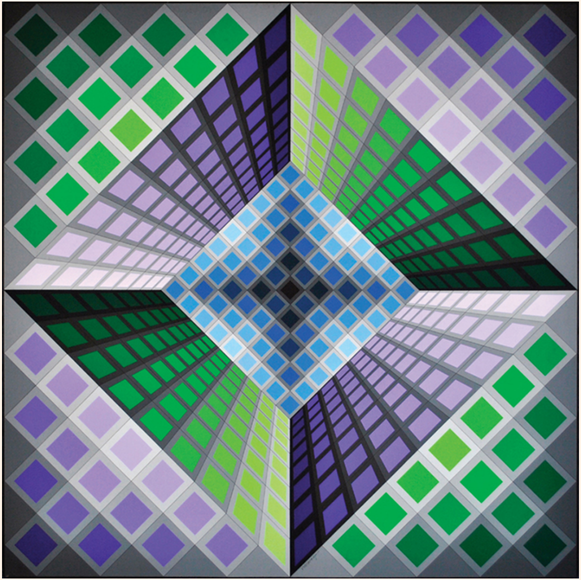
II. PCF-theoretic background



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Theorem (Shelah)

Suppose that μ is a singular cardinal. Then there is an increasing sequence $\langle \mu_i \mid i < \mathrm{cf}(\mu) \rangle$ of regular cardinals, cofinal in μ, such that there is a scale of length μ^+ in $\prod_{i<\mathrm{cf}(\mu)} \mu_i$.

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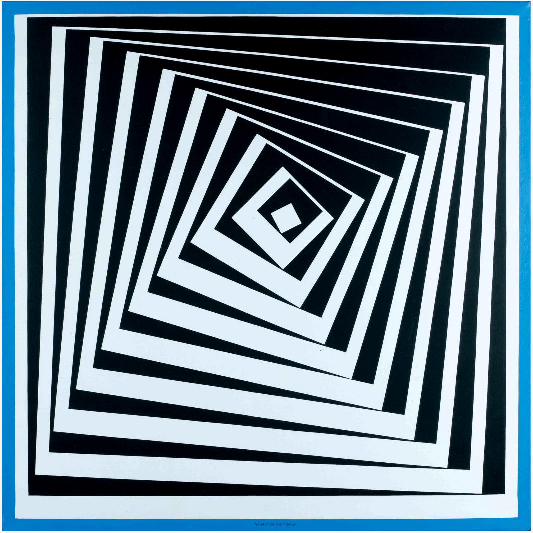
III. Diagonal pseudo-Prikry sequences



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- $(\kappa^+)^W$ is the successor of a regular cardinal in V;
- every V-regular cardinal in the interval $[\kappa, (\kappa^+)^V)$ has countable cofinality in W.

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Suppose one wants to find a model with a singular cardinal κ of countable cofinality at which both SCH and $\Box_{\kappa,\omega}$ fail. The simplest way to obtain a failure of SCH is to start with a measurable cardinal κ at which GCH fails and do Prikry forcing. But we've seen that this necessarily forces $\Box_{\kappa,\omega}$ to hold. By a generalization of this result due to Magidor and Sinapova, $\Box_{\kappa,\omega}$ will hold in any extension W in which

- $(\kappa^+)^W$ is the successor of a regular cardinal in V;
- every V-regular cardinal in the interval $[\kappa, (\kappa^+)^V)$ has countable cofinality in W.

This might lead us to look for Prikry-type extensions W in which $(\kappa^+)^W$ is the successor of a singular cardinal in V.

Diagonal supercompact Prikry forcing

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Theorem (LH, '18, [5])

Suppose that V is an inner model of W and, in V, μ is a singular cardinal, $\mathrm{cf}(\mu) = \theta$, and $\vec{\mu} = \langle \mu_i \mid i < \theta \rangle$ is an increasing sequence of regular cardinals cofinal in μ such that there is a scale of length μ^+ in $\prod \vec{\mu}$.

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Proof

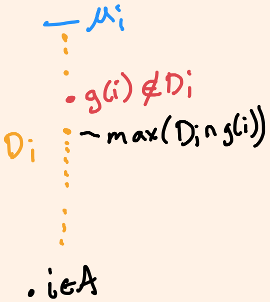
Fix a scale $\vec{f} = \langle f_\alpha \mid \alpha < \mu^+ \rangle$ in $\prod \vec{\mu}$.

Without loss of generality, we may assume that $\vec{f}$ is club-increasing.

In W, let ν be the predecessor of $(\mu^+)^V$ (so $\mathrm{cf}^W(\nu) = \theta$). In W, $\vec{f}$ is still club-increasing, and $\theta^{+3} < \mu^+$, so $\vec{f}$ has an e.u.b., g, such that $\mathrm{cf}(g(i)) > \theta$ for all $i < \theta$. In particular, since $\mathrm{cf}(\mu_i) = \theta$ for all i, we have $g \in \prod \vec{\mu}$.

More proof

We claim that g is as desired. Suppose for sake of contradiction that $\langle D_i \mid i < \theta \rangle \in V$ is a diagonal club in $\vec{\mu}$ and there is an unbounded set $A \subseteq \theta$ such that $g(i) \notin D_i$ for all $i \in A$



Define a function $h \in \prod \vec{\mu}$ by

$$h(i) = \begin{cases} \max(D_i \cap g(i)) & i \in A \\ 0 & i \notin A \end{cases}$$

Then $h < g$, so there is $\alpha < \mu^+$ such that $h <^* f_\alpha$.

Still more proof

Define a function $\hat{h} \in \prod \vec{\mu}$ by letting

$\hat{h}(i) = \min(D_i \setminus f_\alpha(i))$. Since $f_\alpha, \langle D_i \mid i < \theta \rangle \in V$,

we have $\hat{h} \in V$. Therefore, since $\vec{f}$ is a scale,

there is $\beta < \mu^+$ such that $\hat{h} <^* f_\beta$.

Now, for all sufficiently large $i \in A$, we have

$\max(D_i \cap g(i)) = h(i) < f_\alpha(i) \leq \min(D_i \setminus f_\alpha(i)) = \hat{h}(i) < f_\beta(i)$.

Moreover, since $(h(i), g(i)] \cap D_i = \emptyset$, we have

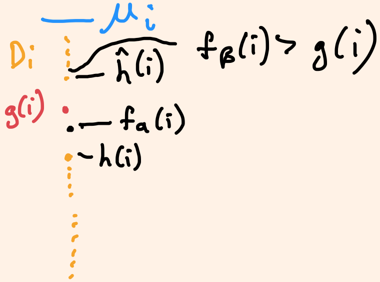
$\hat{h}(i) > g(i)$. Therefore, for unboundedly many $i < \theta$,

we have $f_\beta(i) > g(i)$, contradicting the fact

that g is an e.u.b. for $\vec{f}$. $\boxtimes$

This proof isn't over yet?

A typical $i \in A$:



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Then, *in W, there is a function $g \in \prod \vec{\mu}$ such that, for every diagonal club in $\vec{\mu}$, $\langle D_i \mid i < \theta \rangle \in V$, we have $g(i) \in D_i$ for all sufficiently large $i < \theta$. Also, we can require that* $\sup\{\mathrm{cf}(g(i)) \mid i < \theta\} = \mu$.

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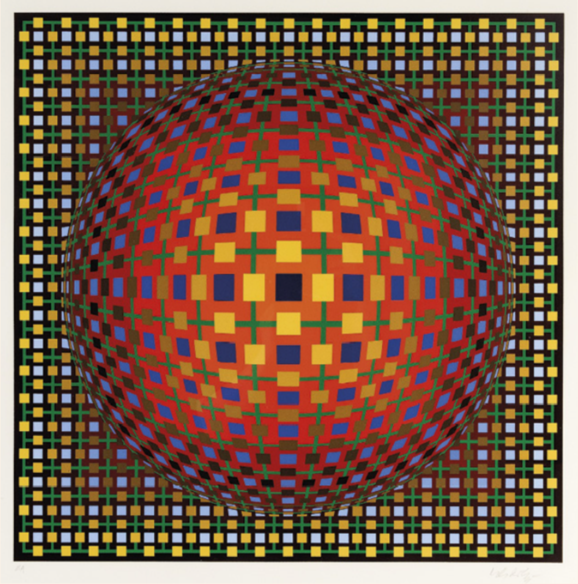
Then, *in* W*, there is* $\langle y_i \mid i < \theta \rangle$ *such that, for all* $\alpha < \mu^+$ *and all sufficiently large* $i < \theta$, $y_i \in D(\alpha)_i$*.*

IV. Fat trees and pseudo-Prikry sequences



A new proof

We prove the Džamonja–Shelah result using the methods of the previous section. So, we have $V \subseteq W$, κ is inaccessible in V, κ is singular of cofinality θ in W, and $(\kappa^+)^V = (\kappa^+)^W$. We are given a sequence $\langle C_\alpha \mid \alpha < \kappa^+ \rangle \in V$ of clubs in κ. In W, we must find a sequence $\langle \gamma_i \mid i < \theta \rangle$ such that, for all $\alpha < \kappa^+$, for all sufficiently large $i < \theta$, we have $\gamma_i \in C_\alpha$.

More new proof

In V, let $\langle D_\eta \mid \eta < \kappa^+ \rangle$ re-enumerate $\langle C_\alpha \mid \alpha < \kappa^+ \rangle$ so that each C_α equals D_η for unboundedly many $\eta < \kappa^+$. Build a $<^*$-increasing sequence $\vec{f} = \langle f_\eta \mid \eta < \kappa^+ \rangle$ of elements of ${}^\kappa\kappa$ by recursion on η as follows:

- for all $\eta < \kappa^+$ and $i < \kappa$, $f_{\eta+1}(i) = \min(D_\eta \setminus (f_\eta(i)+1))$
- if $\eta < \kappa^+$ is a limit ordinal and $\mathrm{cf}(\eta) < \kappa$, then there is a club $D_\eta \subseteq \eta$ such that, for all $i < \kappa$
$$f_\eta(i) = \sup\{f_\xi(i) \mid \xi \in D_\eta\}$$
- if $\eta < \kappa^+$ and $\mathrm{cf}(\eta) = \kappa$, then f_η is any $<^*$-upper bound for $\langle f_\xi \mid \xi < \eta \rangle$.

Still more new proof

Now, in W, let $\langle \kappa_i \mid i<\theta \rangle$ be increasing and cofinal in κ, and define a sequence $\langle \hat{f}_\eta \mid \eta<\kappa^+ \rangle$ of elements of ${}^\theta\kappa$ by letting $\hat{f}_\eta(i) = f_\eta(\kappa_i)$ for all $\eta<\kappa^+$ and $i<\theta$. Then $\langle \hat{f}_\eta \mid \eta<\kappa^+ \rangle$ is $<^*$-increasing and club-increasing at all ordinals $\eta<\kappa^+$ with $cf(\gamma)>\theta$ (since all ordinals formerly of cofinality κ now have cofinality θ). This is enough to imply that $\langle \hat{f}_\eta \mid \eta<\kappa^+ \rangle$ has an e.u.b. $g \in {}^\theta On$ such that $cf(g(i))>\theta$ for all $i<\theta$. Since $cf^W(\kappa)=\theta$, it follows that $g \in {}^\theta\kappa$. Let $\gamma_i = g(i)$ for all $i<\theta$.

We claim that $\langle \gamma_i \mid i<\theta \rangle$ is as desired.

This had better be the end of the new proof

Suppose not. Then there is $\alpha < \kappa^+$ and an unbounded $A \subseteq \theta$ such that $g(i) \notin C_\alpha$ for all $i \in A$. Define $h \in {}^\theta\kappa$ by

$$h(i) = \begin{cases} \max(C_\alpha \cap g(i)) & \text{if } i \in A \\ 0 & \text{if } i \notin A. \end{cases}$$

Then $h < g$, so there is $\xi < \kappa^+$ such that $h <^* \hat{f}_\xi$. Find $\eta \geq \xi$ such that $C_\alpha = D_\eta$. Then, by our construction of $\vec{f}$, for all large enough $i < \theta$, we have $\hat{f}_\xi(i) \leq \hat{f}_\eta(i) < \min(C_\alpha \setminus (\hat{f}_\eta(i)+1)) = \hat{f}_{\eta+1}(i)$.

Since $C_\alpha \cap (h(i), g(i)]$ for all $i \in A$, it follows that $\hat{f}_{\eta+1}(i) > g(i)$ for all large enough $i \in A$, contradicting the fact that g is an e.u.b. for $\langle \hat{f}_\eta \mid \eta < \kappa^+ \rangle$

∎

Fat trees

Definition

Suppose that κ is a regular, uncountable cardinal, $n < \omega$, and, for all $m \leq n$, $\lambda_m \geq \kappa$ is a regular cardinal. Then

$$T \subseteq \bigcup_{k \leq n+1} \prod_{m < k} \lambda_m$$

is a *fat tree* of type $(\kappa, \langle \lambda_0, \ldots, \lambda_n \rangle)$ if:

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2. for all $\sigma \in T$ such that $k := \mathrm{lh}(\sigma) \leq n$, $\mathrm{succ}_T(\sigma) := \{\alpha \mid \sigma^\frown\langle\alpha\rangle \in T\}$ is $(<\kappa)$-club in λ_k.

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Suppose that κ is a regular, uncountable cardinal, $n < \omega$, and, for all $m \leq n$, $\lambda_m \geq \kappa$ is a regular cardinal. Then

$$T \subseteq \bigcup_{k \leq n+1} \prod_{m < k} \lambda_m$$

is a *fat tree* of type $(\kappa, \langle \lambda_0, \ldots, \lambda_n \rangle)$ if:

1. for all $\sigma \in T$ and $\ell < \mathrm{lh}(\sigma)$, we have $\sigma \restriction \ell \in T$;
2. for all $\sigma \in T$ such that $k := \mathrm{lh}(\sigma) \leq n$, $\mathrm{succ}_T(\sigma) := \{\alpha \mid \sigma^\frown\langle\alpha\rangle \in T\}$ is $(< \kappa)$-club in λ_k.

Lemma

If $\mathcal{C}$ is a club in $\mathcal{P}_\kappa(\kappa^{+n})$, then there is a fat tree of type $(\kappa, \langle \kappa^{+n}, \kappa^{+n-1}, \ldots, \kappa \rangle)$ such that, for every maximal $\sigma \in T$, there is $x \in \mathcal{C}$ such that, for all $m \leq n$, $\sup(x \cap \kappa^{+m}) = \sigma(n - m)$.

Outside guessing of fat trees

Theorem (LH, '18, [5])

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4. $(\lambda^+)^V$ *remains a cardinal in* W;
5. $n < \omega$ *and, in* V, $\langle \lambda_i \mid i \leq n \rangle$ *is a sequence of regular cardinals from* $[\kappa, \lambda]$ *and* $\langle T(\alpha) \mid \alpha < \lambda^+ \rangle$ *is a sequence of fat trees of type* $(\kappa, \langle \lambda_0, \ldots, \lambda_n \rangle)$.

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4. *$(\lambda^+)^V$ remains a cardinal in W;*
5. *$n < \omega$ and, in V, $\langle \lambda_i \mid i \leq n \rangle$ is a sequence of regular cardinals from $[\kappa, \lambda]$ and $\langle T(\alpha) \mid \alpha < \lambda^+ \rangle$ is a sequence of fat trees of type $(\kappa, \langle \lambda_0, \ldots, \lambda_n \rangle)$.*

Then, in W, there is a sequence $\langle \sigma_i \mid i < \theta \rangle$ such that, for all $\alpha < \lambda^+$ and all sufficiently large $i < \theta$, σ_i is a maximal element of $T(\alpha)$.

Some remarks

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Conjecture (Gitik)

Suppose that V is an inner model of W, κ is regular in V, $(\mathrm{cf}(\kappa))^W = \omega$, $(\aleph_1)^V = (\aleph_1)^W$, and V and W agree about a final segment of cardinals. Then there is an inner model $V' \subseteq V$ such that W contains a sequence that is generic over V' for Namba forcing, stationary tower forcing, or a Prikry-type forcing.

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All artwork by Victor Vasarely.

Thank you!