A new iteration scheme with applications to singular cardinals combinatorics

Alejandro Poveda

Einstein Institute of Mathematics

Prikry Forcing Online - December 14th

This is based on a joint work with A. Rinot & D. Sinapova

1. **Sigma-Prikry forcing I: The axioms**, Canadian Journal of Mathematics, to appear.
2. **Sigma-Prikry forcing II: Iteration Scheme**, Journal of Mathematical Logic, to appear.
3. **Sigma-Prikry forcing III: Down to $\aleph_\omega$**, Preprint.

Find the papers here!
http://assafrinot.com/t/sigma-prikry

The talk in a nutshell

The three main characters of the talk are:

The talk in a nutshell

The three main characters of the talk are:

1. Stationary reflection at successors of singulars (Compactness).

The talk in a nutshell

The three main characters of the talk are:

1. Stationary reflection at successors of singulars (Compactness).
2. The failure of the SCH (Incompactness).

The talk in a nutshell

The three main characters of the talk are:

1. Stationary reflection at successors of singulars (Compactness).
2. The failure of the SCH (Incompactness).
3. Prikry-type forcings and their iterations.

The talk in a nutshell

The three main characters of the talk are:

1. Stationary reflection at successors of singulars (Compactness).
2. The failure of the SCH (Incompactness).
3. Prikry-type forcings and their iterations.

Goal

Show how the latter can be used to resolve the intrinsic tension between (1) and (2).

An application

The very first application of the Σ-Prikry framework:

Theorem (P., Rinot, Sinapova) (JML-2020)

Assume that $\langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of supercompact cardinals. Then there is a generic extension where $\kappa = \sup_{n<\omega} \kappa_n$ is a strong limit cardinal, SCH_κ fails and $\mathrm{Refl}(<\omega, \kappa^+)$ holds.

An application

The very first application of the Σ-Prikry framework:

Theorem (P., Rinot, Sinapova) (JML-2020)

Assume that $\langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of supercompact cardinals. Then there is a generic extension where $\kappa = \sup_{n<\omega} \kappa_n$ is a strong limit cardinal, SCH_κ fails and $\mathrm{Refl}({<}\omega, \kappa^+)$ holds.

Independently proved by **Ben-Neria, Hayut and Unger**, and shortly after by **Gitik**. Was part of **Sharon**'s Ph.D. thesis ('05), but unfortunately the proof was incomplete.

Stationary Reflection

Compactness principles

Compactness Principle

A **Compactness Principle** for a given property φ is a statement of the form:

"If every *small* substructure has property φ then the structure has property φ, as well"

Compactness principles

Compactness Principle

A **Compactness Principle** for a given property φ is a statement of the form:

"If every *small* substructure has property φ then the structure has property φ, as well"

The dual of **CP** are **Reflection Principles**:

Reflection Principle

A **Reflection Principle** for a given property φ is a statement of the form:

"If a structure has property φ then there is a *small* substructure having property φ"

Compactness principles

Compactness Principle

A **Compactness Principle** for a given property φ is a statement of the form:

"If every *small* substructure has property φ then the structure has property φ, as well"

The dual of **CP** are **Reflection Principles**:

Reflection Principle

A **Reflection Principle** for a given property φ is a statement of the form:

"If a structure has property φ then there is a *small* substructure having property φ"

In practice, ***small*** means "having cardinality $<\kappa$", where κ is some relevant cardinal

Compactness in Logic

Definition

A set of sentences Γ is called κ-satisfiable, if every $T \in [\Gamma]^{<\kappa}$ is satisfiable.

Compactness in Logic

Definition

A set of sentences Γ is called κ-satisfiable, if every $T \in [\Gamma]^{<\kappa}$ is satisfiable.

Theorem (Compactness of First Order Logic)

Any collection $\mathcal{L}_{\omega,\omega}$-sentences which is $\aleph_0$-satisfiable, is satisfiable.

Compactness in Logic

Definition

A set of sentences Γ is called κ-satisfiable, if every $T \in [\Gamma]^{<\kappa}$ is satisfiable.

Theorem (Compactness of First Order Logic)

Any collection $\mathcal{L}_{\omega,\omega}$-sentences which is $\aleph_0$-satisfiable, is satisfiable.

Question

For which cardinals $\kappa \geq \aleph_1$, the logics $\mathcal{L}_{\kappa,\kappa}$ (or $\mathcal{L}^n_{\kappa,\kappa}$) are κ-compact?

Compactness in Logic

Definition

A set of sentences Γ is called κ-satisfiable, if every $T \in [\Gamma]^{<\kappa}$ is satisfiable.

Theorem (Compactness of First Order Logic)

Any collection $\mathcal{L}_{\omega,\omega}$-sentences which is $\aleph_0$-satisfiable, is satisfiable.

Question

For which cardinals $\kappa \geq \aleph_1$, the logics $\mathcal{L}_{\kappa,\kappa}$ (or $\mathcal{L}^n_{\kappa,\kappa}$) are κ-compact?

Theorem (Tarski/Magidor)

The following are equivalent:

1. $\mathcal{L}_{\kappa,\kappa}$ (resp. $\mathcal{L}^2_{\kappa,\kappa}$) is κ-compact.
2. κ is a strongly compact (extendible).

Compactness in Algebra

Definition

Let $\kappa \geq \aleph_0$. An abelian group is called κ-free if all its subgroups of size $<\kappa$ are free.

Compactness in Algebra

Definition

Let $\kappa \geq \aleph_0$. An abelian group is called κ-free if all its subgroups of size $<\kappa$ are free.

Question

For what cardinals κ every κ-free abelian group of cardinality κ is also free?

Compactness in Algebra

Definition

Let $\kappa \geq \aleph_0$. An abelian group is called κ-free if all its subgroups of size $<\kappa$ are free.

Question

For what cardinals κ every κ-free abelian group of cardinality κ is also free?

Shelah's Compactness Theorem

If κ is a **singular** cardinal then every κ-free abelian group of size κ is free

Compactness in Set Theory

Definition

Let κ be a regular uncountable cardinal.

1. A set $C \subseteq \kappa$ is called a **club** if it is closed and unbounded.
2. A set $S \subseteq \kappa$ is called **stationary** if $S \cap C \neq \emptyset$, for every club $C \subseteq \kappa$.

Compactness in Set Theory

Definition

Let κ be a regular uncountable cardinal.

1. A set $C \subseteq \kappa$ is called a **club** if it is closed and unbounded.
2. A set $S \subseteq \kappa$ is called **stationary** if $S \cap C \neq \emptyset$, for every club $C \subseteq \kappa$.

Fact: For every club $C \subseteq \kappa$ there is a club $D \subseteq C$ such that for each $\alpha \in D$, $C \cap \alpha$ is a club.

Definition

Let κ be a regular uncountable cardinal.

1. A set $C \subseteq \kappa$ is called a **club** if it is closed and unbounded.
2. A set $S \subseteq \kappa$ is called **stationary** if $S \cap C \neq \emptyset$, for every club $C \subseteq \kappa$.

Fact: For every club $C \subseteq \kappa$ there is a club $D \subseteq C$ such that for each $\alpha \in D$, $C \cap \alpha$ is a club.

Informally speaking, club sets **reflect**.

Definition

Let κ be a regular uncountable cardinal.

1. A set $C \subseteq \kappa$ is called a **club** if it is closed and unbounded.
2. A set $S \subseteq \kappa$ is called **stationary** if $S \cap C \neq \emptyset$, for every club $C \subseteq \kappa$.

Fact: For every club $C \subseteq \kappa$ there is a club $D \subseteq C$ such that for each $\alpha \in D$, $C \cap \alpha$ is a club.

Informally speaking, club sets **reflect**.

Question (naive version):

Do stationary sets *reflect*?

Compactness in Set Theory: Stationary reflection

Definition

1. A stationary set $S \subseteq \kappa$ **reflects** if there is $\alpha < \kappa$ with $\mathrm{cf}(\alpha) > \aleph_0$ such that $S \cap \alpha$ is stationary in α.

Compactness in Set Theory: Stationary reflection

Definition

1. A stationary set $S \subseteq \kappa$ **reflects** if there is $\alpha < \kappa$ with $\mathrm{cf}(\alpha) > \aleph_0$ such that $S \cap \alpha$ is stationary in α.
2. For a given stationary set $S \subseteq \kappa$, the principle $\mathrm{Refl}(S)$ asserts that every stationary subset $T \subseteq S$ reflects.

Compactness in Set Theory: Stationary reflection

Definition

1. A stationary set $S \subseteq \kappa$ **reflects** if there is $\alpha < \kappa$ with $\mathrm{cf}(\alpha) > \aleph_0$ such that $S \cap \alpha$ is stationary in α.
2. For a given stationary set $S \subseteq \kappa$, the principle $\mathrm{Refl}(S)$ asserts that every stationary subset $T \subseteq S$ reflects.

Question

For which cardinals κ and stationaries $S \subseteq \kappa$ does $\mathrm{Refl}(S)$ hold?

Compactness in Set Theory: Stationary reflection

Definition

1. A stationary set $S \subseteq \kappa$ **reflects** if there is $\alpha < \kappa$ with $\mathrm{cf}(\alpha) > \aleph_0$ such that $S \cap \alpha$ is stationary in α.
2. For a given stationary set $S \subseteq \kappa$, the principle $\mathrm{Refl}(S)$ asserts that every stationary subset $T \subseteq S$ reflects.

Question

For which cardinals κ and stationaries $S \subseteq \kappa$ does $\mathrm{Refl}(S)$ hold?

We need to separate the discussion into three cases:

1. **Limit cardinals:**

Theorem (Tarski (?), Jensen (1972))

If κ is weakly compact then $\mathrm{Refl}(\kappa)$ holds. Under $V = L$, this is an equivalence.

1. **Limit cardinals:**

Theorem (Tarski (?), Jensen (1972))

If κ is weakly compact then $\mathrm{Refl}(\kappa)$ holds. Under $V = L$, this is an equivalence.

2. **Successor of a regular:** Set $\kappa := \lambda^+$, with $\lambda = \mathrm{cf}(\lambda)$.

1. **Limit cardinals:**

Theorem (Tarski (?), Jensen (1972))

If κ is weakly compact then $\mathrm{Refl}(\kappa)$ holds. Under $V = L$, this is an equivalence.

2. **Successor of a regular:** Set $\kappa := \lambda^+$, with $\lambda = \mathrm{cf}(\lambda)$.

$\mathrm{Refl}(\kappa)$ fails

Actually $\mathrm{Refl}(E^\kappa_\lambda)$ fails, where $E^\kappa_\lambda = \{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \lambda\}$.

1. **Limit cardinals:**

Theorem (Tarski (?), Jensen (1972))

If κ is weakly compact then $\mathrm{Refl}(\kappa)$ holds. Under $V = L$, this is an equivalence.

2. **Successor of a regular:** Set $\kappa := \lambda^+$, with $\lambda = \mathrm{cf}(\lambda)$.

$\mathrm{Refl}(\kappa)$ fails

Actually $\mathrm{Refl}(E^\kappa_\lambda)$ fails, where $E^\kappa_\lambda = \{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \lambda\}$. Indeed, let $\alpha < \kappa$ be with $\mathrm{cf}(\alpha) > \aleph_0$ and $C \subseteq \alpha$ be a club of points of cofinality $<\lambda$. Clearly, $C \cap E^\kappa_\lambda = \emptyset$, and so $E^\kappa_\lambda \cap \alpha$ is not stationary.

1. **Limit cardinals:**

Theorem (Tarski (?), Jensen (1972))

If κ is weakly compact then $\mathrm{Refl}(\kappa)$ holds. Under $V = L$, this is an equivalence.

2. **Successor of a regular:** Set $\kappa := \lambda^+$, with $\lambda = \mathrm{cf}(\lambda)$.

$\mathrm{Refl}(\kappa)$ fails

Actually $\mathrm{Refl}(E^\kappa_\lambda)$ fails, where $E^\kappa_\lambda = \{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \lambda\}$. Indeed, let $\alpha < \kappa$ be with $\mathrm{cf}(\alpha) > \aleph_0$ and $C \subseteq \alpha$ be a club of points of cofinality $<\lambda$. Clearly, $C \cap E^\kappa_\lambda = \emptyset$, and so $E^\kappa_\lambda \cap \alpha$ is not stationary.

Despite of this we can still obtain an optimal reflection pattern:

Theorem (Harrington & Shelah) (NDJFL - 1985)

The following are equiconsistent:

- There is a Mahlo cardinal.
- $\mathrm{Refl}(E^\kappa_{<\lambda})$ holds.

3. **Successors of a singular:**

Unlike of successors of regulars now one can arrange full reflection:

Theorem (Magidor) (JSL–1982)

Assume there are ω-many supercompact cardinals and that the GCH holds. Then there is a generic extension where $\mathrm{Refl}(\aleph_{\omega+1})$ holds.

3. **Successors of a singular:**

Unlike of successors of regulars now one can arrange full reflection:

Theorem (Magidor) (JSL–1982)

Assume there are ω-many supercompact cardinals and that the GCH holds. Then there is a generic extension where $\mathrm{Refl}(\aleph_{\omega+1})$ holds.

This strong large-cardinal assumptions do not appear by chance.

The other side of the coin: square principles

Definition (Jensen)

Let κ be an infinite cardinal. A sequence $\langle C_\alpha \mid \alpha < \kappa^+ \rangle$ is called a $\square_\kappa$-sequence if the following are true for each $\alpha < \kappa^+$:

1. $C_\alpha \subseteq \alpha$ is a club set;
2. if $\mathrm{cf}(\alpha) < \kappa$ then $\mathrm{otp}(C_\alpha) < \kappa$;
3. for all $\beta \in \lim(C_\alpha)$, $C_\alpha \cap \beta = C_\beta$.

We say that $\square_\kappa$ holds if there is a $\square_\kappa$-sequence.

The other side of the coin: square principles

Definition (Jensen)

Let κ be an infinite cardinal. A sequence $\langle C_\alpha \mid \alpha < \kappa^+ \rangle$ is called a $\Box_\kappa$-sequence if the following are true for each $\alpha < \kappa^+$:

1. $C_\alpha \subseteq \alpha$ is a club set;
2. if $\mathrm{cf}(\alpha) < \kappa$ then $\mathrm{otp}(C_\alpha) < \kappa$;
3. for all $\beta \in \lim(C_\alpha)$, $C_\alpha \cap \beta = C_\beta$.

We say that $\Box_\kappa$ holds if there is a $\Box_\kappa$-sequence.

$\Box$-sequences are prototypical manifestations of incompactness

If $\Box_\kappa$ holds then there is no club $C \subseteq \kappa^+$ *threading* $\langle C_\alpha \mid \alpha < \kappa^+ \rangle$. In other words, there is no club set $C \subseteq \kappa^+$ that may continue the $\Box_\kappa$-sequence.

1. $\square_\kappa$ is **incompatible** with $\mathrm{Refl}(\kappa^+)$. Specifically, if $\square_\kappa$ holds then $\mathrm{Refl}(S)$ fails, for every stationary set $S \subseteq \kappa^+$.

1. $\Box_\kappa$ is **incompatible** with $\mathrm{Refl}(\kappa^+)$. Specifically, if $\Box_\kappa$ holds then $\mathrm{Refl}(S)$ fails, for every stationary set $S \subseteq \kappa^+$.
2. Avoiding $\Box_\kappa$ is **hard and costly**, and thus so is getting $\mathrm{Refl}(\kappa^+)$:

1. $\square_\kappa$ is **incompatible** with $\mathrm{Refl}(\kappa^+)$. Specifically, if $\square_\kappa$ holds then $\mathrm{Refl}(S)$ fails, for every stationary set $S \subseteq \kappa^+$.
2. Avoiding $\square_\kappa$ is **hard and costly**, and thus so is getting $\mathrm{Refl}(\kappa^+)$:

   (ℵ) **Why is it hard?**

   - If W is L-like, then $W \models$ "$\forall\kappa \geq \aleph_0\ \square_\kappa$".
   - If W is L-like and W *resembles sufficiently* V, then $\square_\kappa$ holds.

1. $\square_\kappa$ is **incompatible** with $\mathrm{Refl}(\kappa^+)$. Specifically, if $\square_\kappa$ holds then $\mathrm{Refl}(S)$ fails, for every stationary set $S \subseteq \kappa^+$.
2. Avoiding $\square_\kappa$ is **hard and costly**, and thus so is getting $\mathrm{Refl}(\kappa^+)$:

   (א) **Why is it hard?**
   - If W is L-like, then $W \models$ "$\forall\kappa \geq \aleph_0\ \square_\kappa$".
   - If W is L-like and W *resembles sufficiently* V, then $\square_\kappa$ holds.

   (ב) **Why is it costly?** The failure of square yields inner models with large cardinals (e.g., if $\square_{\omega_1}$ fails then $\aleph_2$ is Mahlo in L)

1. $\square_\kappa$ is **incompatible** with $\mathrm{Refl}(\kappa^+)$. Specifically, if $\square_\kappa$ holds then $\mathrm{Refl}(S)$ fails, for every stationary set $S \subseteq \kappa^+$.
2. Avoiding $\square_\kappa$ is **hard and costly**, and thus so is getting $\mathrm{Refl}(\kappa^+)$:
   - (א) **Why is it hard?**
     - If W is L-like, then $W \models$ "$\forall \kappa \geq \aleph_0 \, \square_\kappa$".
     - If W is L-like and W *resembles sufficiently* V, then $\square_\kappa$ holds.
   - (ב) **Why is it costly?** The failure of square yields inner models with large cardinals

     (e.g., if $\square_{\omega_1}$ fails then $\aleph_2$ is Mahlo in L)

     This is even more dramatic when the relevant cardinal is singular:

     (e.g., if $\square_\kappa$ fails then there is an inner model with ∞-many Woodin cardinals)

1. $\square_\kappa$ is **incompatible** with $\mathrm{Refl}(\kappa^+)$. Specifically, if $\square_\kappa$ holds then $\mathrm{Refl}(S)$ fails, for every stationary set $S \subseteq \kappa^+$.
2. Avoiding $\square_\kappa$ is **hard and costly**, and thus so is getting $\mathrm{Refl}(\kappa^+)$:

   (א) **Why is it hard?**

   - If W is L-like, then $W \models$ "$\forall\kappa \geq \aleph_0\ \square_\kappa$".
   - If W is L-like and W *resembles sufficiently* V, then $\square_\kappa$ holds.

   (ב) **Why is it costly?** The failure of square yields inner models with large cardinals

   (e.g., if $\square_{\omega_1}$ fails then $\aleph_2$ is Mahlo in L)

   This is even more dramatic when the relevant cardinal is singular:

   (e.g., if $\square_\kappa$ fails then there is an inner model with ∞-many Woodin cardinals)

Arranging $\mathrm{Refl}(\kappa^+)$ is always **hard** and **costly**.
Specially if κ is singular.

The Singular Cardinal Hypothesis

The behaviour of the continuum function

While the behaviour of the continuum function is almost arbitrary at regular cardinals,

Theorem (Easton)

Assume the GCH holds. For every pair of regular cardinals $\kappa < \lambda$ there is a generic extension where $\mathsf{GCH}_{<\kappa}$ holds and $2^\kappa = \lambda$

The behaviour of the continuum function

While the behaviour of the continuum function is almost arbitrary at regular cardinals,

Theorem (Easton)

Assume the GCH holds. For every pair of regular cardinals $\kappa < \lambda$ there is a generic extension where $\mathrm{GCH}_{<\kappa}$ holds and $2^\kappa = \lambda$

The situation is much more restrictive for singulars:

The behaviour of the continuum function

While the behaviour of the continuum function is almost arbitrary at regular cardinals,

Theorem (Easton)

Assume the GCH holds. For every pair of regular cardinals $\kappa < \lambda$ there is a generic extension where $\mathrm{GCH}_{<\kappa}$ holds and $2^\kappa = \lambda$

The situation is much more restrictive for singulars:

Theorem (Silver) - Silver's compactness theorem

For every singular κ of **uncountable cofinality** if $\mathrm{GCH}_{<\kappa}$ holds then GCH_κ also does.

The behaviour of the continuum function

While the behaviour of the continuum function is almost arbitrary at regular cardinals,

Theorem (Easton)

Assume the GCH holds. For every pair of regular cardinals $\kappa < \lambda$ there is a generic extension where $\mathrm{GCH}_{<\kappa}$ holds and $2^\kappa = \lambda$

The situation is much more restrictive for singulars:

Theorem (Silver) - Silver's compactness theorem

For every singular κ of **uncountable cofinality** if $\mathrm{GCH}_{<\kappa}$ holds then GCH_κ also does.

Question

Does Silver's theorem extends for singular cardinals of **countable cofinality**?

The Singular Cardinal Hypothesis

Definition (simplified version)

The **S**ingular **C**ardinal **H**ypothesis (SCH) is the assertion that for every **singular strong limit cardinal** κ, $2^\kappa = \kappa^+$ (i.e., SCH_κ holds).

The Singular Cardinal Hypothesis

Definition (simplified version)

The **S**ingular **C**ardinal **H**ypothesis (SCH) is the assertion that for every **singular strong limit cardinal** κ, $2^\kappa = \kappa^+$ (i.e., SCH_κ holds).

Question

1. Is the failure of the SCH consistent with ZFC?

The Singular Cardinal Hypothesis

Definition (simplified version)

The **S**ingular **C**ardinal **H**ypothesis (SCH) is the assertion that for every **singular strong limit cardinal** κ, $2^\kappa = \kappa^+$ (i.e., SCH_κ holds).

Question

1. Is the failure of the SCH consistent with ZFC? (✓)

- $\neg$SCH is consistent modulo the existence of a κ^{++}-supercompact cardinal κ.
  (Silver & Prikry)

The Singular Cardinal Hypothesis

Definition (simplified version)

The **S**ingular **C**ardinal **H**ypothesis (SCH) is the assertion that for every **singular strong limit cardinal** κ, $2^\kappa = \kappa^+$ (i.e., SCH_κ holds).

Question

1. Is the failure of the SCH consistent with ZFC? (✓)
2. Which is (consistently) the first witness for ¬SCH?

- ¬SCH is consistent modulo the existence of a κ^{++}-supercompact cardinal κ.
  (Silver & Prikry)
- By Silver's it cannot be $\aleph_{\omega_1}$, but could it be $\aleph_\omega$?

The Singular Cardinal Hypothesis

Definition (simplified version)

The **S**ingular **C**ardinal **H**ypothesis (SCH) is the assertion that for every **singular strong limit cardinal** κ, $2^\kappa = \kappa^+$ (i.e., SCH_κ holds).

Question

1. Is the failure of the SCH consistent with ZFC? (✓)
2. Which is (consistently) the first witness for ¬SCH?

- ¬SCH is consistent modulo the existence of a κ^{++}-supercompact cardinal κ. (Silver & Prikry)
- By Silver's it cannot be $\aleph_{\omega_1}$, but could it be $\aleph_\omega$?

Yes

Theorem (Magidor) (Ann. Math –1977)

Assume there is a supercompact cardinal along with a huge cardinal on top. Then there is a generic extension where $\mathrm{GCH}_{<\aleph_\omega}$ holds but $\mathrm{SCH}_{\aleph_\omega}$ fails.

Theorem (Magidor) (Ann. Math –1977)

Assume there is a supercompact cardinal along with a huge cardinal on top. Then there is a generic extension where $\mathrm{GCH}_{<\aleph_\omega}$ holds but $\mathrm{SCH}_{\aleph_\omega}$ fails.

Theorem (Gitik, Woodin) (Optimal assumptions)

If there exists a measurable cardinal κ with Mitchell order κ^{++}, then there is a generic extension where $\mathrm{GCH}_{<\aleph_\omega}$ holds but $\mathrm{SCH}_{\aleph_\omega}$ fails.

There is tension between $\neg\mathsf{SCH}_\kappa$ and $\mathrm{Refl}(\kappa^+)$

There is tension between $\neg\mathsf{SCH}_\kappa$ and $\mathrm{Refl}(\kappa^+)$

- Getting $\neg$SCH usually involves singularizing cardinals;

There is tension between $\neg\mathsf{SCH}_\kappa$ and $\mathrm{Refl}(\kappa^+)$

- Getting $\neg$SCH usually involves singularizing cardinals;
- Singularizing typically yields weak forms of square, which are at odds with reflection.

Theorem (Gitik - Džamonja & Shelah)

Suppose $V \subseteq W$ are two inner models where a cardinal κ is a V-inaccessible but W-singular with $\mathrm{cf}^W(\kappa) = \omega$. If moreover $(\kappa^+)^V = (\kappa^+)^W$ then $W \models \square_{\kappa,\omega}$.

There is tension between $\neg\mathsf{SCH}_\kappa$ and $\mathrm{Refl}(\kappa^+)$

- Getting $\neg$SCH usually involves singularizing cardinals;
- Singularizing typically yields weak forms of square, which are at odds with reflection.

Theorem (Gitik - Džamonja & Shelah)

Suppose $V \subseteq W$ are two inner models where a cardinal κ is a V-inaccessible but W-singular with $\mathrm{cf}^W(\kappa) = \omega$. If moreover $(\kappa^+)^V = (\kappa^+)^W$ then $W \models \Box_{\kappa,\omega}$.

It is both **hard** and **costly** to arrange $\mathrm{Refl}(\kappa^+)$ along with $\neg\mathsf{SCH}_\kappa$.

Σ-Prikry forcings and their iterations

"Is hard to manipulate the combinatorics of singulars", yet again

There are essentially two sorts of obstacles when dealing with singular cardinals:

“Is hard to manipulate the combinatorics of singulars”, yet again

There are essentially two sorts of obstacles when dealing with singular cardinals:

1. **Foundational:** Almost any manipulation requires very large-cardinals.

   e.g., $\neg\mathsf{SCH}_\kappa$ requires $o(\kappa)=\kappa^{++}$ (Gitik & Woodin)
   and $\neg\square_\kappa$ implies $\mathrm{AD}^{L(\mathbb{R})}$ (Steel)

“Is hard to manipulate the combinatorics of singulars”, yet again

There are essentially two sorts of obstacles when dealing with singular cardinals:

1. **Foundational:** Almost any manipulation requires very large-cardinals.

   e.g., $\neg$SCH$_\kappa$ requires $o(\kappa)=\kappa^{++}$ (Gitik & Woodin)
   and $\neg\square_\kappa$ implies $\mathrm{AD}^{L(\mathbb{R})}$ (Steel)

2. **Technical:** Lack of iteration theorems at the level of singular cardinals.

“Is hard to manipulate the combinatorics of singulars”, yet again

There are essentially two sorts of obstacles when dealing with singular cardinals:

1. **Foundational:** Almost any manipulation requires very large-cardinals.

   e.g., $\neg\mathsf{SCH}_\kappa$ requires $o(\kappa) = \kappa^{++}$ (Gitik & Woodin)
   and $\neg\square_\kappa$ implies $\mathrm{AD}^{L(\mathbb{R})}$ (Steel)

2. **Technical:** Lack of iteration theorems at the level of singular cardinals.

Iteration Theorem

Every κ^{++}-length and κ-supported iteration of κ^{++}-cc forcing is again κ^{++}-cc.

“Is hard to manipulate the combinatorics of singulars”, yet again

There are essentially two sorts of obstacles when dealing with singular cardinals:

1. **Foundational:** Almost any manipulation requires very large-cardinals.

   e.g., $\neg$SCH$_\kappa$ requires $o(\kappa) = \kappa^{++}$ (Gitik & Woodin)
   and $\neg\square_\kappa$ implies $\mathrm{AD}^{L(\mathbb{R})}$ (Steel)

2. **Technical:** Lack of iteration theorems at the level of singular cardinals.

Iteration Theorem

Every κ^{++}-length and κ-supported iteration of κ^{++}-cc forcing is again κ^{++}-cc.

Goal

Prove an iteration theorem for singular cardinals and apply it to combine $\mathrm{Refl}(\kappa^+)$ with $\neg$SCH$_\kappa$.

Iteration Theorem

Every κ^{++}-length and κ-supported iteration of κ^{++}-cc forcing has the κ^{++}-cc.

Obstacle

Some additional properties over the forcings are required. A crucial one is κ-**closedness**, which is not prevalent enough when κ is singular.

Iteration Theorem

Every κ^{++}-length and κ-supported iteration of κ^{++}-cc forcing has the κ^{++}-cc.

Obstacle

Some additional properties over the forcings are required. A crucial one is κ-**closedness**, which is not prevalent enough when κ is singular.

Shooting a club trough a stationary subset

Let κ with $\mathrm{cf}(\kappa) = \omega$ and $S \subseteq E_{\omega}^{\kappa^+}$ stationary. Then the typical forcing using bounded closed sets is not even $\aleph_1$-closed!

Iteration Theorem

Every κ^{++}-length and κ-supported iteration of κ^{++}-cc forcing has the κ^{++}-cc.

Obstacle

Some additional properties over the forcings are required. A crucial one is κ-**closedness**, which is not prevalent enough when κ is singular.

Shooting a club trough a stationary subset

Let κ with $\mathrm{cf}(\kappa) = \omega$ and $S \subseteq E^{\kappa^+}_{\omega}$ stationary. Then the typical forcing using bounded closed sets is not even $\aleph_1$-closed!

Shortage of iteration theorems when κ singular

Question

Is there any hope to succeed without κ-closedness?

Question

Is there any hope to succeed without κ-closedness?

An alternative: The Prikry workaround

Look at forcings $\mathbb{P}$ which have the Prikry property and are "layered-closed":

1. $\mathbb{P}$ can be written as $\bigcup_{n<\omega} \mathbb{P}_n$, according to some reasonable notion of length.
2. The layers $\mathbb{P}_n$ are "eventually as closed as necessary".

Question

Is there any hope to succeed without κ-closedness?

An alternative: The Prikry workaround

Look at forcings $\mathbb{P}$ which have the Prikry property and are “layered-closed":

1. $\mathbb{P}$ can be written as $\bigcup_{n<\omega} \mathbb{P}_n$, according to some reasonable notion of length.
2. The layers $\mathbb{P}_n$ are “eventually as closed as necessary”.

Example: Prikry forcing

Let $\mathbb{P}$ be Prikry forcing. Then,

1. $\mathbb{P} = \bigcup_{n<\omega} \mathbb{P}_n$, where $\mathbb{P}_n := \{(s, A) \mid (s, A) \in P, |s| = n\}$;
2. $\mathbb{P}_n$ is κ-directed closed.

Question

Is there any hope to succeed without κ-closedness?

An alternative: The Prikry workaround

Look at forcings $\mathbb{P}$ which have the Prikry property and are “layered-closed":

1. $\mathbb{P}$ can be written as $\bigcup_{n<\omega} \mathbb{P}_n$, according to some reasonable notion of length.
2. The layers $\mathbb{P}_n$ are “eventually as closed as necessary”.

Example: Prikry forcing

Let $\mathbb{P}$ be Prikry forcing. Then,

1. $\mathbb{P} = \bigcup_{n<\omega} \mathbb{P}_n$, where $\mathbb{P}_n := \{(s, A) \mid (s, A) \in P, |s| = n\}$;
2. $\mathbb{P}_n$ is κ-directed closed.

Revised Strategy

Find an iteration theorem for κ^{++}-length and κ-supported iterations of κ^{++}-cc **Prikry-type forcings**.

Iteration schemes for Prikry-type forcings already exist (**Magidor and Gitik iterations**) and they have been shown to be very successful. But, they **seem to be useful to change the universe below a given cardinal.**

(e.g., Magidor's proof of the Identity crises phenomenon)

Iteration schemes for Prikry-type forcings already exist (**Magidor and Gitik iterations**) and they have been shown to be very successful. But, they **seem to be useful to change the universe below a given cardinal.**

(e.g., Magidor's proof of the Identity crises phenomenon)

As a result, two crucial features of these iterations are:

1. The chain condition of the iterates grows progressively.
2. The degree "layer-closedness" of the iterates increases along the iteration.

Iteration schemes for Prikry-type forcings already exist (**Magidor and Gitik iterations**) and they have been shown to be very successful. But, they **seem to be useful to change the universe below a given cardinal.**

(e.g., Magidor's proof of the Identity crises phenomenon)

As a result, two crucial features of these iterations are:

1. The chain condition of the iterates grows progressively.

   We want to keep the same chain condition

2. The degree "layer-closedness" of the iterates increases along the iteration.

   We want to maintain the degree of "layered-closedness"

Iteration schemes for Prikry-type forcings already exist (**Magidor and Gitik iterations**) and they have been shown to be very successful. But, they **seem to be useful to change the universe below a given cardinal.**

(e.g., Magidor's proof of the Identity crises phenomenon)

As a result, two crucial features of these iterations are:

1. The chain condition of the iterates grows progressively.

   We want to keep the same chain condition

2. The degree "layer-closedness" of the iterates increases along the iteration.

   We want to maintain the degree of "layered-closedness"

Comparing both schemata

Magidor & Gitik iterations $\cong$ Easton-style iteration to force $\neg \mathrm{GCH}_\kappa$ at a supercompact κ

Our iterations $\cong$ Forcing iteration to obtain $\mathrm{FA}_{2^{\kappa^+}}(\Gamma)$, for κ singular

Σ-Prikry forcings in a nutshell

$\Sigma = \langle \kappa_n \mid n < \omega \rangle$ is non-decreasing seq. of regular uncountable cardinals. Set $\kappa := \sup(\Sigma)$.

A Σ-Prikry poset is a triple $(\mathbb{P}, \ell, c)$ such that:

Σ-Prikry forcings in a nutshell

$\Sigma = \langle \kappa_n \mid n < \omega \rangle$ is non-decreasing seq. of regular uncountable cardinals. Set $\kappa := \sup(\Sigma)$.

A Σ-Prikry poset is a triple $(\mathbb{P}, \ell, c)$ such that:

1. $\ell \colon P \to \omega$ is a "canonical notion of length";

Σ-Prikry forcings in a nutshell

$\Sigma = \langle \kappa_n \mid n < \omega \rangle$ is non-decreasing seq. of regular uncountable cardinals. Set $\kappa := \sup(\Sigma)$.

A Σ-Prikry poset is a triple $(\mathbb{P}, \ell, c)$ such that:

1. $\ell\colon P \to \omega$ is a "canonical notion of length";
2. $c\colon P \to \mu$ witnesses a **strong form of μ^+-Linkness**, where $\mathbb{1} \Vdash_{\mathbb{P}} \check{\mu} = \check{\kappa}^+$.

$$c(p) = c(q) \implies P_0^p \cap P_0^q \neq \emptyset.$$

Σ-Prikry forcings in a nutshell

$\Sigma = \langle \kappa_n \mid n < \omega \rangle$ is non-decreasing seq. of regular uncountable cardinals. Set $\kappa := \sup(\Sigma)$.

A Σ-Prikry poset is a triple $(\mathbb{P}, \ell, c)$ such that:

1. $\ell \colon P \to \omega$ is a "canonical notion of length";
2. $c \colon P \to \mu$ witnesses a **strong form of μ^+-Linkness**, where $\mathbb{1} \Vdash_{\mathbb{P}} \check{\mu} = \check{\kappa}^+$.
$$c(p) = c(q) \Longrightarrow P_0^p \cap P_0^q \neq \emptyset.$$
3. $\mathbb{P}$ is a forcing poset such that:
   - For each $n < \omega$, $\mathbb{P}_n$ is κ_n-directed-closed;
   - $\mathbb{P}$ has the **Complete Prikry Property**.

Σ-Prikry forcings in a nutshell

$\Sigma = \langle \kappa_n \mid n < \omega \rangle$ is non-decreasing seq. of regular uncountable cardinals. Set $\kappa := \sup(\Sigma)$.

A Σ-Prikry poset is a triple $(\mathbb{P}, \ell, c)$ such that:

1. $\ell \colon P \to \omega$ is a "canonical notion of length";
2. $c \colon P \to \mu$ witnesses a **strong form of μ^+-Linkness**, where $\mathbb{1} \Vdash_{\mathbb{P}} \check{\mu} = \check{\kappa}^+$.
$$c(p) = c(q) \Longrightarrow P_0^p \cap P_0^q \neq \emptyset.$$
3. $\mathbb{P}$ is a forcing poset such that:
    - For each $n < \omega$, $\mathbb{P}_n$ is κ_n-directed-closed;
    - $\mathbb{P}$ has the **Complete Prikry Property**.

Prikry forcing

$\Sigma := \langle \kappa \rangle$, $\ell(s, A) := |s|$, $c(s, A) := s$, $\mu = (\kappa^+)^V$.

Σ-Prikry forcings in a nutshell

$\Sigma = \langle \kappa_n \mid n < \omega \rangle$ is non-decreasing seq. of regular uncountable cardinals. Set $\kappa := \sup(\Sigma)$.

A Σ-Prikry poset is a triple $(\mathbb{P}, \ell, c)$ such that:

1. $\ell \colon P \to \omega$ is a "canonical notion of length";
2. $c \colon P \to \mu$ witnesses a **strong form of μ^+-Linkness**, where $\mathbb{1} \Vdash_{\mathbb{P}} \check{\mu} = \check{\kappa}^+$.
$$c(p) = c(q) \Longrightarrow P_0^p \cap P_0^q \neq \emptyset.$$
3. $\mathbb{P}$ is a forcing poset such that:
    - For each $n < \omega$, $\mathbb{P}_n$ is κ_n-directed-closed;
    - $\mathbb{P}$ has the **Complete Prikry Property**.

Prikry forcing

$\Sigma := \langle \kappa \rangle$, $\ell(s, A) := |s|$, $c(s, A) := s$, $\mu = (\kappa^+)^V$.

The class of Σ-Prikry forcing is quite broad

(e.g., Gitik-Sharon, **Extender Based Prikry**, etc).

Towards a model of $\neg\mathsf{SCH}_\kappa + \mathrm{Refl}(<\omega, \kappa^+)$

Set up

1. Let $\Sigma := \langle \kappa_n \mid n < \omega \rangle$ be strictly increasing, where each κ_n is Laver indestructible supercompact. Set $\kappa := \sup(\Sigma)$;
2. Let $\mathbb{P}$ be the Extender-Based Prikry forcing with respect to $\mathcal{E} = \langle E_n \mid n < \omega \rangle$, where E_n is a $(\kappa_n, \kappa^{++} + 1)$-extender;
3. Assuming $2^{2^\kappa} = \kappa^{++}$, we fix a bookkeeping function $\psi : \kappa^{++} \to H_{\kappa^{++}}$.

The first step: Which stationary sets reflects?

Proposition (P., Rinot & Sinapova - (2020))

Let $\mathbb{Q}$ be a Σ-Prikry forcing not collapsing κ^+. Then $V^{\mathbb{Q}} \models \mathrm{Refl}(<\omega, \kappa^+ \cap \mathrm{cf}^V(> \omega))$.

The first step: Which stationary sets reflects?

Proposition (P., Rinot & Sinapova - (2020))

Let $\mathbb{Q}$ be a Σ-Prikry forcing not collapsing κ^+. Then $V^{\mathbb{Q}} \models \mathrm{Refl}({<}\omega, \kappa^+ \cap \mathrm{cf}^V(>\omega))$.

Strategy

Define a forcing iteration $\mathbb{P}_{\kappa^{++}}$ such that

1. $\mathbb{P}_{\kappa^{++}}$ is Σ-Prikry and does not collapse κ^+,
2. $V^{\mathbb{P}_{\kappa^{++}}} \models \mathrm{Refl}(\kappa^+ \cap \mathrm{cf}^V(\omega))$,
3. $\mathbb{P}_{\kappa^{++}}$ projects to $\mathbb{P}$.

Provided $\mathbb{P}_{\kappa^{++}}$ fulfills the above conditions it yields the desired generic extension.

Iterating Σ-Prikry forcings

The slogan of our iterations

Let $\mathbb{Q}$ be a Σ-Prikry forcing and a problem $\sigma \in V^{\mathbb{Q}}$. We want a Σ-Prikry forcing $\mathbb{A}$ that projects onto $\mathbb{Q}$ and settles the problem raised by σ.

Iterating Σ-Prikry forcings

The slogan of our iterations

Let $\mathbb{Q}$ be a Σ-Prikry forcing and a problem $\sigma \in V^{\mathbb{Q}}$. We want a Σ-Prikry forcing $\mathbb{A}$ that projects onto $\mathbb{Q}$ and settles the problem raised by σ.

The above is achieved by invoking a solving-problem functor $\mathbb{A}(\cdot, \cdot)$ such that

0. $\mathbb{A} := \mathbb{A}(\mathbb{Q}, \sigma)$, “solves the problem raised by σ”

Iterating Σ-Prikry forcings

The slogan of our iterations

Let $\mathbb{Q}$ be a Σ-Prikry forcing and a problem $\sigma \in V^{\mathbb{Q}}$. We want a Σ-Prikry forcing $\mathbb{A}$ that projects onto $\mathbb{Q}$ and settles the problem raised by σ.

The above is achieved by invoking a solving-problem functor $\mathbb{A}(\cdot,\cdot)$ such that

0. $\mathbb{A} := \mathbb{A}(\mathbb{Q},\sigma)$, "solves the problem raised by σ"

and for which there are maps $(\pi, \pitchfork)$ such that:

1. **There is a projection** π between $\mathbb{A}$ and $\mathbb{Q}$
2. **There is a canonical operation** $\pitchfork$ to move from $\mathbb{Q}$ to $\mathbb{A}$, which coheres with π

Iterating Σ-Prikry forcings

The slogan of our iterations

Let $\mathbb{Q}$ be a Σ-Prikry forcing and a problem $\sigma \in V^{\mathbb{Q}}$. We want a Σ-Prikry forcing $\mathbb{A}$ that projects onto $\mathbb{Q}$ and settles the problem raised by σ.

The above is achieved by invoking a solving-problem functor $\mathbb{A}(\cdot, \cdot)$ such that

0. $\mathbb{A} := \mathbb{A}(\mathbb{Q}, \sigma)$, "solves the problem raised by σ"

and for which there are maps $(\pi, \pitchfork)$ such that:

1. **There is a projection** π between $\mathbb{A}$ and $\mathbb{Q}$
2. **There is a canonical operation** $\pitchfork$ to move from $\mathbb{Q}$ to $\mathbb{A}$, which coheres with π

Upshot

Provided (1) & (2) of the above hold then $\mathbb{A}$ **is not so far from being Σ-Prikry.**

The iteration scheme

1. Set $\mathbb{P}_0 := (\{\emptyset\}, \leq)$ and $\mathbb{P}_1 := {}^1\mathbb{P}$;
2. $\mathbb{P}_{\alpha+1}$: If $\psi(\alpha) = (\beta, r, \sigma)$ with $\beta < \alpha$, $r \in P_\beta$, $\sigma \in V^{\mathbb{P}_\beta}$ and

$$r \Vdash_{\mathbb{P}_\beta} \sigma \text{ is a non-reflecting stationary set of } \kappa^+ \cap \mathrm{cf}^V(\omega),$$

then $\mathbb{P}_{\alpha+1} := \mathbb{A}(\mathbb{P}_\alpha, \sigma)$, where $\mathbb{A}(\cdot, \cdot)$ is a functor that *destroys the stationarity of* σ.

The iteration scheme

1. Set $\mathbb{P}_0 := (\{\emptyset\}, \leq)$ and $\mathbb{P}_1 := {}^1\mathbb{P}$;
2. $\mathbb{P}_{\alpha+1}$: If $\psi(\alpha) = (\beta, r, \sigma)$ with $\beta < \alpha$, $r \in P_\beta$, $\sigma \in V^{\mathbb{P}_\beta}$ and

   $$r \Vdash_{\mathbb{P}_\beta} \sigma \text{ is a non-reflecting stationary set of } \kappa^+ \cap \mathrm{cf}^V(\omega),$$

   then $\mathbb{P}_{\alpha+1} := \mathbb{A}(\mathbb{P}_\alpha, \sigma)$, where $\mathbb{A}(\cdot,\cdot)$ is a functor that *destroys the stationarity of* σ.
3. $\mathbb{P}_\alpha$ is the κ-supported inverse limit of $\langle \mathbb{P}_\beta \mid \beta < \alpha \rangle$.

The above iteration scheme is successful

Fact

1. $\mathbb{P}_{\kappa^{++}}$ is Σ-Prikry and does not collapse κ^+.

Proof

1. Corollary of our iteration theorem.

The above iteration scheme is successful

Fact

1. $\mathbb{P}_{\kappa^{++}}$ is Σ-Prikry and does not collapse κ^+.
2. $V^{\mathbb{P}_{\kappa^{++}}} \models \mathrm{Refl}(\kappa^+ \cap \mathrm{cf}^V(\omega))$.

Proof

1. Corollary of our iteration theorem.
2. By the κ^{++}-cc of $\mathbb{P}_{\kappa^{++}}$ and the usual "catch our tail" argument.

The above iteration scheme is successful

Fact

1. $\mathbb{P}_{\kappa^{++}}$ is Σ-Prikry and does not collapse κ^+.
2. $V^{\mathbb{P}_{\kappa^{++}}} \models \mathrm{Refl}(\kappa^+ \cap \mathrm{cf}^V(\omega))$.
3. $\mathbb{P}_{\kappa^{++}}$ projects to $\mathbb{P}$.

Proof

1. Corollary of our iteration theorem.
2. By the κ^{++}-cc of $\mathbb{P}_{\kappa^{++}}$ and the usual "catch our tail" argument.
3. Essentially, by our assumption over the functors.

A recent discovery

In recent joint work we have found a tweaking of Σ-Prikryness that encompasses forcings with interleaved collapses. A remarkable forcing captured by this framework is **Gitik's Extender Based Prikry forcing with interleaved collapses**.

A recent discovery

In recent joint work we have found a tweaking of Σ-Prikryness that encompasses forcings with interleaved collapses. A remarkable forcing captured by this framework is **Gitik's Extender Based Prikry forcing with interleaved collapses**.

As an application of this new framework we prove the following:

A recent discovery

In recent joint work we have found a tweaking of Σ-Prikryness that encompasses forcings with interleaved collapses. A remarkable forcing captured by this framework is **Gitik's Extender Based Prikry forcing with interleaved collapses**.

As an application of this new framework we prove the following:

Theorem (P., Rinot & Sinapova) (2020)

Assuming the consistency of infinitely many supercompact cardinals, it is consistent that all of the following hold:

1. $\mathrm{GCH}_{<\aleph_\omega}$ holds.
2. $2^{\aleph_\omega} = \aleph_{\omega+2}$, hence $\mathrm{SCH}_{\aleph_\omega}$ fails.

} Magidor - Ann. Math. (1977)

3. $\mathrm{Refl}(\aleph_{\omega+1})$. } Magidor - JSL (1982)

Thank you very much for your attention!