Prikry forcing and universal collapse

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Prikry Forcing Online

In this talk, all ideal on λ are λ-complete and non-principal. For an ideal I on λ, Notion of forcing $\mathcal{P}(\lambda)/I$ is $I^+(=\mathcal{P}(\lambda) \setminus I)$ ordered by inclusion.

We will consider an ideal on κ^+.

For a poset $\mathbb{P}$, $\mathbb{P}$ has the (α, β, γ)-c.c. if, for every $X \in [\mathbb{P}]^\alpha$, there is a $Y \in [X]^\beta$ such that

$$\forall Z \in [Y]^\gamma (Z \text{ has a common extension}).$$

Of course, usual κ-c.c. is the $(\kappa, 2, 2)$-c.c., and κ-Knaster is the $(\kappa, \kappa, 2)$-c.c.

Definition

For an ideal I on κ^+,

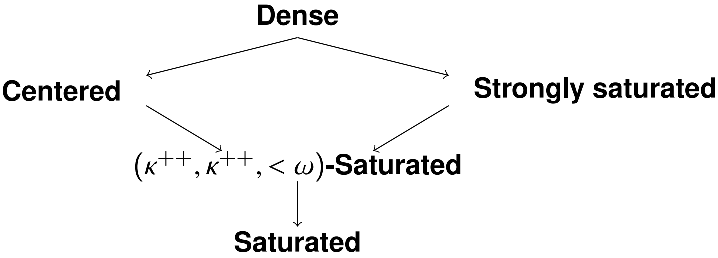
- *I is saturated if $\mathcal{P}(\kappa^+)/I$ has the κ^{++}-c.c.*
- *I is strongly saturated if $\mathcal{P}(\kappa^+)/I$ has the $(\kappa^{++}, \kappa^{++}, \kappa)$-c.c.*

Remark

Strongly saturation is not the strongest saturation property of ideals. In facts, dense ideal is strongly saturated. Here, I is **dense** *if $\mathcal{P}(\kappa^+)/I$ has a dense subset of size κ^+.*

Definition

Ideal I over κ^+ is **centered** *if $\mathcal{P}(\kappa^+)/I$ is an union of κ^+-many filters.*



Each consistency is

Propety for ideal on κ^+	κ is regular	κ is singular
Dense	Woodin	Inconsistent
Centered	Foreman–Laver	Foreman
Strongly saturated	Laver	?
Saturated	Kunen	Foreman

Theorem (Sakai–Eskew[1])

If κ is a singular cardinal, then there is no dense ideal on κ^+.

Question

Is it consistent that κ^+ carries a strongly saturated ideal for singular κ?

We assume that κ is indestructibly supercompact and $\lambda > \kappa$ is huge from here.

Theorem (Laver[3],Universal Collapse)

If $j : V \to M$ is a huge embedding (i.e. ${}^{j(\lambda)}M \subseteq M$ holds) with critical point λ, then there is a poset $\mathbf{P}$ such that

- $\mathbf{P} \subseteq V_\lambda$ *has the λ-c.c. and it is κ-directed closed.*
- $\mathbf{P}$ *collapses λ to κ^+.*
- *There is a complete embedding $\tau : \mathbf{P} * \dot{\mathbb{L}}(\lambda, j(\lambda)) \to j(\mathbf{P})$ with $\tau(p, \dot{1}) = p$ for all $p \in \mathbf{P}$.*
- $\mathbf{P} * \dot{\mathbb{L}}(\lambda, j(\lambda)) \Vdash \kappa^+ = \lambda$ *carries an strongly saturated ideal $\dot{I}$.*

Here, $\mathbb{L}(\mu, \gamma)$ is the set of all partial function $p : \mu \times \gamma \to \mu$ s.t.

- $\mathrm{supp}(p) = \{\xi < \gamma \mid \exists \alpha < \mu(\langle \alpha, \xi \rangle \in \mathrm{dom}(p))\}$ is μ-Easton. i.e. $\forall \zeta > \mu$:Regular($|\mathrm{supp}(p) \cap \zeta| < \zeta$).
- $\exists \alpha < \mu(\mathrm{dom}(p) \subseteq \alpha \times \gamma)$.

In the proof of Laver's theorem, Laver essentially showed the following lemma.

Lemma (Laver)

$\mathbf{P} * \dot{\mathbb{L}}(\lambda, j(\lambda)) \Vdash j(\mathbf{P})/\dot{G} * \dot{H}$ *has the* $(j(\lambda), j(\lambda), \kappa)$*-c.c.*

Note that, in $V^{\mathbf{P} * \dot{\mathbb{L}}(\lambda, j(\lambda))}$, some cardinals are collapsed as follows

$$\kappa^+ = \lambda, j(\lambda) = \lambda^+ = \kappa^{++}.$$

More precisely, the following holds.

Lemma

$\mathbf{P} * \dot{\mathbb{L}}(\lambda, j(\lambda)) \Vdash \mathcal{B}(\mathcal{P}(\kappa^+)/\dot{I}) \simeq \mathcal{B}(j(\mathbf{P})/\dot{G} * \dot{H})$.

This lemma shown by using similar method in Foreman–Magidor–Shelah's paper part II[2].

As an approach, we combined the Prikry forcing as follows:

Lemma

Suppose that

- $j : V \to M$ *is a huge embedding with critical point* λ*.*
- $\dot{U}$ *is a normal ultrafilter on* κ *in* $V^{\mathbf{P}}$*.*
- $\tau : \mathbf{P} * \dot{\mathbb{L}}(\lambda, j(\lambda)) \to j(\mathbf{P})$ *is complete embedding in Laver's theorem.*

Then the following map is complete embedding:

$$\begin{array}{ccc} \mathbf{P} * (\dot{\mathbb{L}}(\lambda, j(\lambda)) \times \mathcal{P}_{\dot{U}}) & \longrightarrow & j(\mathbf{P}) * \mathcal{P}_{j(\dot{U})} \\ \Psi & & \Psi \\ \langle p, \langle q, r \rangle \rangle & \longmapsto & \langle \tau(p, q), r \rangle \end{array}$$

Here, $\mathcal{P}_U$ *is the Prikry forcing defined by* U*.*

We proved that

Theorem (T.)

Suppose $j : V \to M$ is huge embedding with critical point $\lambda > \kappa$ and $\dot{U}$ is normal ultrafilter on κ in $V^{\mathbf{P}}$. Then there is a $\mathbf{P} * (\dot{\mathbb{L}}(\lambda, j(\lambda)) \times \mathcal{P}_{\dot{U}})$-name $\dot{I}$ such that

- *$\mathbf{P} * (\dot{\mathbb{L}}(\lambda, j(\lambda)) \times \mathcal{P}_{\dot{U}})$ forces that*
  - $\mathrm{cf}(\kappa) = \omega$, $\kappa^{+} = \lambda$, $\kappa^{++} = \lambda^{+} = j(\lambda)$
  - *$\dot{I}$ is an ideal on λ.*
  - *$\mathcal{B}(\mathcal{P}(\lambda)/\dot{I}) \simeq \mathcal{B}(j(\mathbf{P}) * \mathcal{P}_{j(\dot{U})}/\dot{G} * (\dot{H} \times \dot{K}))$ has the $j(\lambda)$-c.c..*

To know the saturation property of I, it is enough to study the quotient forcing.

Question

In $V^{\mathbf{P} * (\dot{\mathbb{L}}(\lambda, j(\lambda)) \times \mathcal{P}_{\dot{U}})}$, Does $j(\mathbf{P}) * \mathcal{P}_{j(\dot{U})}/\dot{G} * (\dot{H} \times \dot{K})$ have the $(j(\lambda), j(\lambda), \kappa)$-c.c.?

More general, we can show

Lemma

Suppose that 6 tuple $\langle \mathbb{P}, \dot{\mathbb{Q}}, \mathbb{R}, \tau, \dot{U}, \dot{W} \rangle$ *satisfying that*

- $\mathbb{P} \Vdash \dot{\mathbb{Q}}$ *is poset,* $\mathbb{P} \Vdash \dot{U}$ *is normal ultrafilter over* κ*.*
- $\mathbb{P} \lessdot \mathbb{R}$*,* $\mathbb{R} \Vdash \dot{U} \subseteq \dot{W}$ *is ultrafilter over* κ*.*
- $\tau : \mathbb{P} * \dot{\mathbb{Q}} \to \mathbb{R}$ *is complete embedding with* $\tau(p, \dot{1}) = p$ *for every* $p \in \mathbb{P}$*.*

Then the follwing map is complete embedding:

$$\begin{array}{ccc} \mathbb{P} * (\dot{\mathbb{Q}} \times \mathcal{P}_{\dot{U}}) & \longrightarrow & \mathbb{R} * \mathcal{P}_{\dot{W}} \\ \Cup & & \Cup \\ \langle p, \langle q, r \rangle \rangle & \longmapsto & \langle \tau(p, q), r \rangle \end{array}$$

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