Minimal Magidor-type Forcing (Countable Case)

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Prikry Forcing

Suppose κ is a measurable cardinal, U is a normal measure over κ.

Definition

Let $\mathcal{P} = \{\langle p, A\rangle \mid p \in [\kappa]^{<\omega}$ and $A \in U$ with $min(A) > max(p)\}$, and for $\langle p, A\rangle, \langle q, B\rangle \in \mathcal{P}$, $\langle p, A\rangle \leq \langle q, B\rangle$ iff

1. p is a end extension of q, i.e., $p \cap (max(q) + 1) = q$;
2. $A \subseteq B$;
3. $p \setminus q \subseteq B$.

Intermediate Models of Prikry Forcing

Theorem (Gitik-Kanovei-Koepke, 2010)

Suppose $V[G]$ is a generic extension given by Prikry forcing. Then for any $X \in V[G]$, there is a subset $C' \subseteq C_G$ such that $V[X] = V[C']$. Furthermore, the order structure of intermediate models under inclusion is isomorphic to $\mathcal{P}(\omega)/$finite.

Several forcings adding reals are minimal, i.e. there are only trivial intermediate transitive models of *ZFC* between the ground model and its generic extension.

Question (Koepke-Rasch-Schlicht)

Is there any minimal Prikry-type forcing, which preserves all cardinals while singularizing a measurable cardinal?

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Theorem (Koepke-Rasch-Schlicht, 2013)

There is a minimal Prikry-type forcing $\mathbb{P}_{\mathcal{U}}$.

Definition of $\mathbb{P}_{\mathcal{U}}$

Fix a sequence $\mathcal{U} = \langle U_\alpha \mid \alpha < \kappa \rangle$ of pairwise disjoint normal measures over κ, and a sequence $\langle A_\alpha \mid \alpha < \kappa \rangle$ of pairwise disjoint subsets of κ such that $A_\alpha \in U_\alpha$ for any $\alpha < \kappa$.

Definition

A set $T \subseteq [\kappa]^{<\omega}$ is called a $\mathcal{U}-$tree with trunk t iff

1. $\langle T, \trianglelefteq \rangle$ is a tree;
2. $t \in T$ and for any $s \in T$, $s \trianglelefteq t$ or $t \trianglelefteq s$;
3. For any $s \in T$ with $t \trianglelefteq s$, $Succ_T(s) = \{\alpha \mid s^\frown \langle \alpha \rangle \in T\} \in U_{max(s)}$.

For simplicity, assume $Succ_T(s) = \{\alpha \mid s^\frown \langle \alpha \rangle \in T\} \subseteq A_{max(s)}$.

Definition

Let $\mathbb{P}_{\mathcal{U}} = \{\langle t, T\rangle \mid T \text{ is a } \mathcal{U}\text{-tree with trunk } t\}$, and for any $\langle s, S\rangle, \langle t, T\rangle \in \mathbb{P}_{\mathcal{U}}$,

1. $\langle s, S\rangle \leq \langle t, T\rangle \iff S \subseteq T$;
2. $\langle s, S\rangle \leq^* \langle t, T\rangle \iff S \subseteq T$ and $s = t$.

Combinatorial Property

Lemma

For a condition $\langle t, T\rangle \in \mathbb{P}_{\mathcal{U}}$, *for any* $s \in T$ *with* $t \trianglelefteq s$ *and* $n > 0$, *if* $c : Lev_{T\restriction s}(|s| + n) \to \kappa$, *where* $T \restriction s = \{r \in T \mid s \trianglelefteq r \text{ or } r \trianglelefteq s\}$, *there is an* $\mathcal{U}-$*tree* $S \subseteq T$ *with the trunk* t, *and an* $m \leq n$ *such that for any* $u, v \in Lev_{S\restriction s}(|s| + n)$,

$$c(u) = c(v) \text{ iff } u \restriction (|s| + m) = v \restriction (|s| + m).$$

Theorem (Koepke-Rasch-Schlicht, 2013)

Suppose $V[G]$ is a $\mathbb{P}_{\mathcal{U}}-$generic extension, then for any $X \in V[G]$, either $X \in V$ or $V[X] = V[G]$.

Sketch of proof:We only consider the case: $X \subseteq \kappa$, $X \notin V$ and $X \cap \alpha \in V$ for any $\alpha < \kappa$. Fix a condition $\langle t, T \rangle$.
Claim: $V[X] \models cf(\kappa) = \omega$.

Theorem (Koepke-Rasch-Schlicht, 2013)

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Claim: $V[X] \models cf(\kappa) = \omega$.
There is a direct extension $\langle t, T' \rangle \leq^* \langle t, T \rangle$ such that for any $r \in T'$ with $t \trianglelefteq r$, there is an $A(r)$ such that for any $r' \in Succ'_T(r)$, $\langle r', T' \upharpoonright r' \rangle \Vdash \dot{X} \cap max(r') = A(r) \cap max(r')$. Since $X \notin V$, $X \neq A(r)$. Let α_r be the minimal α such that $X \cap \alpha \neq A(r) \cap \alpha$. Level by level, we can get a cofinal sequence of κ using the information of X.

Let $\langle x_m \mid m < \omega \rangle \in V[X]$ witnesses this;
For each x_m, by the combinatorial property, we can thin out T' to S, so that there is a $n_m < \omega$ such that for any $u, v \in S^{|t|+n_m}$, $u = v$ iff the values of $\dot{x}_m$ decided by $\langle u, S \upharpoonright u \rangle$ and $\langle v, S \upharpoonright v \rangle$ are the same one, which means that the value of $\dot{x}_m$ and $C_G \upharpoonright (|t| + n_m)$ are mutually computable.
Since $X \notin V$, so from X, we can compute the whole C_G.

Theorem (Benhamou and Gitik)

Let $\vec{U}$ be a coherent sequence such that $o^{\vec{U}}(\kappa) < \kappa$. Then for every V-generic filter $G \subseteq \mathbb{M}[\vec{U}]$, and every $A \in V[G]$, there is $C_0 \subseteq C_G$ such that $V[A] = V[C_0]$.

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1. *(Weak form) If $X \in V[G]$, and in $V[X]$, $C_G(\delta)$ is singularized, then $V[C_G \restriction \delta] \subseteq V[X]$;*

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Question

Is there any Magidor-type forcing $\mathbb{M}$, which satisfies some similar minimal property?

1. *(Weak form) If $X \in V[G]$, and in $V[X]$, $C_G(\delta)$ is singularized, then $V[C_G \restriction \delta] \subseteq V[X]$;*
2. *For any $X \in V[G]$, there is some limit δ such that $V[X] = V[C_G \restriction \delta]$.*

Theorem

Suppose δ is a countable limit ordinal, $V[G]$ is a generic extension given by a minimal Magidor-type forcing $\mathbb{M}_\delta[\vec{U}]$. Then for any $X \in V[G]$, there is a $\delta' \leq \delta$ such that $V[X] = V[C_G \restriction \delta']$.

Let us fix a sequence of pairwise disjoint normal measures $\vec{U} = \langle U^\alpha_x \mid \alpha < \omega_1, x \in [\kappa]^{<\omega} \rangle$ over κ, a sequence of pairwise disjoint subset $\vec{A} = \langle A^\alpha_x \mid \alpha < \omega_1, x \in [\kappa]^{<\omega} \rangle$ of κ and a sequence of function $\langle h_\alpha \mid \alpha < \omega_1 \rangle$ such that

1. $U^\alpha_x \lhd U^\beta_y$ for any $\alpha < \beta < \omega_1$ and $x, y \in [\kappa]^{<\omega}$, where $U^\alpha_x \lhd U^\beta_y$ means $U^\alpha_x \in Ult(U^\beta_y, V)$;
2. For any $\alpha < \omega_1$ and $x, y \in [\kappa]^{<\omega}$, $A^\alpha_x \in U^\alpha_x$ and $min(A^\alpha_x) > max(x \cup \{\omega_1\})$;
3. For any $\alpha < \beta < \omega_1$ and $x \in [\kappa]^{<\omega}$, $[h_\alpha]_{U^\beta_x} = \langle U^\alpha_y \mid y \in [\kappa]^{<\omega} \rangle$, and if $\gamma \in A^\beta_x$, $h_\alpha(\gamma) = \langle u^\alpha_y \mid y \in [\gamma]^{<\omega} \rangle$ is a sequence of normal measures over γ, and $A^\alpha_y \cap \gamma \in u^\alpha_y$ for any $y \in [\gamma]^{<\omega}$.

For simplicity, let $h_\alpha(\kappa) = \langle U^\alpha_x \mid x \in [\kappa]^{<\omega} \rangle$ for every $\alpha < \omega_1$.

Fix a countable limit ordinal δ.

Definition

A function $l : \delta \to \omega$ is a level function of δ iff for any $\alpha = \omega^{\alpha_0} + ... + \omega^{\alpha_k} < \delta$ with $\alpha_0 \geq ... \geq \alpha_k$ and $k > 0$,

1. $l(\omega^{\alpha_0} + ... + \omega^{\alpha_k}) > l(\omega^{\alpha_0} + ... + \omega^{\alpha_{k-1}})$;
2. For any $n < \omega$, $|l^{-1}(n)| < \omega$.

Definition

Suppose l is a level function of δ, let $K_l = \langle \delta, \preceq \rangle$, where $\forall \alpha, \beta < \delta \; \alpha \preceq \beta$ iff $\alpha \leq \beta$ and $l(\alpha) \leq l(\beta)$.

Definition

l is a minimal level function of δ iff K_l satisfies that for any unbounded subset x of some limit ordinal $\beta \leq \delta$, $\bar{x} = \{\alpha < \beta \mid \exists \gamma \in x \, (\alpha \preceq \gamma)\} = \beta$, i.e., the downward closure of x in K_l is β.

For any such l, we will define a minimal Magidor-type forcing $M_l[\vec{U}]$.

Definition

We define a function l with $dom(l) = \{(\alpha, \beta) \mid \alpha$ is a countable limit ordinal and $\beta < \alpha\}$ by induction.

1. $l(\omega, k) = k$ for $k < \omega$;
2. If $\alpha = \omega^{\gamma}$, where γ is a successor ordinal, let $\alpha_k = \omega^{\gamma-1} \cdot k$ for $k < \omega$;
   If $\alpha = \omega^{\gamma}$, where γ is a limit ordinal, let $\langle \gamma_k \mid 0 < k < \omega \rangle$ be a cofinal increasing sequence of γ.
   Let $\alpha_k = \omega^{\gamma_k}$ if $0 < k < \omega$, and $\alpha_0 = 0$.
   For $i < \alpha_1$, let $l(\alpha, i) = l(\alpha_1, i)$;
   For $\alpha_k \leq i < \alpha_{k+1}$, let $l(\alpha, i) = l(\alpha_{k+1} - \alpha_k, i - \alpha_k) + k - 1$;
3. If $\alpha = \omega^{\gamma_1} + ... + \omega^{\gamma_n}$ with $\gamma_1 \geq \gamma_2 \geq ... \geq \gamma_n > 0$. Let $\alpha_k = \sum_{0<j<k+1} \omega^{\gamma_j}$ for $0 \leq k < n + 1$.
   For $\alpha_k \leq i < \alpha_{k+1}$ with some $0 \leq k < n + 1$, let $l(\alpha, i) = l(\alpha_{k+1} - \alpha_k, i - \alpha_k) + k - 1$.

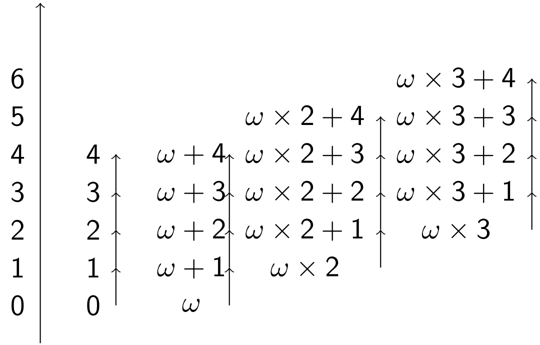
Lemma

Let $l_\delta(\alpha) = l(\delta, \alpha)$ for $\alpha < \delta$. Then l_δ is a minimal level function of δ.

We can prove this by induction. For $\alpha = \omega^{\alpha_0} + ... + \omega^{\alpha_k} < \delta$ with $i_0 \geq ... \geq i_k$ and $k > 0$, α and $\alpha' = \omega^{i_0} + ... + \omega^{i_{k-1}}$ is in the same part, or $\alpha' = \alpha_m$ and $\alpha = \alpha_{m+1}$. In both cases, $l(\alpha') < l(\alpha)$;
For $\alpha_k \leq i < \alpha_{k+1}$, $l(\alpha, i) = l(\alpha_{k+1} - \alpha_k, i - \alpha_k) + k - 1 \geq k - 1$, so $|l^{-1}(n)| < \omega$ by induction.

Example $\delta = \omega^2$

$l^{-1}(n) = \{\omega \times (n+1), \omega \times n + 1, ..., \omega + n, n\}$.
In K_l, $\alpha \prec \beta$ iff $\alpha < \beta$ and $l(\alpha) < l(\beta)$.



Notation

Fix a level function l of δ, and for simplicity, we will omit the l, i.e. we will say K rather than K_l and $\mathbb{M}[\vec{U}]$ rather than $\mathbb{M}_l[\vec{U}]$.
We say $x \in [\kappa]^{<\omega}$, means that $x \in [\kappa \setminus \omega_1]^{<\omega}$, and $|x \cap A^{\alpha}_{y}| \leq 1$ for any $\alpha < \omega_1$, $y \in [\kappa]^{<\omega}$ with $min(y) < \delta$. So we can view x as a partial function from a finite initial subset of K to κ, where for $\alpha \in x$ with $\alpha \in A^{\beta}_{y}$, $x(\min(y)) = \alpha$.
For $a \subset K$, we mean a is a finite initial subset of K. Fix a $x \in [\kappa]^{<\omega}$, $a \subset K$ and $\alpha < \delta$.

1. $x^{-} = x \setminus \{max(x)\}$;
2. $\bar{\alpha} = \{\beta \leq \alpha \mid \beta \preceq \alpha\}$;
3. $\alpha^{-} = (\bar{\alpha})^{-}$;
4. $a^{-} = a \setminus \{max(a)\}$;
5. $a_s = \{\beta < \delta \mid \forall \alpha \prec \beta \ \alpha \in a\}$.

Definition

For an $x \in [\kappa]^{<\omega}$ with $dom(x) = b$, and $b \subseteq a \subseteq K$, suppose for any $c \subsetneq a$, we have defined D^c_x. Let $\alpha = max(a \setminus b)$.
If $\alpha = max(a)$, for $X \subseteq [\kappa]^{<\omega}$, $X \in D^a_x$ iff

$$\{y \mid \{\bar{\beta} \mid \beta < \kappa, y \cup \{\beta\} \in X\} \in U^{o(\alpha)}_{\{\alpha\} \cup x \restriction \alpha^-}\} \in D^{a^-}_x$$

If $\alpha < max(a)$, let $\gamma = min(b \setminus (\alpha + 1))$. For $X \subseteq [\kappa]^{<\omega}$, $X \in D^a_x$ iff

$$\{y \mid \{\bar{\beta} \mid \beta < \kappa, y \cup \{\beta\} \in X\} \in h_{o(\alpha)}(x(\gamma))(\{\alpha\} \cup y \restriction \beta^-)\} \in D^{a^-}_x$$

Definition

A set $T \subseteq [\kappa]^{<\omega}$ is a $\vec{U}$–tree with trunk t iff

1. $(T, \trianglelefteq)$ is a tree;
2. For every $s \in T$, $s \restriction dom(t) \subseteq t$;
3. For $t \trianglelefteq u$ with $a = dom(u)$, $Succ_T(u) = \{r \in T \mid dom(r) = a_s, r \restriction a = u\} \in D_u^{a_s}$.

Definition

Let $\mathbb{M}[\vec{U}] = \{\langle t, T\rangle \mid T$ is a $\vec{U}$−tree with trunk $t\}$, and for $\langle s, S\rangle, \langle t, T\rangle \in \mathbb{M}[\vec{U}]$,

1. $\langle s, S\rangle \leq \langle t, T\rangle$ iff $S \subseteq T$;
2. $\langle s, S\rangle \leq^* \langle t, T\rangle$ iff $S \subseteq T$ and $s = t$.

In the latter case, we call $\langle s, S\rangle$ is a direct extension of $\langle t, T\rangle$.

Combinatorial Properties

Definition

For $u, v \in [\kappa]^{<\omega}$, let $type(u, v) \in 3^{|u \cup v|}$ denote the order configuration of u and v, i.e., if $s = \langle \xi_i \mid i < k \rangle$ is the strict increasing sequence with $u \cup v = s$,

$$type(u, v)(i) = \begin{cases} 0, & if\ \xi_i \in u - v, \\ 1, & if\ \xi_i \in u \cap v, \\ 2, & if\ \xi_i \in v - u. \end{cases}$$

Combinatorial Properties

Lemma

For $i = 1, 2$, if $t_i \subseteq a_i$, $X_i \in D^{a_i}_{t_i}$, and $F : X_1 \times X_2 \to \lambda$ for some λ less than the least measurable cardinal, then there are some $Y_i \in D^{a_i}_{t_i}$ such that for any $u \in Y_1$ and $v \in Y_2$, the value of F only depends on type(u, v).

Lemma

Let $X \in D^a_t$, where $dom(t) \subset a$, and $F : X \to \kappa$. Then we have a $Y \in D^a_t$ and a $dom(t) \subseteq b \subseteq a$ such that

$$\forall x, y \in Y\,(F(x) = F(y) \leftrightarrow x \restriction b = y \restriction b). \tag{1}$$

Lemma

Let $X \in D_t^a$, *where* $dom(t) \subset a$, *and* $F : X \to \kappa$. *Then we have a* $Y \in D_t^a$ *and a* $dom(t) \subseteq b \subseteq a$ *such that*

$$\forall x, y \in Y \, (F(x) = F(y) \leftrightarrow x \restriction b = y \restriction b). \tag{1}$$

We prove this by induction. Thin out X to X', so that there is an $I \subseteq a_t = \{\alpha \in a \mid \text{ there is no } \beta \in a \text{ such that } \alpha \prec \beta\}$ such that

$$\forall u, v \in X' (u \restriction (a - a_t) = v \restriction (a - a_t) \rightarrow (F(u) = F(v) \leftrightarrow u \restriction I = v \restriction I)$$

Lemma

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$$\forall x, y \in Y \, (F(x) = F(y) \leftrightarrow x \upharpoonright b = y \upharpoonright b). \tag{1}$$

We prove this by induction. Thin out X to X', so that there is an $I \subseteq a_t = \{\alpha \in a \mid \text{ there is no } \beta \in a \text{ such that } \alpha \prec \beta\}$ such that

$$\forall u, v \in X' (u \upharpoonright (a - a_t) = v \upharpoonright (a - a_t) \to (F(u) = F(v) \leftrightarrow u \upharpoonright I = v \upharpoonright I)$$

If $I \neq a_t$, by induction; if $I = a_t$, define

$$G(x, y) = \begin{cases} 0, & \textit{if } F(x) = F(y); \\ 1, & \textit{otherwise.} \end{cases}$$

By the above lemma, we can thin out X' to Y, so that for any $u, v \in Y$, if $u = v$ iff $F(u) = F(y)$.

Lemma

Suppose that Δ *is an open dense subset of* $M[\vec{U}]$, $p = \langle t, T \rangle \in M[\vec{U}]$. *Then there is a direct extension* $\langle t, T' \rangle \leq^* \langle t, T \rangle$ *such that for any finite subset* $dom(t) \subseteq a \subset K$, *if some* $\langle s, S \rangle \leq \langle t, T' \rangle$ *with* $t \trianglelefteq s$, $a = dom(s)$ *and* $\langle s, S \rangle \in \Delta$, *then for any* $\langle r, R \rangle \leq \langle t, T' \rangle$ *with* $a \subseteq dom(r)$, *we have* $\langle r, R \rangle \in \Delta$.

Lemma

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Proof.

Let $F : T^a \to 2$ such that for any $s \in T^a$,

$$F(s) = \begin{cases} 0, & \text{if } \exists \langle s, T_s \rangle \leq^* \langle s, T \restriction s \rangle \text{ such that } \langle s, T_s \rangle \in \Delta, \\ 1, & \text{otherwise.} \end{cases}$$

Lemma

Suppose that Δ *is an open dense subset of* $M[\vec{U}]$*,* $p = \langle t, T\rangle \in M[\vec{U}]$*. Then there is a direct extension* $\langle t, T'\rangle \leq^* \langle t, T\rangle$ *such that for any finite subset* $dom(t) \subseteq a \subset K$*, if some* $\langle s, S\rangle \leq \langle t, T'\rangle$ *with* $t \trianglelefteq s$*,* $a = dom(s)$ *and* $\langle s, S\rangle \in \Delta$*, then for any* $\langle r, R\rangle \leq \langle t, T'\rangle$ *with* $a \subseteq dom(r)$*, we have* $\langle r, R\rangle \in \Delta$*.*

Proof.

Let $F : T^a \to 2$ such that for any $s \in T^a$,

$$F(s) = \begin{cases} 0, & \text{if } \exists\langle s, T_s\rangle \leq^* \langle s, T \restriction s\rangle \text{ such that } \langle s, T_s\rangle \in \Delta, \\ 1, & \text{otherwise.} \end{cases}$$

Then there is a $Y \in D^a_t$ such that $F \restriction Y$ is a constant function. Let T_a be the amalgamation of T_s for every $s \in Y$. Let T' be the intersection of all these T_a. □

Proposition

Suppose $\langle t, T\rangle \Vdash \dot{x} \in Ord$, then there exists a direct extension $\langle t, T'\rangle$ of $\langle t, T\rangle$ and a $dom(t) \subseteq a \subset K$ such that for every $\langle s, S\rangle \leq \langle t, T'\rangle$ with $dom(s) \supseteq a$, we have $\langle s, S\rangle \parallel \dot{x}$.

Proof.

Let $\Delta = \{p \in M[\vec{U}] \mid p \parallel \dot{x}\}$, we know Δ is a dense open set of $M[\vec{U}]$. So by the above Lemma, we are done. □

Lemma

If $|A| < \lambda_0$*, where* λ_0 *is the least measurable cardinal and* A *is a set of ordinals, then* $V[A] = V[C_G \restriction \delta']$ *for some limit ordinal* $\delta' \leq \delta$*.*

Proof.

Lemma

If $|A| < \lambda_0$, where λ_0 is the least measurable cardinal and A is a set of ordinals, then $V[A] = V[C_G \restriction \delta']$ for some limit ordinal $\delta' \leq \delta$.

Proof.

By the above Lemma, for any $\langle t, T \rangle \Vdash \dot{x} \in Ord$, there exists a direct extension $\langle t, T' \rangle$ of $\langle t, T \rangle$ and a $dom(t) \subseteq a \subset K$ such that for every $\langle s, S \rangle \leq \langle t, T' \rangle$ with $dom(s) \supseteq a$, we have $\langle s, S \rangle \parallel \dot{x}$;

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If $|A| < \lambda_0$*, where* λ_0 *is the least measurable cardinal and* A *is a set of ordinals, then* $V[A] = V[C_G \upharpoonright \delta']$ *for some limit ordinal* $\delta' \leq \delta$*.*

Proof.

By the above Lemma, for any $\langle t, T\rangle \Vdash \dot{x} \in Ord$, there exists a direct extension $\langle t, T'\rangle$ of $\langle t, T\rangle$ and a $dom(t) \subseteq a \subset K$ such that for every $\langle s, S\rangle \leq \langle t, T'\rangle$ with $dom(s) \supseteq a$, we have $\langle s, S\rangle \parallel \dot{x}$;
Now for any $u \in T'^a$, let $F(u) = \gamma$ iff $\langle u, T' \upharpoonright u\rangle \Vdash \dot{x} = \check{\gamma}$. We can thin out T'^a to X such that there is a $dom(t) \subseteq I \subseteq a$, for any $u, v \in X$, $F(u) = F(v)$ iff $u \upharpoonright I = v \upharpoonright I$.
So if $A = \langle x_\xi \mid \xi < \eta\rangle$, A and $C_G \upharpoonright \cup_{\xi<\eta} I_\xi$ can be mutually computable. □

Lemma

Suppose $A \subseteq \kappa$, $A \cap \alpha \in V$ *for any* $\alpha < \kappa$ *and* $A \notin V$. *For any* $p = \langle s, S \rangle \in \mathbb{M}[\vec{U}]$, *we can find a direct extension* $p^* = \langle s, T \rangle \leq^* p$, *and for any* $\alpha < \delta$, *a finite subset* $\bar{\alpha} \subseteq a_\alpha \subseteq \alpha + 1$ *such that*

$$\forall r \in T^{a_\alpha} \; \langle r, T \restriction r \rangle || \dot{A} \cap max(r). \tag{2}$$

From the Lemma, we can prove that in $V[A]$, there is a cofinal sequence $\langle \gamma_\xi \mid \xi < \eta \rangle$ of κ, where $\eta \leq \delta$ is limit, such that $\gamma_\xi \geq C_G(\xi)$ for any $\xi < \eta$.

Generalization: level function

For minimal level function l, we have the following correspondence:

$$\text{Initial segments of } K_l/\mathit{finite} \xleftrightarrow{\text{Combinatorial Property}} \text{Intermediate models}$$

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It seems that we can generalize this correspondence to level function of l.

Generalization: uncountable case

We can't directly generalize to uncountable case, since if so, the height of K will be uncountable, or at some level, there are uncountable elements.

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Thank You!