Lecture Notes:
Forcing
&
Symmetric Extensions

Asaf Karagila

Last Update:
February 11, 2026

Contents

Part I

A Generic Introduction to Forcing

Chapter 1

Introduction

The technique of forcing was introduced by Paul J. Cohen in 1963 in order to show that ZFC does not prove that $V = L$, the Continuum Hypothesis, and that ZF does not prove the Axiom of Choice.

The idea behind forcing is fairly simple. Start with a reasonable model of ZFC, M, and adjoin new sets to it in a manner that preserves ZFC and preferably preserves "niceness". We have some immediate restrictions here. For example, the new set must be a subset of some V_α^M, in order to not add ordinals.

Cohen identified a condition, now called "genericity", which we will study extensively over the next few weeks. The idea is to identify a partial order inside of M which will "approximate" larger and larger parts of the new set that we want to add. If we are lucky, then we can find a set that is approximated by our partial order and satisfies this genericity condition over M.[1]

Example 1.1. Let M be a countable transitive model of ZFC. Since M is countable, it does not contain all the real numbers in V. So we wish to adjoin a new real number to this model. One simple way of doing that is by approximating its decimal expansion by finite means. In other words, we want to approximate the real by the finite initial segments of its decimal expansion.

Now we run into an obvious problem: we don't know the real, and we don't even know a priori *which real* we want to add. So instead of committing to a specific real, we will simply consider all the possible finite initial segments of decimal expansion.

If r is the real number that we ended up adding, we want to avoid it having properties that can be recognised by M in advance. Clearly, it will have some specified properties. Is the first digit 0 or 5, or something else? When is the first 9 going to appear? All of these will have to be specified a posteriori, but we want to avoid specifying them in advance if we can.

But what we can notice, for example, is that any finite approximation will have some extension that contains the digit 6, or any finite sequence of digits for that matter, so in being a very generic real number, we expect that the real we are approximating will eventually have 6, and 0, and the sequence 14159, and really all of these sequences. On the other hand, we want to limit the amount of information "coded" into this new real number. For example, since M is countable, we can enumerate it (externally) and encode this enumeration somehow into this real number while also having included, eventually, every finite sequence.

The way we control this "level of genericity" is by looking at properties of our partial order and its subsets in M. Specifically, we are interested in dense sets. So our generic real will be one

[1] Assuming that M is countable will be enough to prove that we are always lucky. Because who needs luck when you can use diagonalisation?

that meets every dense set that M knows about. This will turn out to be the correct definition, as we will see later.

Definition 1.2. We say that $\langle \mathbb{P}, \leq \rangle$ is a *notion of forcing* if it is a preordered set (reflexive and transitive) with a maximum element denoted by $\mathbb{1}_{\mathbb{P}}$, or $\mathbb{1}$ when the context is clear. The elements of $\mathbb{P}$ will be called *conditions*. If $q \leq p$, we will say that q is a *stronger* condition than, or that it is an *extension* of, p. Two conditions p, q are *compatible* if they have a common extension (denoted by $p \parallel q$), and they are *incompatible* otherwise (denoted by $p \perp q$).

Remark. We will see later on that we can make some assumptions on $\mathbb{P}$ without losing any generality. The simplest one, which will be the common one, is that $\mathbb{P}$ is in fact a separative partial order, but these assumptions may sometimes be strengthened to the statement that $\mathbb{P}$ is a complete Boolean algebra.

Definition 1.3. We say that a preordered set is *separative* if whenever p, q are two distinct conditions the exactly one of these two statements hold:

1. $q < p$ and $p \nleq q$,
2. There is some $r \leq q$ such that r is incompatible with p.

Exercise 1.4. If $\mathbb{P}$ is a separative preorder, then $\mathbb{P}$ is a partial order. In other words, $\mathbb{P}$ is antisymmetric.

Proposition 1.5. *Let $\mathbb{P}$ be a preordered set. Define $p \sim q \iff \forall r, r \parallel p \iff r \parallel q$. Then $\mathbb{P}/{\sim}$ has a natural partial order defined on it which is separative, and $[p] \perp [q]$ if and only if p and q are incompatible in $\mathbb{P}$. Moreover, if $\mathbb{P}$ has a maximum element, so will $\mathbb{P}/{\sim}$.*

Proof. Let $\mathbb{P}_* = \mathbb{P}/{\sim}$ and define

$$[q] \leq_* [p] \iff \{r \in \mathbb{P} \mid r \parallel q\} \subseteq \{r \in \mathbb{P} \mid r \parallel p\}.$$

Easily, $\leq_*$ is a partial order and $[\mathbb{1}_{\mathbb{P}}]$, if it exists, is a maximum.

To see that $\leq_*$ is separative, suppose that $[q] \nleq_* [p]$, then by definition there is some $r \in \mathbb{P}$ such that $r \parallel q$ and $r \perp p$. But now, if s is stronger than both r and q, then $[s] \leq_* [r], [q]$, witnessing their compatibility. And since r is incompatible with p, $\{s \in \mathbb{P} \mid s \parallel r\} \nsubseteq \{s \in \mathbb{P} \mid s \parallel p\}$, as the former set contains r itself and the latter does not.

Finally, suppose that $[p] \parallel [q]$ and let $[r]$ be a joint extension of them. Then, if $s \parallel r$, then $s \parallel p, q$. Let $s \leq p, r$ be some extension, then s is certainly compatible with r and p, but it also implies that $s \parallel q$. So there is a joint extension of s and q, which is therefore a joint extension of p, q, and so $p \parallel q$. In the other direction, if $p \parallel q$, and r is a joint extension of both, then $[r] \leq_* [p], [q]$. □

Exercise 1.6. Suppose that $\mathbb{P}$ is a preordered set and $\mathbb{Q}$ is a separative partial order such that there is some $s \colon \mathbb{P} \to \mathbb{Q}$ which is a surjective homomorphism of preordered sets that preserves incompatibility, then $\mathbb{Q} \cong \mathbb{P}_*$. In other words, $\mathbb{P}_*$ is **the** separative quotient of $\mathbb{P}$.

Definition 1.7. Let $\mathbb{P}$ be a notion of forcing and let $D \subseteq \mathbb{P}$ be a set.

1. We say that D is *predense* if for every $p \in \mathbb{P}$ there is some $q \in D$ such that $p \parallel q$.
2. We say that D is *dense* if for every $p \in \mathbb{P}$ there is some $q \in D$ such that $q \leq p$.
3. We say that D is *open* if for every $p \in D$ and $q \leq p$, $q \in D$.

4. We say that D is *an antichain* if for any two distinct $p, q \in D$, $p \perp q$.[2]

5. We say that D is *a filter* if for any $p, q \in D$ there is some $r \in D$ such that $r \leq p, q$, and if $q \in D$ and $q \leq p$, then $p \in D$.

The definitions of predense, dense, and open can also be relativised to be taken below a fixed condition p.

Exercise 1.8. D is a maximal antichain if and only if D is a predense antichain.

Definition 1.9. Let M be a model of ZFC and let $\mathbb{P} \in M$ be a notion of forcing. We say that a filter $G \subseteq \mathbb{P}$ is an *M-generic filter* if for any dense open $D \subseteq \mathbb{P}$ such that $D \in M$, $G \cap D \neq \varnothing$.

Exercise 1.10. We can replace "dense open" by "dense", "predense", or "maximal antichain" in the definition of generic filter.

Exercise 1.11. If D is a dense set, then D contains a maximal antichain. Find a counterexample in the case of a predense set.

Exercise 1.12. Suppose that $\mathbb{P} \in M$ is a notion of forcing, and let $\mathbb{P}_*$ be its separative quotient. If $G \subseteq \mathbb{P}$ is an M-generic filter, then $G_* = \{[p] \mid p \in G\}$ is an M-generic filter for $\mathbb{P}_*$. Similarly, if G_* is an M-generic filter for $\mathbb{P}_*$, then $G = \{p \mid [p] \in G_*\}$ is an M-generic filter.

Theorem 1.13. *Suppose that $\mathbb{P}$ is a separative notion of forcing in M, and G is an M-generic filter. Then either $\mathbb{P}$ has a minimal element, or $G \notin M$.*

Proof. Suppose that $\mathbb{P}$ does not have any minimal elements. If $p \in \mathbb{P}$ is any condition, then there is some $q < p$, and by separativity, since $p \nleq q$, there is some $r < p$ which is incompatible with q. In other words, any condition in $\mathbb{P}$ has two incompatible extensions.

If $F \in M$ is any filter, let $D_F = \mathbb{P} \setminus F$. We claim that D_F is dense. If it is, then $G \cap D_F \neq \varnothing$, so $G \neq F$.

If $p \in F$, then it has two incompatible extension, q_0, q_1. Since any two conditions in F must be compatible, at most one of q_0, q_1 can be in F, so the other must be in D_F. If $p \notin F$, then $p \in D_F$. □

Theorem 1.14. *Let M be a countable transitive model of* ZFC *and let $\mathbb{P} \in M$ be a forcing notion. Then there is an M-generic filter $G \subseteq \mathbb{P}$.*

Proof. We enumerate the dense open subsets of $\mathbb{P}$ inside M as $\{D_n \mid n < \omega\}$. Let G_* be defined recursively: First choose $p_0 \in D_0$. Suppose p_n was chosen from D_n, then let $p_{n+1} \in D_{n+1}$ be an extension of p_n; this p_{n+1} can be found since D_{n+1} is dense.

Let $G = \{p \in \mathbb{P} \mid \exists n, p_n \leq p\}$. We claim that G is an M-generic filter. If $p, q \in G$, then there is some n such that $p_n \leq p, q$, and if $q \in G$ and $q \leq p$, then there is some $p_n \leq q \leq p$, so $p \in G$. The genericity easily follows from the definition of G. □

Corollary 1.15. *If M is a countable transitive model, $\mathbb{P}$ is a notion of forcing, then any $p \in \mathbb{P}$ is contained in some M-generic filter.*

Proof. Consider $\mathbb{P} \restriction p = \{q \in \mathbb{P} \mid q \leq p\}$, with the order it inherits from $\mathbb{P}$. □

[2] This is different from the order-theoretic definition of an antichain.

If we go back to our example from earlier, approximating a new real number by its decimal expansion, we see that if G is an M-generic filter, then it is a sequence of longer and longer finite sequences of digits, and the limit of this sequence is some uniquely determined real number r_G. Moreover, given any real number, r, in M or otherwise, it defines a filter for our forcing, G, and $r_G = r$. So if $r \in M$, G_r is not M-generic. So the real we add using an M-generic filter is truly a new real number, from the perspective of M.

Let us state some theorems which we will prove in the next section, but explain why we are choosing countable transitive models, as well as the incredible strength of the method.

Theorem 1.16 (The Generic Model Theorem). *Let M be a countable transitive model of* ZFC*, and let G be an M-generic filter (for some $\mathbb{P} \in M$). Then there is a countable transitive model $M[G]$ satisfying:*

1. $M[G] \models \mathsf{ZFC}$.
2. $\mathrm{Ord}^M = \mathrm{Ord}^{M[G]}$.
3. $M \subseteq M[G]$.
4. *$M[G]$ is the smallest countable transitive model extending M and containing G as a set.*

The model $M[G]$ is called a *generic extension* of M, indeed by the last property it is *the* generic extension of M by G. Much like in the case of field extensions, where we can provide a formal expression definable over our field that will be evaluated to a member of the extension (a polynomial or a rational function, for example), we can also define *names* in M for the members of $M[G]$, and we have a relationship between the conditions of the forcing notions and statements about these names. We will use the dot-notation to denote general names, so $\dot{x}$ and $\dot{y}$ will be names. Given a generic filter G, we will write $\dot{x}^G$ to mean the interpretation of this name by this filter, which is an object in $M[G]$.

We write $p \Vdash \varphi(\dot{x})$, where p is a condition in some forcing notion $\mathbb{P}$ and $\dot{x}$ is some $\mathbb{P}$-name, to mean that p forces that the object evaluated from $\dot{x}$ will have the property φ.

Theorem 1.17 (The Forcing Theorem). *Suppose that $\dot{x} \in M$ is a name, then*

$$M[G] \models \varphi(\dot{x}^G) \iff \exists p \in G, p \Vdash \varphi(\dot{x}).$$

Now is a good a time as any to discuss the philosophical foundations of forcing. Namely, what gives with the countable models of ZFC, or even more, the countable *transitive* models?

The real answer is that these are not necessary. We can use the Reflection theorem to extract some finite fragment of ZFC which is necessary for the machinery of forcing and use countable and transitive models of that finite fragment; we can use the type omitting theorem to show there are generic filters; we can use all manners of Boolean-valued models defined in the meta-theory, which may very well be some weak theory of arithmetic, to argue the truth, relative truth, and independence of statements from ZFC. In fact, for the most part, the Axiom of Choice is not needed at all for the basic machinery of forcing to work.

However, forcing is meant to be a technique that is simple to apply, since it can be complicated enough on its own. Since moving our proofs from one form of formalisation to the other is going to be a tedious, but mechanical work, some of which we will see play out here, we will stick with the countable transitive models of ZFC for our forcing concerns. At least for the beginning of this course. We will later leave all the presumptions behind, and simply force "over the universe", that is, we will be working inside what may-very-well be a countable transitive model of ZFC.

Chapter 2

The basic mechanics of forcing

Much like when describing a field extension we can describe the elements of the extension "formally" as evaluations of rational functions or polynomials, one of the greatest strengths of forcing is that we can describe the elements of the extension by the generic filter inside the starting model, or ground model. For the rest of this chapter we fix a countable transitive model M.

2.1 To force is to name names

Definition 2.1. Let $\mathbb{P}$ be a notion of forcing, we define the class of $\mathbb{P}$-names, $M^{\mathbb{P}}$ by recursion.

1. $M_0^{\mathbb{P}} = \varnothing$,
2. $M_{\alpha+1}^{\mathbb{P}} = \mathcal{P}^M(\mathbb{P} \times M_\alpha^{\mathbb{P}})$,
3. $M_\alpha^{\mathbb{P}} = \bigcup_{\beta<\alpha} M_\beta^{\mathbb{P}}$ for a limit ordinal α,
4. $M^{\mathbb{P}} = \bigcup_{\alpha\in\mathrm{Ord}} M_\alpha^{\mathbb{P}}$.

An element of $M^{\mathbb{P}}$ is called a $\mathbb{P}$-name and will usually be denoted by $\dot{x}$.

The definition lends itself to a natural notion of $\mathbb{P}$-name rank, or $\mathbb{P}$-rank of a name, which is the least α such that $\dot{x} \subseteq \mathbb{P} \times M_\alpha^{\mathbb{P}}$. This allows us to define all sort of things by recursion on the rank of a name.

Definition 2.2. Suppose $x \in M$. The *canonical name* for x is denoted by $\check{x}$ and it is defined recursively as $\check{x} = \{\langle \mathbb{1}, \check{y}\rangle \mid y \in x\}$.

Definition 2.3. Suppose that $\{\dot{x}_i \mid i \in I\}$ is a family of $\mathbb{P}$-names. We want to turn them into a name for the set in the most obvious way.

$$\{\dot{x}_i \mid i \in I\}^\bullet = \{\langle \mathbb{1}, \dot{x}_i\rangle \mid i \in I\}.$$

This extends to ordered pairs, sequences, functions, etc. We can now recast $\check{x}$ as $\{\check{y} \mid y \in x\}^\bullet$.

Definition 2.4. The canonical name for the generic filter is $\dot{G} = \{\langle p, \check{p}\rangle \mid p \in \mathbb{P}\}$.

Definition 2.5. Let $\dot{x}$ be a $\mathbb{P}$-name and let G be an M-generic filter. The interpretation of $\dot{x}$, denoted by $\dot{x}^G$, is defined recursively, $\{\dot{y}^G \mid \exists p \in G, \langle p, \dot{y}\rangle \in \dot{x}\}$.

Exercise 2.6. For all $x \in M$ and for any M-generic G, $\check{x}^G = x$.

Exercise 2.7. If H is an M-generic filter, then $\dot{G}^H = H$.

Definition 2.8. $M[G] = \{\dot{x}^G \mid \dot{x} \in M^{\mathbb{P}}\}$.

Proposition 2.9. *$M \subseteq M[G]$, $G \in M[G]$, and $M[G]$ is a transitive set.*

Proof. For any $x \in M$, $\check{x}$ is a $\mathbb{P}$-name, so $\check{x}^G \in M[G]$, and therefore $M \subseteq M[G]$, similarly $\dot{G}^G = G \in M[G]$. Suppose that $x \in y \in M[G]$, then $y = \dot{y}^G$ for some $\dot{y}$, and by definition the elements of y are of the form $\dot{z}^G$. In particular, $x = \dot{x}^G$ for some $\dot{x} \in M^{\mathbb{P}}$, so $x \in M[G]$ as well. $\square$

We want to define a relation between the conditions and the names, which will help us analyse the truth of $M[G]$. What properties are we looking for in our forcing relation, if it is to predict correctly the properties of $M[G]$?

1. Certainly, if $\langle p, \dot{x}\rangle \in \dot{y}$ and $p \in G$, then by definition we want to have that $p \Vdash \dot{x} \in \dot{y}$.
2. If $q \in G$ and $q \leq p$, then $p \in G$. So that means that if $p \Vdash \varphi$, then q must also force it.
3. If $p \Vdash \varphi$, then p cannot force $\neg\varphi$ as well.
4. If p forces something, then it should be true in any generic extension, so long p is in the generic filter.
5. And ideally, if $M[G] \models \varphi(x)$, we want to know there is some $q \in G$ and $\dot{x}$ such that $\dot{x}^G = x$ and $q \Vdash \varphi(\dot{x})$.

Naively, we can define that $p \Vdash \dot{x} \in \dot{y}$ if and only if $\langle p, \dot{x}\rangle \in \dot{y}$, we can then continue from this definition to $=$ and other formulas, requiring that $p \Vdash \exists x\varphi(x)$ if and only there is some $\dot{x}$ such that $p \Vdash \varphi(\dot{x})$. But this is too simplistic.

Consider the name $\dot{x} = \{\langle p, \check{\varnothing}\rangle\}$, where p is some condition different than $\mathbb{1}$. Then clearly, the only way we can preserve the coherence of $p \Vdash \check{\varnothing} \in \dot{x}$, that is extending p will not change this fact, is by insisting that the names are defined differently, i.e. if $\langle p, \dot{u}\rangle \in \dot{v}$ and $q \leq p$, then we want to require that $\langle q, \dot{u}\rangle \in \dot{v}$ as well. This is not necessarily a bad idea, and we can modify the definition of names so that it is "almost true". But this still does not solve the problem that if $q \perp p$, then $q \Vdash \dot{x} = \check{\varnothing}$, or at least this should be the case, and so now $q \Vdash \dot{x} \in \check{1}$, despite the fact that $\langle q, \dot{x}\rangle$ is not in $\check{1}$.

We know that being generic means meeting every predense set, so being a generic filter that contains a particular condition p we simply require that we meet any *predense set below* p. But we can say a bit more, since the forcing relation should be coherent with the order of the forcing notion (this is property (2) in the above list), we can define our forcing relation using dense, and indeed dense open, sets.

We define the forcing relation by induction on the complexity of φ (in the syntax with $\neg, \wedge, \exists$ and $=$ as logical symbols). For atomic formulas we will do this simultaneously. And to do this, we use recursion on the $\mathbb{P}$-rank of the names involved.

Remark. This allows us to define the forcing relation, uniformly, for any finite number of formulas without contradiction Tarski's theorem about the undefinability of the truth. The reason, of course, is that if we can define this for all formulas, then simply consider the forcing $\{\mathbb{1}\}$, and we can ask which statements are forced to be true, but that must be the truth in the model. This is a fine and very subtle point which is worth taking the time to fully internalise.

Definition 2.10 (The Forcing Relation). Let p be a condition and $\dot{x}, \dot{y}$ be two names.

1. $p \Vdash \dot{x} = \dot{y}$ if and only if the two conditions below hold:
   (a) For any $\langle p', \dot{z}\rangle \in \dot{x}$, $\{q \leq p \mid q \leq p' \to \exists \langle q', \dot{w}\rangle \in \dot{y}, q \leq q' \wedge q \Vdash \dot{z} = \dot{w}\}$ is dense below p.
   (b) For any $\langle q', \dot{w}\rangle \in \dot{y}$, $\{q \leq p \mid q \leq q' \to \exists \langle p', \dot{z}\rangle \in \dot{x}, q \leq p' \wedge q \Vdash \dot{z} = \dot{w}\}$ is dense below p.
2. $p \Vdash \dot{x} \in \dot{y}$ if and only if the set $\{q \leq p \mid \exists \langle p', \dot{z}\rangle \in \dot{y}, q \leq p' \wedge q \Vdash \dot{z} = \dot{x}\}$ is dense below p.
3. $p \Vdash \varphi \wedge \psi$ if and only if $p \Vdash \varphi$ and $p \Vdash \psi$.
4. $p \Vdash \neg\varphi$ if and only if there is no $q \leq p$ such that $q \Vdash \varphi$.
5. $p \Vdash \exists x \varphi(x)$ if and only if the set $\{q \leq p \mid \exists \dot{x} : q \Vdash \varphi(\dot{x})\}$ is dense below p.

Remark. Cohen's original forcing relation was different. Instead of $\wedge$, Cohen used $\vee$, but with the same template for his definition, and he defined $p \Vdash \exists x\varphi$ if and only if there is some name $\dot{x}$ such that $p \Vdash \varphi(\dot{x})$. These conditions are far more in line with computational and intuitionistic approach, but turned out to be slightly harder to work with (along with other limitations of Cohen's original definition). Luckily, Cohen's forcing relation turned out to be equivalent, in a sense, to the modern one we just defined. That is, both relations have ultimately the same "relationship" with the generic extension.

Proposition 2.11. *1. If $p \Vdash \varphi$ and $q \leq p$, then $q \Vdash \varphi$.*

2. $p \Vdash \varphi$ if and only if $\{q \leq p \mid q \Vdash \varphi\}$ is dense below p.

Proof. We prove these by induction on φ and the rank of the names. Let us deal with (1) first, noting that it also implies the $\Longrightarrow$ direction of (2).

If φ is atomic, say $\dot{x} \in \dot{y}$, the set $\{q \leq p \mid \exists \langle p', \dot{z}\rangle \in \dot{y}, q \leq p' \wedge q \Vdash \dot{z} = \dot{x}\}$ is dense below p, in particular if $q \leq p$, then the set is also dense below q. The case for $\dot{x} = \dot{y}$ is similar. The case for connectives and quantifiers is similar. So indeed, if $q \leq p$, then $q \Vdash \varphi$ as well.

In the case of (2) it remains to show that if $\{q \leq p \mid q \Vdash \varphi\}$ is dense below p, then $p \Vdash \varphi$ as well. The proof is similar. Since for each $q \leq p$ the set of conditions in the definition of $q \Vdash \dot{x} = \dot{y}$ is dense below q, the union of these dense sets is dense below p. □

Exercise 2.12. Show that $p \Vdash \varphi \vee \psi$ if and only if the set $\{q \leq p \mid q \Vdash \varphi \vee q \Vdash \psi\}$ is dense below p.

Exercise 2.13. Show that $p \Vdash \varphi \to \psi$ and $p \Vdash \varphi$ imply $p \Vdash \psi$.

Exercise 2.14. Show that $p \Vdash \forall x \varphi(x)$ if and only if for all $\dot{x}$, $p \Vdash \varphi(\dot{x})$.

Exercise 2.15. Show that if φ is a formula, then $\{p \in \mathbb{P} \mid p \Vdash \varphi \vee p \Vdash \neg\varphi\}$ is a dense open set.

Exercise 2.16. There is no p and φ such that $p \Vdash \varphi \wedge \neg\varphi$.

Exercise 2.17. Suppose that $p \Vdash \varphi(\dot{x}) \wedge \dot{x} = \dot{y}$, then $p \Vdash \varphi(\dot{y})$.

Theorem 2.18 (The Mixing Lemma). *Suppose that $p \Vdash \exists x \varphi(x)$, then there is some $\dot{x}$ such that $p \Vdash \varphi(\dot{x})$.*

Proof. The set $\{q \leq p \mid \exists \dot{x}_q, q \Vdash \varphi(\dot{x}_q)\}$ is dense below p. So it contains a maximal antichain D, for each $q \in D$ let $\dot{x}_q$ be a name such that $q \Vdash \varphi(\dot{x}_q)$, we may assume without loss of generality that if $\langle q', \dot{y}\rangle \in \dot{x}_q$, then $q' \leq q$:

1. If $q' \perp q$, then we may omit the pair entirely.

2. If q' is compatible with q, then we may replace q' by all of their joint extensions.

In particular, if $q, r \in D$ are distinct, then $r \Vdash \dot{x}_q = \check{\varnothing}$. Finally, let $\dot{x} = \bigcup\{\dot{x}_q \mid q \in D\}$.

Claim. *If $q \in D$, then $q \Vdash \dot{x} = \dot{x}_q$.* □

Finally, $p \Vdash \varphi(\dot{x})$, otherwise there is some $r \leq p$ such that $r \Vdash \neg\varphi(\dot{x})$. Since D was a maximal antichain, there is some $q \in D$ such that q is compatible with r, but since $q \Vdash \varphi(\dot{x})$, this is impossible. □

Exercise 2.19. For an atomless forcing notion $\mathbb{P}$, $\mathbb{P}$ is separative if and only if $q \leq p \iff q \Vdash \check{p} \in \dot{G}$.

2.2 The generic model

Theorem 2.20 (The Forcing Theorem).

1. *If $p \Vdash \varphi(\dot{x})$, and G is M-generic such that $p \in G$, then $M[G] \models \varphi(\dot{x}^G)$.*

2. *If $M[G] \models \varphi(\dot{x}^G)$, then there is some $p \in G$ such that $p \Vdash \varphi(\dot{x})$.*

Proof. We begin by proving both of these for atomic formulas, by induction on $\mathbb{P}$-ranks, for both $\dot{x} \in \dot{y}$ and $\dot{x} = \dot{y}$ simultaneously. In the case of (1), suppose that $p \Vdash \dot{x} \in \dot{y}$, then there is some $q \in G$ such that $q \leq p$ and some $\langle p', \dot{z}\rangle \in \dot{y}$ for which $q \leq p'$ and $q \Vdash \dot{x} = \dot{z}$. By the induction hypothesis, $M[G] \models \dot{x}^G = \dot{z}^G$, and since $q \leq p'$, we have that $p' \in G$ so by the very definition of $\dot{y}^G$ we have that $\dot{x}^G = \dot{z}^G \in \dot{y}^G$. The proof for equality is similar.

In the case of (2), suppose that $M[G] \models \dot{x}^G \in \dot{y}^G$, then by definition there is some $p \in G$ and $\dot{z}$ such that $\langle p, \dot{z}\rangle \in \dot{y}$ and $\dot{x}^G = \dot{z}^G$. By the induction hypothesis, there is some $q \in G$ such that $q \Vdash \dot{x} = \dot{z}$, and without loss of generality we can assume $q \leq p$, so $q \Vdash \dot{x} \in \dot{y}$. And again for equality the proof is similar.

We can now prove the theorem for more complicated formulas. $M[G] \models \varphi \wedge \psi$ if and only if $M[G] \models \varphi$ and $M[G] \models \psi$ if and only if there are $p_0, p_1 \in G$ such that $p_0 \Vdash \varphi$ and $p_1 \Vdash \psi$ if and only if there is $q \in G$ such that $q \Vdash \varphi \wedge \psi$. Negation and $\exists x\varphi$ are proved similarly. □

We can characterise the forcing relation externally to our model as well. This will require us to know all the generic filters, which we can do in V, since M is a countable model. This semantic approach to the forcing relation is not very different from the syntactic definition above, which also takes place in the meta-theory.

Corollary 2.21. $p \Vdash \varphi \iff$ *for any M-generic G, such that $p \in G$, $M[G] \models \varphi$.*

Proof. In the forward direction, this is clause (1) of The Forcing Theorem. In the other direction, if $p \nVdash \varphi$, then there is some $q \leq p$ such that $q \Vdash \neg\varphi$. Let G be a generic filter such that $q \in G$, then $p \in G$, but $M[G] \models \neg\varphi$. □

Theorem 2.22. $M[G] \models \mathsf{ZFC}$*, or in other words* $\mathbb{1} \Vdash \mathsf{ZFC}$.

Proof. Using Proposition 2.9, $M[G]$ is transitive so Extensionality and Foundation hold immediately, and since $M \subseteq M[G]$ we also get $\omega \in M[G]$ so Infinity holds as well.

Suppose that $x \in M[G]$, let $\dot{x}$ be a name such that $\dot{x}^G = x$. We define a name $\dot{y}$ as follows:

$$\dot{y} = \{\langle p, \dot{u}\rangle \mid \exists \langle p', \dot{z}\rangle \in \dot{x}, p \leq p' \wedge p' \Vdash \dot{u} \in \dot{z}\}.$$

The problem is that this may very well be a proper class, since we can find $\dot{u}$ satisfying the condition of arbitrarily high $\mathbb{P}$-rank. The solution is to not that we can require the $\mathbb{P}$-rank of $\dot{u}$ to be smaller than the $\mathbb{P}$-rank of $\dot{x}$, which will guarantee it is a set.

Claim. $\dot{y}^G = \bigcup x$.

Proof. Suppose that $u \in \bigcup x$, then there is some $z \in x$ such that $u \in z$. Therefore, there is some $\langle p, \dot{z}\rangle \in \dot{x}$ such that $p \in G$ and $\dot{z}^G = z$, moreover there is some $q \in G$ and $\dot{u}$ such that $\langle q, \dot{u}\rangle \in \dot{z}$. Since the $\mathbb{P}$-rank of $\dot{u}$ is below that of $\dot{x}$, and G is a filter, there is some $r \leq p, q$ such that $r \in G$ and $r \Vdash \dot{u} \in \dot{z}$, so $\langle q, \dot{u}\rangle \in \dot{y}$. Therefore, $\bigcup x \subseteq \dot{y}^G$.

The other direction is trivial: if $u \in \dot{y}^G$, then there is $\langle p, \dot{u}\rangle \in \dot{y}$ with $p \in G$ and so by definition, $p \Vdash \exists z(z \in \dot{x}, \dot{u} \in \dot{z})$, $M[G] \models \dot{y}^G \subseteq \bigcup x$. □

Therefore, from the claim we have that $M[G]$ satisfies Union.

For Power Set it is not hard to verify that if $\dot{x}$ is any name, $\{\langle p, \dot{y}\rangle \mid p \Vdash \dot{y} \subseteq \dot{x}\}$ is a name for the power set of $\dot{x}^G$, and if we cut the $\mathbb{P}$-ranks of these $\dot{y}$s to that of $\dot{x}$ itself, then the argument will also be valid.

For Replacement, suppose that $M[G] \models \forall u \in \dot{x}^G \exists! v \varphi(u, v)$, then there is some $p \in G$ which forces that. Consider the name $\dot{y} = \{\langle q, \dot{v}\rangle \mid \exists \langle p', \dot{u}\rangle \in \dot{x}, q \leq p, p' \wedge q \Vdash \varphi(\dot{u}, \dot{v})\}$. If we can show that $\dot{y}$ is a set, subjected to restricting $\dot{v}$ to those of minimal rank satisfying the property for a fixed $\dot{u}$, then this is certainly enough to prove Replacement in $M[G]$. But, of course, for each $\langle p', \dot{u}\rangle$ and $q \leq p, p'$ the minimal ranked $\dot{v}$s form a set, so there is only a set of potential candidates.

Finally, for the Axiom of Choice, note that $\dot{x}$ is well-orderable and G is well-orderable, and there is, in $M[G]$, a definable function from $\dot{x} \times G \to \dot{x}^G$, with the trivial[3] exception for when $\dot{x}^G = \varnothing$ but $\dot{x}$ is not the empty name. So $\dot{x}^G$ is the image of a well-orderable set, and therefore it is well-orderable as well. □

Remark. Note that we only used the Axiom of Choice to prove that $M[G] \models$ AC. And indeed, if we only had assumed that $M \models$ ZF, we can still prove $M[G] \models$ ZF. On the other hand, it is quite possible that $M[G] \models$ AC while $M \models \neg$AC.

Theorem 2.23. *$M[G]$ and M have the same ordinals.*

Proof. Since $\mathrm{Ord}^M \subseteq \mathrm{Ord}^{M[G]}$ it is enough to show the other inclusion. Suppose that α was the least ordinal in $M[G] \setminus M$, then $\alpha = \mathrm{Ord}^M$, and let $\dot{\alpha}$ be a name for α. Then for each $\beta < \alpha$, there is some $\langle p, \check{\beta}\rangle \in \dot{\alpha}$. However, $\dot{\alpha} \in M$ is a set in M, so this is impossible. □

[3] But equally annoying.

2.3 Morphisms of forcing notions

Definition 2.24. Let $\mathbb{P}$ and $\mathbb{Q}$ be two forcing notions. We say that $\pi\colon \mathbb{Q} \to \mathbb{P}$ is a *projection (of forcing notions)* if:

1. $\pi(\mathbb{1}_{\mathbb{Q}}) = \mathbb{1}_{\mathbb{P}}$.
2. If $q_1 \leq_{\mathbb{Q}} q_0$, then $\pi(q_1) \leq_{\mathbb{P}} \pi(q_0)$.
3. For any $p \in \mathbb{P}$ and $q \in \mathbb{Q}$, if $p \leq_{\mathbb{P}} \pi(q)$, then there is some $q' \leq_{\mathbb{Q}} q$ such that $\pi(q') \leq_{\mathbb{P}} p$.

Remark. We sometimes require that π is surjective, in which case we can require that $\pi(q') = p$ in the third condition and omit the first one.

Proposition 2.25. *Suppose that $\mathbb{P}$ and $\mathbb{Q}$ are two forcing notions and $\pi\colon \mathbb{Q} \to \mathbb{P}$ is a projection. Then for any dense open subset $D \subseteq \mathbb{P}$, $\pi^{-1}(D)$ is a dense subset of $\mathbb{Q}$.*

Proof. Suppose that $q \in \mathbb{Q}$, we want to find some $q' \leq_{\mathbb{Q}} q$ such that $\pi(q') \in D$. Since D is a dense open subset of $\mathbb{P}$, there is some $p \in D$ such that $p \leq_{\mathbb{P}} \pi(q)$. By condition (3) in the definition of a projection, there is some $q' \leq_{\mathbb{Q}} q$ such that $\pi(q') \leq_{\mathbb{P}} p$. Since D is a dense open set, $\pi(q') \in D$ as wanted. □

Corollary 2.26. *Suppose that $\mathbb{P}$ and $\mathbb{Q}$ are two forcing notions and $\pi\colon \mathbb{Q} \to \mathbb{P}$ is a projection. If $G \subseteq \mathbb{Q}$ is a generic filter, then $H = \{p \in \mathbb{P} \mid \exists q \in \mathbb{Q}, \pi(q) \leq_{\mathbb{P}} p\}$ is a generic filter.*

Proof. To see that H is a filter it is enough to show that if $p_0, p_1 \in H$, then there is some $p \in H$ such that $p \leq_{\mathbb{P}} p_0, p_1$. And we may assume that $p_i = \pi(q_i)$ for $q_i \in G$, since we can replace each p_i by a stronger condition which is in $\pi``G$. Since G is a filter, there is some $q \in G$ such that $q \leq_{\mathbb{Q}} q_0, q_1$. Therefore $\pi(q) \in H$ and $\pi(q) \leq_{\mathbb{P}} p_0, p_1$ as wanted.

To see that H is generic, let D be a dense open subset of $\mathbb{P}$, then $\pi^{-1}(D)$ is a dense subset of $\mathbb{Q}$, and therefore there is some $q \in G$ such that $\pi(q) \in D$, so $H \cap D \neq \varnothing$. □

Example 2.27. Let $\mathbb{P}$ be $2^{<\omega}$ and let $\mathbb{Q}$ be the tree $\omega^{<\omega}$, both ordered by reverse inclusion. Consider the function $\pi\colon \mathbb{Q} \to \mathbb{P}$ defined by $\pi(s) = \langle s(i) \pmod 2 \mid i < |s|\rangle$.

Much less trivially, we can find a projection in the other direction, at least if we are willing to restrict to a dense subset of $2^{<\omega}$. Let D be the set of all those sequences which has 1 as their last coordinate (and the empty sequence). Then $\sigma\colon D \to \omega^{<\omega}$, which is defined as follows, is a projection. Given $s \in D$, let $1_s = \{i < \omega \mid s(i) = 1\}$ and for $i \in 1_s \setminus \{|s| - 1\}$ let $i^* = \min\{j \in 1_s \mid j < i\}$. Then we simply define $\sigma(s) = \langle i^* - i \mid i \in 1_s\rangle$.

In simpler words, we consider the letter 1 as a separator, and count the sizes of blocks of consecutive 0s in our sequence.

Definition 2.28. We say that $\pi\colon \mathbb{P} \to \mathbb{Q}$ is a *complete embedding* if:

1. For all $p_0, p_1 \in \mathbb{P}$, $p_1 \leq_{\mathbb{P}} p_0 \iff \pi(p_1) \leq_{\mathbb{Q}} \pi(p_0)$.
2. For all $p_0, p_1 \in \mathbb{P}$, $p_1 \perp_{\mathbb{P}} p_0 \iff \pi(p_1) \perp_{\mathbb{Q}} \pi(p_0)$.
3. For all $q \in \mathbb{Q}$ there is some $p \in \mathbb{P}$ such that whenever $p' \leq_{\mathbb{P}} p$, then $\pi(p') \mathrel{||}_{\mathbb{Q}} q$.

Exercise 2.29. Show that condition (3) can be replaced by "the image of a dense/predense/maximal antichain in $\mathbb{P}$ is predense in $\mathbb{Q}$".

Proposition 2.30. *Suppose that $\pi\colon \mathbb{P} \to \mathbb{Q}$ is a complete embedding, if $G \subseteq \mathbb{Q}$ is a generic filter, then $H = \pi^{-1}(G) \subseteq \mathbb{P}$ is a generic filter as well.*

Proof. We first verify that H is generic, but indeed if $D \subseteq \mathbb{P}$ is a dense set, by the exercise, $\pi``D$ is predense in $\mathbb{Q}$, so there is some $p \in D$ such that $\pi(p) \in G$, and therefore $p \in H$.

Let $p \in H$ and let $p' \in \mathbb{P}$ such that $p \leq_{\mathbb{P}} p'$. Then by definition $\pi(p) \leq_{\mathbb{Q}} \pi(p')$ and $\pi(p) \in G$, therefore $\pi(p') \in G$ and so $p' \in H$. Suppose now that $p, p' \in H$. Then $\pi(p)$ and $\pi(p')$ are both in G, these are compatible and therefore p and p' are compatible in $\mathbb{P}$, however we want to find such a witness inside H.

Consider the set $D = \{r \in \mathbb{P} \mid r \perp_{\mathbb{P}} p \vee r \perp_{\mathbb{P}} p' \vee r \leq_{\mathbb{P}} p, p'\}$. It is not hard to verify that this set is indeed dense, since p and p' are compatible. Therefore $G \cap \pi``D \neq \varnothing$, so there is some $r \in \mathbb{P}$ such that $r \in D$ and $\pi(r) \in G$. However, since $\pi(p)$ and $\pi(p')$ are both in G it has to be the case that $r \leq p, p'$, and so $r \in H$ as well. □

What we see is that if $\mathbb{P}$ embeds into $\mathbb{Q}$ or $\mathbb{Q}$ projects onto $\mathbb{P}$, then any generic extension by $\mathbb{Q}$ will contain a generic extension by $\mathbb{P}$.

Definition 2.31. We say that $\pi\colon \mathbb{P} \to \mathbb{Q}$ is a *dense embedding* if:

1. For all $p_0, p_1 \in \mathbb{P}$, $p_1 \leq_{\mathbb{P}} p_0 \iff \pi(p_1) \leq_{\mathbb{Q}} \pi(p_0)$.
2. For all $q \in \mathbb{Q}$ there is some $p \in \mathbb{P}$ such that $\pi(p) \leq_{\mathbb{Q}} q$.

Proposition 2.32. *Suppose that $\pi\colon \mathbb{P} \to \mathbb{Q}$ is a dense embedding if and only if π is both a projection and a complete embedding.*

Proof. Assume that π is a dense embedding. The fact that π is a projection is trivial to verify. To see that it is a complete embedding, $p \perp_{\mathbb{P}} p'$ implies that there is no $r \leq_{\mathbb{P}} p, p'$, but if there was $q \leq_{\mathbb{Q}} \pi(p), \pi(p')$, then by (2) we had some $\pi(r) \leq_{\mathbb{Q}} q$ and by (1) we would have that $r \leq_{\mathbb{P}} p, p'$. The other direction is similar. Lastly, let $q \in \mathbb{Q}$ be a condition and let $p \in \mathbb{P}$ be such that $\pi(p) \leq_{\mathbb{Q}} q$. Then if $p' \leq_{\mathbb{P}} p$, then not only $\pi(p')$ is compatible with q, it is in fact an extension of q.

In the other direction, if π is a projection and a complete embedding, (1) holds by virtue of π being an embedding. Moreover, since $\pi``\mathbb{P}$ is a predense subset of $\mathbb{Q}$, given any $q \in \mathbb{Q}$, $q \mathrel{||_{\mathbb{Q}}} \pi(p')$ for some p'. Therefore, there is some $q' \leq_{\mathbb{Q}} q, \pi(p')$ and by the definition of a projection we have that there is some $p \in \mathbb{P}$ such that $\pi(p) \leq_{\mathbb{Q}} q'$, and so we have density as well. □

Definition 2.33. We say that $\langle B, \leq \rangle$ is a *Boolean algebra* if it is a partially ordered set with a minimum (denoted by 0_B), maximum (denoted by $\mathbb{1}_B$), every two elements p, q have a least upper bound ($p + q$ or $p \vee q$) and a greatest lower bound ($p \cdot q$ or $p \wedge q$), and for each $p \in B$ there is a unique q such that $p + q = \mathbb{1}_B$ and $p \cdot q = 0_B$.

We say that B is *complete* if every subset of B has a least upper bound ($\sum$ or sup) and a greatest lower bound ($\prod$ or inf).

When we consider a Boolean algebra as a forcing notion we will implicitly omit 0_B from consideration. In particular, when referring to dense embeddings, the meaning will always be to the partial order that is B without 0_B.

Theorem 2.34. *If $\mathbb{P}$ is a separative partial order, then there is a unique (up to isomorphism) complete Boolean algebra, called "the Boolean completion of $\mathbb{P}$" and denoted by $B(\mathbb{P})$, such that $\mathbb{P}$ admits a dense embedding into $B(\mathbb{P})$.*

Proof. Consider the function $i(p) = \{q \in \mathbb{P} \mid q \leq p\}$. We say that a subset U of $\mathbb{P}$ is *regular open* if it satisfies that $p \in U$ if and only if $i(p) \cap U$ is dense open below p. This is equivalent to endowing $\mathbb{P}$ with the downwards-cone topology and requiring that U is equal to $\operatorname{int}(\operatorname{cl}(U))$. We claim that $B(\mathbb{P}) = \{U \subseteq \mathbb{P} \mid U \text{ is regular open}\}$, ordered by inclusion is a complete Boolean algebra.[4] Note that quite trivially, $i(p)$ is always regular since $\mathbb{P}$ is separative.[5]

Suppose that $\mathcal{A} \subseteq B(\mathbb{P})$, then there is a smallest regular open set which contains $\bigcup \mathcal{A}$, namely the interior of the closure of the union. This is easily the sup of our Boolean algebra; and it is enough to show the complement operation exists. And indeed, since $U \in B(\mathbb{P})$ is open, its complement is closed, so we can take its interior, which is a regular open set, which we can check is the complement of U.

Finally, to see that the embedding is dense, simply note that if $A \in B$, then there is some $p \in \mathbb{P}$ such that $i(p) \subseteq A$.

It remains to show that B is unique up to an isomorphism. However, note that for any $A \in B$ we have that $A = \sup\{i(p) \mid p \in A\}$. So if $j \colon \mathbb{P} \to B'$ is a dense embedding to a complete Boolean algebra, we can extend it to a unique isomorphism $j^+ \colon B \to B'$ by that very definition. $\square$

Exercise 2.35. Show that U is a regular open subset of $\mathbb{P}$ if and only if there is some φ_U such that $p \in U$ if and only if $p \Vdash \varphi_U$.

We will tacitly assume that $\mathbb{P} \subseteq B(\mathbb{P})$ by identifying p and $i(p)$, unless stated otherwise.

Theorem 2.36.

1. *Suppose that there is a complete embedding $\mathbb{P} \to \mathbb{Q}$, then there is a projection $\mathbb{Q} \to B(\mathbb{P})$.*
2. *Suppose that there is a projection $\mathbb{Q} \to \mathbb{P}$, then there is a complete embedding $\mathbb{P} \to B(\mathbb{Q})$.*

Proof. Suppose that $\pi \colon \mathbb{P} \to \mathbb{Q}$ is a complete embedding. Let $T(q) = \{p \in \mathbb{P} \mid \pi(p) \perp_{\mathbb{Q}} q\}$, for $q \in \mathbb{Q}$, and let $\tau(q) = -\sup T(q)$, which is well-defined in $B(\mathbb{P})$, and we claim is a projection. First note that $T(\mathbb{1}_{\mathbb{Q}}) = \varnothing$, and so $\tau(\mathbb{1}_{\mathbb{Q}}) = \mathbb{1}_{B(\mathbb{P})}$. Next, note that if $q' <_{\mathbb{Q}} q$, then $T(q) \subseteq T(q')$, and therefore $\sup T(q) \leq_{B(\mathbb{P})} \sup T(q')$, and since negation is order-reversing, $\tau(q') \leq_{B(\mathbb{P})} \tau(q)$. Finally, suppose that $p \leq_{B(\mathbb{P})} \tau(q)$, we need to find some $q' \leq_{\mathbb{Q}} q$ such that $\tau(q') \leq_{B(\mathbb{P})} p$. We may assume, since $\mathbb{P}$ is dense in $B(\mathbb{P})$, that $p \in \mathbb{P}$. As since π is a complete embedding, $\sup(T(\pi(p)) = -p$, and so $\tau(\pi(p)) = p$. If $p \leq_{B(\mathbb{P})} \tau(q)$, then it must be the case that $\pi(p)$ is compatible with q. Let $r \leq_{\mathbb{Q}} \pi(p), q$, then $\tau(r) \leq_{B(\mathbb{P})} \tau(\pi(p)) = p$, as wanted.

Suppose that $\pi \colon \mathbb{Q} \to \mathbb{P}$ is a projection, and again assume that $\mathbb{Q} = B(\mathbb{Q})$, we will deal with the general case at the end. We define $\tau(p) = \sup\{q \mid \pi(q) \leq_{\mathbb{P}} p\}$. We first need to check that τ is an order embedding that preserves incompatibility. Quite immediately from the definition, if $p_1 \leq_{\mathbb{P}} p_0$, we get that $\tau(p_1) \leq_{\mathbb{Q}} \tau(p_0)$. Conversely, suppose that $p_1 \nleq_{\mathbb{P}} p_0$, then by separativity there is some $r \leq p_1$ which is incompatible with p_0. Since $\pi(\mathbb{1}_{\mathbb{Q}}) = \mathbb{1}_{\mathbb{P}}$, there is some q_1 such that $\pi(q_1) \leq_{\mathbb{P}} r$. It is not hard to show that if q_0 is such that $\pi(q_0) \leq_{\mathbb{P}} p_0$, then $q_1 \perp_{\mathbb{Q}} q_0$. At the same time, however, $q_1 \leq_{\mathbb{Q}} \tau(p_1)$, so it is impossible that $\tau(p_1) \leq_{\mathbb{Q}} \tau(p_0)$, as in that case q_1 would have to be compatible with some q_0 such that $\pi(q_0) \leq_{\mathbb{P}} p_0$. The same argument also shows that incompability is preserved by τ.

Finally, we need to verify that if $q \in \mathbb{Q}$, then there is some p such that for all $p' \leq_{\mathbb{P}} p$, $\tau(p')$ is compatible with q. We set $p = \pi(q)$, then if $p' \leq_{\mathbb{P}} \pi(q)$, by definition of projection there is

[4] Note that the Boolean complement is not the set theoretic complement, but rather the interior of the complement!

[5] Indeed, $\mathbb{P}$ is separative if and only if $i(p)$ is a regular open set.

some $q' \leq_{\mathbb{Q}} q$ such that $\pi(q') \leq_{\mathbb{P}} p'$, which means that $q' \leq_{\mathbb{Q}} \tau(p')$ and so the two conditions are compatible.

In the case that $\mathbb{Q}$ is not a complete Boolean algebra, it is still the case that $\mathbb{Q}$ is dense in $B(\mathbb{Q})$, so the checking that τ is an embedding is the same as before. The problem is in the previous paragraph, since $\pi(q)$ might not be well-defined if $q \in B(\mathbb{Q}) \setminus \mathbb{Q}$. However, by density we can take any $q' \in \mathbb{Q}$ which extends q and use $\pi(q')$ for that; or we can instead define the embedding from $B(\mathbb{P})$ instead. □

Corollary 2.37. *If $\mathbb{P}$ and $\mathbb{Q}$ are complete Boolean algebras, then the existence of a projection is equivalence to the existence of an embedding in the other direction.* □

We will write $\mathbb{P} \lessdot \mathbb{Q}$ to mean that there is a complete embedding from $\mathbb{P}$ into $B(\mathbb{Q})$ and $\mathbb{P} \cong \mathbb{Q}$ to mean that $B(\mathbb{P})$ and $B(\mathbb{Q})$ are isomorphic.

Exercise 2.38. Show that $\mathbb{P} \lessdot \mathbb{Q}$ if and only if there is a projection from a dense subset of $\mathbb{Q}$ to $\mathbb{P}$ if and only if there is a complete embedding $\mathbb{P}$ into $\mathbb{Q}$.

Exercise 2.39. Show that $\mathbb{P} \cong \mathbb{Q}$ if and only if there is a dense embedding between them (in one direction or the other).

Remark. Armed with our new understanding of how embeddings behave, we can go back to our projection from $2^{<\omega}$ to $\omega^{<\omega}$. Or at least the one we defined on the dense subset of sequences which end with 1. We could try and extend this embedding to $2^{<\omega}$ by simply omitting the last block of 0s assuming the sequence does not end with a 1; or by adding a "phantom 1" to the sequence.

Neither option works. In the first case, where we trim our sequence, consider $\langle 0, 0, 0\rangle$, which is mapped to $\langle\rangle$ in $\omega^{<\omega}$, then $\langle 2\rangle \leq_{\omega^{<\omega}} \langle\rangle$, but there is no extension of $\langle 0, 0, 0\rangle$ in $2^{<\omega}$ which will be mapped to $\langle 2\rangle$ in $\omega^{<\omega}$, since that would require the third element to be 1 (i.e., the first block of 0s has to be have two elements).

In the other option of adding a "phantom 1" to the end of the sequence we get the dual problem. $\langle 0\rangle$ is mapped to $\langle 1\rangle$, but $\langle 0, 0\rangle \leq_{2^{<\omega}} \langle 0\rangle$ now has to be mapped to $\langle 2\rangle$, so this is no longer (weakly) order preserving.

Instead, if we understand the map into the Boolean completion, we can map $\langle 0\rangle$ into the condition which guarantees that the first coordinate of the generic sequence added by $\omega^{<\omega}$ is non-zero. Similarly, $\langle 0, 1, 0, 0, 0\rangle$ is mapped to the one forcing that the first coordinate is 1, while the second is "at least 3". This can be seen, just as well, as the completion from the dense subset to the entire Boolean completion, and then restricted back to the original $2^{<\omega}$.

Definition 2.40. We say that a condition $p \in \mathbb{P}$ is an *atom* if it does not have any two incompatible extensions.

Exercise 2.41. Suppose that $\mathbb{P}$ is separative, then p is an atom if and only if p is a minimal element.

Theorem 2.42. *Suppose that $\mathbb{P}$ and $\mathbb{Q}$ are separative countable notions of forcings, then $\mathbb{P} \cong \mathbb{Q}$ if and only if $|\{p \in \mathbb{P} \mid p \text{ is an atom}\}| = |\{q \in \mathbb{Q} \mid q \text{ is an atom}\}|$.*

Proof. Using a back-and-forth argument we can show that assuming that $\mathbb{P}$ and $\mathbb{Q}$ are countable Boolean algebras, then they are isomorphic if and only if they have the same number of atoms.. If $\mathbb{P}$ is a countable separative forcing, look at the Boolean algebra it generates inside $B(\mathbb{P})$, and similarly for $\mathbb{Q}$, and apply the back-and-forth argument as necessary. □

Remark. We can provide a robust foundation for forcing, as being done internal to the universe, by defining a "Boolean-valued model" which behaves, in a way, like the class of $\mathbb{P}$-names where the truth of statements is considered in the Boolean completion (rather than the usual $\{\top, \bot\}$). This is a story for a different day.

Chapter 3

Examples: Adding real numbers

We are now ready to ignore the foundations of forcing, and simply, and always, work internal to our universe. This means that we will now talk about V-generic filters, as though V was a countable transitive model in a larger universe. We will also omit V when it will be clear from context.

3.1 Cohen forcing

Definition 3.1. Cohen forcing is the unique atomless countable forcing. We will write $\mathrm{Add}(\omega, 1)$ to denote it, and we will usually consider it as the partial order that consists of $p\colon \omega \to 2$ with $\operatorname{dom} p$ being finite, ordered by reverse inclusion.

As we saw we can represent this forcing as $2^{<\omega}$ or $\omega^{<\omega}$ or $n^{<\omega}$ for any countable $n > 1$. Indeed, any countable partial order will do. And we will see how this can be useful.

Theorem 3.2. *Let $G \subseteq \mathrm{Add}(\omega, 1)$ be a generic filter, and let $c = \bigcup G$, then $c\colon \omega \to 2$ is a new real number. Moreover, $G \in V[c]$, that is, we can reconstruct G from c.*

Proof. It is clear that $c \subseteq \omega \times 2$. To see that $\operatorname{dom} c = \omega$, define for any fixed $n < \omega$ the set $D_n = \{p \in \mathrm{Add}(\omega, 1) \mid n \in \operatorname{dom} p\}$. This is a dense (and open) set, since given any condition p, either n is in the domain of p and $p \in D_n$, or $p \cup \{\langle n, 0\rangle\} \in D_n$ is an extension of p. Therefore, by genericity $\operatorname{dom} c = \omega$. To see now that c is a function, if $n < \omega$ and $\langle n, i\rangle, \langle n, j\rangle \in c$, then there are $p_i, p_j \in G$ such that $p_i(n) = i$ and $p_j(n) = j$. However, G is a filter, so p_i and p_j are compatible and there is some $p \in G$ such that $p_i, p_j \subseteq p$. In particular it must be that $i = j$, since p is a function. As we saw before, given any real number $f \in V$, the set $D_f = \{p \in \mathrm{Add}(\omega, 1) \mid p \nsubseteq f\}$ is dense, and so there is a condition in G which disagrees with f, so $c \neq f$. Finally, working in $V[c]$, it is not hard to see that $G = \{p \in \mathrm{Add}(\omega, 1) \mid p \subseteq c\} = \{c \restriction E \mid E \in [\omega]^{<\omega}\}$. □

Remark. It is often the case that we are adding a particular function or a real and consider $\bigcup G$, where G is our generic filter. We call this function or real number "the associated real".

We say that a real number r is Cohen generic if there a V-generic filter $H \subseteq \mathrm{Add}(\omega, 1)$ such that $r = \bigcup H$.

Theorem 3.3. *Suppose that $x \in V[c]$ is a real number, then either $x \in V$ or there is a Cohen generic r such that $V[x] = V[r]$.*

Proof. Assume that $x \notin V$ and let $\dot{x}$ be a name for x. Let us define a projection from $\mathrm{Add}(\omega, 1)$ by describing an equivalence relation: $p \sim_x q$ if and only if

$$\forall n \in \omega \forall \varepsilon \in 2, \quad p \Vdash \dot{x}(\check{n}) = \check{\varepsilon} \iff q \Vdash \dot{x}(\check{n}) = \check{\varepsilon}.$$

We let $\mathbb{P} = \mathrm{Add}(\omega, 1)/{\sim_x}$ with the ordering given by $[q] \leq_{\mathbb{P}} [p]$ if and only if there is some $q' \in [q]$ and $p' \in [p]$ such that $q' \leq p'$. We claim that this is forcing notion is without atoms. If p is any condition, since $x \notin V$, it is impossible that p have decided all the information about $\dot{x}$, so there is some n such that $p \not\Vdash \dot{x}(\check{n}) = \check{0}$ and $p \not\Vdash \dot{x}(\check{n}) = \check{1}$. Let q_0, q_1 be suitable extensions of p such that $q_\varepsilon \Vdash \dot{x}(\check{n}) = \check{\varepsilon}$, then $q_0 \nsim_x q_1$, and moreover it is impossible for $[q_0]$ and $[q_1]$ to have a compatible extension in $\mathbb{P}$.

We next claim that $p \mapsto [p]$ is a projection. It is easy to see that $[\mathbb{1}_{\mathrm{Add}}] = \mathbb{1}_{\mathbb{P}}$ as the maximum, and by the very definition we have that if $q \leq p$, then $[p] \leq_{\mathbb{P}} [q]$. It remains to verify that if $[p'] \leq_{\mathbb{P}} [p]$ then there is some $q \leq p$ such that $[q] \leq_{\mathbb{P}} [p']$. But indeed, the very definition of $[p'] \leq_{\mathbb{P}} [p]$ was that there is some $q \leq p$ such that $q \sim_x p'$. We can simplify $\mathbb{P}$ by considering instead of $[p]$, the set of partial functions $f\colon \omega \to 2$ such that there is some $p \in \mathrm{Add}(\omega, 1)$ for which $f(n) = \varepsilon \iff p \Vdash \dot{x}(\check{n}) = \check{\varepsilon}$, and ordering these functions by reverse inclusion.

We therefore have that $\mathbb{P}$ is isomorphic to the Cohen forcing. It is enough to show that x is the generic real associated with $\mathbb{P}$ when using the generic H obtained from G under the projection map above. And indeed, if $H \subseteq \mathbb{P}$ is a generic filter, we can define a real $x_H = \bigcup H$ when considering $\mathbb{P}$ presented as partial functions. It is not hard to see, by the definition of this presentation, that $x_H = x$, and that indeed $H = \{f \in \mathbb{P} \mid f \subseteq x\}$. □

Theorem 3.4. *Suppose that $A \in V[c]$ is a set of ordinals, then $A \in V$ or there is a Cohen generic r such that $V[A] = V[r]$.*

Proof. Assume that $A \notin V$ and that $\dot{A}$ is a name for A. We define for each $p \in \mathrm{Add}(\omega, 1)$ the set of ordinals $A_p = \{\xi \in \mathrm{Ord} \mid p \Vdash \check{\xi} \in \dot{A}\}$. The collection $\{A_p \mid p \in \mathrm{Add}(\omega, 1)\}$ is certainly countable, and it is not hard to verify that $A = \bigcup_{p \in G} A_p$. Proceeding in a similar fashion as in the previous proof, we can construct a countable forcing whose generic function is A, or rather its characteristics function, as wanted. □

We will generalise this proof and finish this section by studying an important property of the Cohen forcing: the size of its antichains.

Definition 3.5. We say that a forcing $\mathbb{P}$ satisfies the *countable chain condition (c.c.c.)* if every antichain in $\mathbb{P}$ is countable.

Trivially, since $\mathrm{Add}(\omega, 1)$ is countable, it is c.c.c. Let us prove a general theorem.

Exercise 3.6. $\mathbb{P}$ is c.c.c. if and only if $B(\mathbb{P})$ is c.c.c.

Theorem 3.7. *Suppose that $\mathbb{P}$ is a c.c.c. forcing, if G is a generic filter, then V and $V[G]$ agree on cofinalities (and therefore cardinals).*

Proof. Let α be an ordinal in V. It is enough to argue that if α was regular in V, it remains regular in $V[G]$. If α was singular and $\mathrm{cf}^V(\alpha) \neq \mathrm{cf}^{V[G]}(\alpha)$, then $\mathrm{cf}^V(\alpha)$ must have changed its cofinality as well. And in the case that α was singular and is simply no longer a cardinal, we note that a singular cardinal is a limit cardinal, if $\mu = |\alpha|^{V[G]}$, then $\alpha = (\mu^+)^V$ is a regular cardinal which must also have cardinality μ in $V[G]$ and changed its cofinality.

Suppose that $\mu = \mathrm{cf}^{V[G]}(\alpha)$ and $f\colon \mu \to \alpha$ is a cofinal and increasing function in $V[G]$, for each $\xi < \mu$ there is a maximal antichain D_ξ of conditions which decide the value of $f(\xi)$. Formally, we fix $\dot{f}$ to be a name for f, and without loss of generality we can assume that $\mathbb{1} \Vdash$ "$\dot{f}\colon \check{\mu} \to \check{\alpha}$ is cofinal and increasing" (otherwise, find some $p \in G$ forcing that and consider $\mathbb{P} \restriction p$).

We choose a maximal antichain, D_ξ, inside the dense open set $\{p \mid \exists\beta, p \Vdash \dot{f}(\check{\xi}) = \check{\beta}\}$. By c.c.c., each such antichain must be countable. Let $A_\xi = \{\beta < \alpha \mid \exists p \in D_\xi, p \Vdash \dot{f}(\check{\xi}) = \check{\beta}\}$, then each A_ξ is countable, and since D_ξ is a maximal antichain, these are in fact *all* the possible values for $f(\xi)$. In other words, $\mathbb{1} \Vdash \dot{f}(\check{\xi}) \in \check{A}_\xi$.[6]

Let $g(\xi) = \sup A_\xi$. Since α was uncountable and regular, $g(\xi) < \alpha$. Since f is increasing, it must be that g is non-decreasing; since f was forced to be cofinal, it must also be that g cofinal as well, as clearly $f(\xi) \leq g(\xi)$ for all ξ. But since $g \in V$, and α was regular, and $\mu \leq \alpha$, then it must be that $\mu = \alpha$ as wanted. □

Exercise 3.8. Rewrite the above proof entirely internally to V using the forcing relation.

3.2 Hechler forcing

Definition 3.9. Let $f, g \in \omega^\omega$, we write $f \leq^* g$ to denote that there is some m such that for all $n \geq m$, $f(n) \leq g(n)$. We say, in this case, that g *(eventually) dominates* f.

Exercise 3.10. If $\{f_n \mid n < \omega\} \subseteq \omega^\omega$, then there is some f such that $f_n <^* f$ for all n.

We define the $\mathbb{H}$ forcing as the partial order given by ordered pairs, $p = \langle s, F\rangle$ where $s \in \omega^{<\omega}$ and $F \subseteq \omega^\omega$ is a finite set. The ordering of Hechler forcing is defined as $\langle s_q, F_q\rangle \leq \langle s_p, F_p\rangle$ if and only if

1. $s_p \subseteq s_q$ and $F_p \subseteq F_q$.
2. For any $n \in \mathrm{dom}(s_q \setminus s_p)$ and any $f \in F_p$, $f(n) < s_q(n)$.

We will often refer to s_p *as the stem of* p.

Proposition 3.11. *The Cohen forcing is a projection of* $\mathbb{H}$. □

Let G be a generic for the Hechler forcing, and let d be $\bigcup\{s_p \mid p \in G\}$.

Proposition 3.12. $d\colon \omega \to \omega$. □

Exercise 3.13. Show that $G \in V[d]$. Namely, that we can reconstruct G from the real it defines.

Theorem 3.14. *d dominates any $f\colon \omega \to \omega$ such that $f \in V$.*

Proof. Let f be a function in V, define $D_f = \{\langle s, F\rangle \mid f \in F\}$. We first show that D_f is dense. Indeed, if $\langle s_p, F_p\rangle$ is a condition in $\mathbb{H}$, simply take $\langle s_p, F_p \cup \{f\}\rangle$ as a condition in D_f. Now, if $\langle s, F\rangle \in D_f$, then it must be that any condition extending it must dominate f from $|s|$ onwards. In particular, $f <^* d$. □

Theorem 3.15. *$\mathbb{H}$ is a c.c.c. forcing.*

Proof. It is enough to show that any two conditions with the same stem are compatible, but this is trivial by the definition of the order. □

[6] This is actually a characterisation of c.c.c., in a sense, as we will see in the future.

Theorem 3.16. $\mathbb{H}$ *is not isomorphic to the Cohen forcing.*

Proof. It is enough to show that $\mathrm{Add}(\omega, 1)$ does not add any single function which dominates all the ground model reals. Suppose that $\dot{r}$ was an $\mathrm{Add}(\omega, 1)$-name which was forced to dominate any ground model function.

For each $p \in \mathrm{Add}(\omega, 1)$ define, in the ground model, a decreasing sequence q^p_n and a function f_p such that $q^p_n \Vdash \dot{r} \upharpoonright \check{n} = \check{f}_p \upharpoonright \check{n}$. Let f be a function which dominates all of $\{f_p \mid p \in \mathrm{Add}(\omega, 1)\}$. Suppose that for some n, $p \Vdash \forall m \geq n(\check{f}(\check{m}) < \dot{r}(\check{m}))$, take $m > n$ to be such that $f_p(m) < f(m)$, then we have that $p \Vdash \check{f}_p(\check{m}) < \check{f}(\check{m}) < \dot{r}(\check{m})$. However, $q^p_{m+1} \leq p$ and $q_{m+1} \Vdash \dot{r}(\check{m}) = \check{f}_p(\check{m})$ which is impossible. □

3.3 Collapsing cardinals

We saw two examples of forcings which preserve cardinals, i.e., if κ is a cardinal in V, then it is also a cardinal in $V[G]$, and indeed they have the same cofinality as well.[7] Let us see the most basic example for this failure to fail. We say that a forcing $\mathbb{P}$ *collapses* a cardinal κ, if κ is no longer a cardinal after forcing with $\mathbb{P}$.[8]

Definition 3.17. Let κ be an infinite ordinal, the partial order $\mathrm{Col}(\omega, \kappa)$ is given by finite partial functions $p \colon \omega \to \kappa$, ordered by reverse inclusion.

Exercise 3.18. Show that $\mathrm{Col}(\omega, \kappa) \cong \kappa^{<\omega}$, where the latter is ordered by reverse inclusion; conclude that if κ is a countable ordinal, then $\mathrm{Col}(\omega, \kappa) \cong \mathrm{Add}(\omega, 1)$.

Theorem 3.19. $\mathbb{1} \Vdash |\check{\kappa}| = \aleph_0$.

Proof. Let G be a V-generic filter and let $g = \bigcup G$, we claim that $g \colon \omega \to \kappa$ is a surjection. First, note that for any $n < \omega$, $D_n = \{p \in \mathrm{Col}(\omega, \kappa) \mid n \in \operatorname{dom} p\}$ is a dense open set, and therefore g is defined on all of ω. It is a function by a similar density argument. Finally, for any $\alpha < \kappa$, let $D_\alpha = \{p \in \mathrm{Col}(\omega, \kappa) \mid \alpha \in \operatorname{rng} p\}$, then D_α is a dense open set, as for any p, $p \cup \{\langle n, \alpha\rangle\} \in D_\alpha$ for $n \notin \operatorname{dom} p$. Therefore $\alpha \in \operatorname{rng} g$ for all $\alpha < \kappa$. □

Definition 3.20. We say that a forcing notion $\mathbb{P}$ satisfies the *κ-chain condition* (κ-c.c.) if there are no antichains in $\mathbb{P}$ of size κ.

Exercise 3.21. Suppose that λ is singular, if $\mathbb{P}$ is λ-c.c., then it is κ-c.c. for some $\kappa < \lambda$.

Theorem 3.22. *A λ-c.c. forcing preserves cofinalities $\geq \lambda$. In particular,* $\mathbb{1} \Vdash_{\mathrm{Col}(\omega,\kappa)} \check{\kappa}^+ = \dot{\omega}_1$.

The proof of this theorem is the same as the proof of Theorem 3.7.

Definition 3.23. We say that $\mathbb{Q}$ *absorbs* $\mathbb{P}$ if $\mathbb{1}_\mathbb{Q} \Vdash_\mathbb{Q}$ "$\exists H \subseteq \mathbb{P}, V$-generic".

It is true that if $\mathbb{P} \lessdot \mathbb{Q}$, then $\mathbb{P}$ is absorbed by $\mathbb{Q}$, but the converse need not be true. For example, consider the atomic partial order which is simply an antichain of size $2^{\aleph_1}$ (and a maximum). Since the generic filter is just one of the elements in the antichain, it gets absorbed by every forcing, even by the Cohen forcing which is much smaller. We will see later, however, that if $\mathbb{Q}$ absorbs $\mathbb{P}$, then there is some $p \in \mathbb{P}$ such that $\mathbb{P} \upharpoonright p \lessdot \mathbb{Q}$.

[7] It is possible to change a regular cardinal into a singular cardinal, but this requires large cardinals.

[8] We will sometimes, quite confusingly so, use the term "collapse" to mean that κ became the successor of some other cardinal, that is, in our terminology, we have collapsed all the cardinal in a certain interval.

Exercise 3.24. If $\mathbb{P}$ is a forcing notion of size $|\kappa|$, then $\mathrm{Col}(\omega, 2^\kappa)$ absorbs $\mathbb{P}$.

Theorem 3.25. *Let $\mathbb{P}$ be a forcing notion, then $\mathrm{Col}(\omega, |\mathbb{P}|)$ absorbs $\mathbb{P}$.*

Proof. Let $G \subseteq \mathrm{Col}(\omega, |\mathbb{P}|)$ be a V-generic filter and let $g = \bigcup G$ be the generic surjection it defines. We will use it to define a V-generic filter for $\mathbb{P}$. Simply consider $f(0) = g(0)$ and $f(n+1) = g(m)$ where $m = \min\{k \mid g(k) \leq_{\mathbb{P}} f(n)\}$. We claim that $F = \{p \in \mathbb{P} \mid \exists n, f(n) \leq p\}$ is a V-generic filter. The only non-trivial part is genericity.

Note that if $p \in \mathbb{P}$ and $D \subseteq \mathbb{P}$ is a dense open set (in V), then for any $c \in \mathrm{Col}(\omega, |\mathbb{P}|)$ and for any n, there is some $c' \leq c$ and some $m \geq n$ such that $c'(m) \in D$ and $c'(m) \leq_{\mathbb{P}} p$. Therefore, by the genericity of G we are guaranteed that $F \cap D \neq \varnothing$ for any dense open $D \in V$. □

Theorem 3.26. *Suppose that $\mathbb{P}$ is a forcing notion without atoms such that $|\mathbb{P}| = \kappa$ and $\mathbb{1} \Vdash |\check{\kappa}| = \aleph_0$. Then $\mathbb{P} \cong \mathrm{Col}(\omega, \kappa)$.*

Proof. If $\kappa = \aleph_0$, then we already know that $\mathbb{P} \cong \mathrm{Col}(\omega, \omega) \cong \mathrm{Add}(\omega, 1)$. So we may assume that κ is uncountable. We claim that below any condition p there is a maximal antichain of size κ. Otherwise, there will be some p such that $\mathbb{P} \restriction p$ has κ-c.c., in which case it is impossible that $p \Vdash |\check{\kappa}| = \aleph_0$.

Let $\dot{g}$ be a name such that $\mathbb{1} \Vdash$ "$\dot{g}\colon \check{\omega} \to \dot{G}$ is a surjection".[9] We define a sequence of antichains, C_n, such that if $p \in C_n$, then p decides $\dot{g} \restriction \check{n}$. We let $C_0 = \{\mathbb{1}\}$; suppose that C_n was defined, find below each $p \in C_n$ a maximal antichain of size κ, C^p_{n+1}, such that $q \in C^p_{n+1}$ have decided the value of $\dot{g}(\check{n})$, finally, let C_{n+1} be $\bigcup\{C^p_{n+1} \mid p \in C_n\}$. It is easy to see that C_{n+1} satisfies the recursion hypothesis.

Finally, it is not hard to see that $C = \bigcup\{C_n \mid n < \omega\}$ is a partial order which is isomorphic to $\kappa^{<\omega}$. We claim that it is in fact dense. Since $\mathbb{1}$ forced that $\dot{g}$ is onto $\dot{G}$ and $p \Vdash \check{p} \in \dot{G}$, it must be the case that $p \Vdash \exists n, \dot{g}(n) = \check{p}$. Letting $q \leq p$ be an extension of p which decides the value of such n. There is some $r \in C_{n+1}$ which is compatible with q, since C_{n+1} is a maximal antichain, but that means that $r \Vdash \dot{g}(\check{n}) = \check{p}$ and in particular $r \Vdash \check{p} \in \dot{G}$. But this can only happen if $r \leq p$, and so C is dense. □

Of course, if we collapsed any cardinal, and ω_1 in particular, we have added new real numbers to the universe. First by the virtue of adding a generic for the Cohen forcing, but also if we consider the fact that we now have a real number which codes a well-ordering of ω which is isomorphic to ω_1^V.

We will soon see some conditions that guarantee no new real numbers are added. But first we want to utilise what we have so far and finally prove an important result.

3.4 The failure of the Continuum Hypothesis

Definition 3.27. Let κ be a cardinal. The partial order $\mathrm{Add}(\omega, \kappa)$ is given by finite partial functions $p\colon \kappa \times \omega \to 2$.

Theorem 3.28. *$\mathrm{Add}(\omega, \kappa)$ is a c.c.c. forcing.*

Proof. If κ is finite or countable, then the partial order is countable and there is nothing to check. We can assume, therefore, that κ is uncountable. Let D be a maximal antichain in $\mathrm{Add}(\omega, \kappa)$ and let θ be a large enough regular cardinal (we mainly want $|\mathcal{P}(\mathrm{Add}(\omega, \kappa))| < \theta$ in

[9] Here $\dot{G}$ is the canonical name of the generic filter.

this case). We take $M \prec H(\theta) = \{x \mid |\operatorname{tcl}(x)| < \theta\}$ to be a countable elementary submodel such that $\operatorname{Add}(\omega, \kappa), D \in M$.[10] If p is a condition, let $p_M = p \cap M$, since p is finite to begin with, $p_M \in M$ for all p. We call p_M the "projection" of p into M.

Suppose that $D \nsubseteq M$, which would be the case if it were uncountable. Let $p \in D \setminus M$ be a condition. Since $p_M \in M$, it must be that there is some $q \in D \cap M$ which is compatible with p_M, since M thinks that D is a maximal antichain, and therefore M knows about some $q \in D$ which is compatible with p_M. However, in this case $\operatorname{dom}(p \setminus p_M) \cap M = \varnothing$ and $\operatorname{dom} q \subseteq M$, which means that q is in fact compatible with p to begin with. Therefore it must be that $D \subseteq M$ and therefore countable. □

Remark. Normally, the proof goes through a combinatorial lemma called the "Δ-system lemma" (or the sunflower lemma) which is used to show that given any uncountable set of conditions, there is a finite "root" which uncountably many are disjoint outside that root. This would imply that not only every antichain is countable, but that given any uncountably many conditions, there are uncountably many of them which are pairwise compatible. This property is known as *"Knaster property"*.

Exercise 3.29 (**). Extend the above proof to a proof that $\operatorname{Add}(\omega, \kappa)$ has the Knaster property.

Corollary 3.30. *$\operatorname{Add}(\omega, \kappa)$ is a cofinality-preserving forcing.* □

Theorem 3.31. *If G is a V-generic filter for $\operatorname{Add}(\omega, \kappa)$, then $V[G] \models (\kappa^\omega)^V \geq 2^{\aleph_0} \geq \kappa$.*

Proof. Let G be a V-generic filter and let $g = \bigcup G$, then $g \colon \kappa \times \omega \to 2$. We claim that if $\alpha \neq \beta$, then $g(\alpha, \cdot) \neq g(\beta, \cdot)$ as two real numbers, and that both are not in V. We use the standard density argument. If p is any condition, then there is some large enough n such that $\langle \alpha, n\rangle, \langle \beta, n\rangle$ are both not in the domain of p. Then $p \cup \{\langle\langle \alpha, n\rangle, 0\rangle, \langle\langle \beta, n\rangle, 1\rangle\}$ is a stronger condition which forces that these two are different. Similarly, the proof that $g(\alpha, \cdot) \neq f$ for all $f \in V$ is the same as we have seen before.

It follows now that $2^{\aleph_0} \geq \kappa$. We will show that there are at most κ^ω reals in $V[G]$. If $x \in V[G]$ is a real number, fix a name $\dot{x} \in V$ for x. By c.c.c., for each n there is a countable and maximal antichain $D_{x,n}$ such that $p \in D_{x,n}$ decides the value of $\dot{x}(\check{n})$. Let us define

$$\dot{x}_* = \{\langle p, \langle \check{n}, \check{\varepsilon}\rangle^\bullet\rangle \mid p \in D_{x,n}, p \Vdash \dot{x}(\check{n}) = \check{\varepsilon}\}.$$

Claim. $\mathbb{1} \Vdash \dot{x} = \dot{x}_*$.

Proof of Claim. Suppose not, then there is some $p \Vdash \dot{x} \neq \dot{x}_*$, and by extending if necessary, we can assume there is some $n < \omega$ such that $p \Vdash \dot{x}(\check{n}) \neq \dot{x}_*(\check{n})$. Since $D_{x,n}$ was a maximal antichain, there is some $q \in D_{x,n}$ which is compatible with p. However, by definition we have that $q \Vdash \dot{x}(\check{n}) = \dot{x}_*(\check{n})$. So this is impossible, since if $r \leq p, q$ then $r \Vdash \dot{x}(\check{n}) = \dot{x}_*(\check{n}) \neq \dot{x}(\check{n})$. □

In V, each $\dot{x}_*$ is a countable set. Each name is easily identified with a subset of $\kappa \times \omega \times 2$, so we have that there are at most κ^ω distinct names of this form in V, and therefore the left inequality holds. □

Corollary 3.32. *If $G \subseteq \operatorname{Add}(\omega, \kappa)$ is V-generic, then every real is in V or Cohen over V.* □

Corollary 3.33. *Assume that $V \models \mathsf{GCH}$ and let κ be any cardinal of uncountable cofinality. If $G \subseteq \operatorname{Add}(\omega, \kappa)$ is V-generic, then $V[G] \models 2^{\aleph_0} = \kappa$.* □

[10] It is enough to ask that $\kappa \in M$ in this case, and if κ is definable, e.g. ω_1, this will always be true.

Chapter 4

Combinatorial properties and forcing above the reals

4.1 Closure

Definition 4.1. We say that a partial order $\mathbb{P}$ is *κ-closed* if for any $\gamma < \kappa$ and any descending sequence $\langle p_\alpha \mid \alpha < \gamma\rangle$ of conditions in $\mathbb{P}$, there is some $p \in \mathbb{P}$ such that $p \leq_{\mathbb{P}} p_\alpha$ for all α. In the case where $\kappa = \omega_1$, we will often use "σ-closed".

Theorem 4.2. *If $\mathbb{P}$ is σ-closed, then $\mathbb{P}$ does not add new functions with domain ω into the ground model.*[11] *In particular, no new reals are added and ω_1 is not collapsed.*

Proof. Suppose that $\dot{f}$ is a $\mathbb{P}$-name and $p \Vdash \dot{f}\colon \check{\omega} \to \check{X}$ for some set X. We define by recursion a descending sequence, $p_0 = p$ and $p_{n+1} \leq p_n$ is some extension such that $p_{n+1} \Vdash \dot{f}(\check{n}) = \check{x}_n$ for some $x_n \in X$. We have a countable descending sequence, so by σ-closure we have some q which stronger than all of the conditions on the sequence. Therefore, for all n, $q \Vdash \dot{f}(\check{n}) = \check{x}_n$. In other words, $q \Vdash \dot{f} = \check{g}$, where $g(n) = x_n$. □

Exercise 4.3. Show that if $\mathbb{P}$ is κ-closed, then $\mathbb{P}$ does not add sequences of ground model elements of length $< \kappa$. In particular, if $\mathbb{P}$ is κ-closed, it will not add any bounded subsets to κ.

Exercise 4.4. Show that if $\mathbb{P}$ is κ-closed and has the κ-c.c. then $\mathbb{P}$ is trivial, i.e. the set of atoms is dense. Conclude that if $\mathbb{P}$ is ω-c.c., then it is trivial.

Exercise 4.5. Show that an atomless complete Boolean algebra is *never* κ-closed for $\kappa > \omega$.[12]

Definition 4.6. Let κ and λ be two cardinals with κ infinite. Then $\mathrm{Add}(\kappa, \lambda)$ is the partial order whose conditions are partial functions $p\colon \lambda \times \kappa \to 2$ with $|p| < \kappa$, ordered by reverse inclusion.

Exercise 4.7. If $0 < \lambda \leq \kappa$, then $\mathrm{Add}(\kappa, \lambda) \cong \mathrm{Add}(\kappa, 1) \cong \kappa^{<\kappa}$.

Exercise 4.8. Suppose that κ is a singular cardinal, then $\mathrm{Add}(\kappa, 1)$ collapses κ to its cofinality.

Theorem 4.9. *Let κ be an infinite regular cardinal and let $\lambda > 0$. Then $\mathrm{Add}(\kappa, \lambda)$ is κ-closed and it has $(\kappa^{<\kappa})^+$-c.c. In fact, if G is a V-generic filter, then $V[G] \models \kappa^{<\kappa} = \kappa$.*

[11] We will refer to such a function as a "sequence of ground model elements".
[12] In this case we will usually talk about a "κ-closed dense subset" instead.

Proof. We can split this into two cases. The first where $\lambda \leq \kappa$. In this case, we can assume that $\lambda = 1$. Note that $|\operatorname{Add}(\kappa, 1)| \leq \kappa^{<\kappa}$, since any condition is a subset of size $< \kappa$ of $\kappa \times 2$. And so in this case, the chain condition is trivially true. To see that the forcing is κ-closed, we use the regularity of κ to observe that if $\gamma < \kappa$ and $\{p_\alpha \mid \alpha < \gamma\}$ is a descending sequence of conditions, then $p = \bigcup_{\alpha<\gamma} p_\alpha$ is a condition.

Finally, in this case, take $f \colon \gamma \to 2$ for some $\gamma < \kappa$, coding a bounded subset of κ, then given any $p \in \operatorname{Add}(\kappa, 1)$ we can extend p by defining first the shift of f by an ordinal α, namely, $f_\alpha = \{\langle \alpha+\beta, f(\beta)\rangle \mid \beta < \gamma\}$, and then simply taking $\alpha = \sup \operatorname{dom} p + 1$ and note that $p \cup f_\alpha$ is a condition extending p. Therefore, by a density argument, if G is a V-generic filter and $g = \bigcup G$, then for any $\gamma < \kappa$ and $f \colon \gamma \to 2$ there is some $\alpha < \kappa$ such that $g \restriction [\alpha, \alpha + \gamma) = f_\alpha$.

In the case where $\lambda > \kappa$, it remains to only prove the forcing is still $(\kappa^{<\kappa})^+$-c.c., which can be done exactly the same proof as Theorem 3.28, this time taking models of size κ instead of countable models. □

Corollary 4.10. $\operatorname{Add}(\omega_1, 1)$ *forces that the Continuum Hypothesis is true.*

Proof. Since $\operatorname{Add}(\omega_1, 1)$ is σ-closed, it does not add new real numbers. However, it does force that $\aleph_1^{\aleph_0} = \aleph_1$ and therefore it forces that CH holds. □

Definition 4.11. Let $\kappa \leq \lambda$ be two infinite cardinals, $\operatorname{Col}(\kappa, \lambda)$ is the forcing whose conditions are partial functions $p \colon \kappa \to \lambda$ with $|\operatorname{dom} p| < \kappa$.

Exercise 4.12. $\operatorname{Col}(\kappa, \kappa) \cong \operatorname{Add}(\kappa, 1)$. In particular, if κ is singular, $\operatorname{Col}(\kappa, \kappa)$ collapses κ.

Exercise 4.13. If κ is regular, then $\operatorname{Col}(\kappa, \lambda)$ is κ-closed and has $(\lambda^{<\kappa})^+$-c.c.

Remark. Unlike the case for ω, $\operatorname{Col}(\kappa, \lambda)$ need not absorb other forcings or be the unique collapse of λ.

4.2 Distributivity

Definition 4.14. We say that $\mathbb{P}$ is *κ-distributive* if whenever $\gamma < \kappa$ and $\{D_\alpha \mid \alpha < \gamma\}$ is a family of dense open subsets of $\mathbb{P}$, $\bigcap_{\alpha<\gamma} D_\alpha$ is a dense open subset of $\mathbb{P}$.[13]

Exercise 4.15. Given a family of maximal antichains $\mathcal{A}$ in a forcing $\mathbb{P}$ a *refinement* is a maximal antichain A such that for all $D \in \mathcal{A}$, if $q \in A$ is compatible with $p \in D$, then $q \leq p$. Show that $\mathbb{P}$ is κ-distributive if and only if every family of $< \kappa$ maximal antichains has a refinement. Conclude that every finite family of maximal antichains has a refinement.

Theorem 4.16. $\mathbb{P}$ *is κ-distributive if and only if* $\mathbb{P}$ *does not add sequences of ground model elements of length γ for any $\gamma < \kappa$.*

Proof. Suppose that $\mathbb{P}$ is distributive and let $\dot{f}$ be a name such that $\mathbb{1} \Vdash \dot{f} \colon \check{\gamma} \to \check{X}$ for some $\gamma < \kappa$ and $X \in V$. For every $\alpha < \gamma$, the set $D_\alpha = \{q \leq p \mid \exists x \in X, q \Vdash \dot{f}(\check{\alpha}) = \check{x}\}$ is a dense open set below p. By κ-distributivity, $D = \bigcap_{\alpha<\gamma} D_\alpha$ is dense. Defining for $q \in D$, $f_q(\alpha) = x$ if and only if $q \Vdash \dot{f}(\check{\alpha}) = \check{x}$ provides us with a function in the ground model such that $q \Vdash \dot{f} = \check{f}_q$. Therefore, $\mathbb{1}$ must have already forced that $\dot{f}$ is going to be in the ground model.

[13]Note that the intersection of open sets is always open in the context of forcing. It is enough to talk about density.

In the other direction, suppose that $\gamma < \kappa$ and $D_\alpha = \{p_i^\alpha \mid i < \eta_\alpha\}$ is a maximal antichain for $\alpha < \gamma$. Define the following name,

$$\dot{f} = \{\langle p_i^\alpha, \langle \check{\alpha}, \check{p}_i^\alpha \rangle^\bullet \rangle \mid i < \eta_\alpha, \alpha < \gamma\}.$$

If G is a V-generic filter, since each D_α is a maximal antichain, $\dot{f}^G(\alpha)$ is the unique element of $D_\alpha \cap G$. In particular, $\dot{f}$ is a name for a sequence of ground model elements of length γ. By the assumption, $\dot{f}^G$ is in V for any generic G. In other words, $\mathbb{1}$ must force this name interprets as a ground model function. Let D be a maximal antichain such that for any $q \in D$ there is some f_q such that $q \Vdash \dot{f} = \check{f}_q$. It is not hard to verify that D is a refinement of $\{D_\alpha \mid \alpha < \gamma\}$. □

Corollary 4.17. *If $\mathbb{P}$ is κ-closed, then it is κ-distributive.* □

Remark. We used the Axiom of Choice quite heavily in this proof. The statement "$\mathbb{P}$ is κ-closed implies $\mathbb{P}$ is κ-distributive" is itself equivalent to a choice principle known as $\mathsf{DC}_{<\kappa}$. Moreover, $\mathsf{ZF} + \mathsf{DC}_\lambda$ cannot prove the equivalence in Theorem 4.16 (for any κ!!!).

4.3 Club shooting

Recall that a subset $A \subseteq \kappa$ is closed if for all $\beta < \kappa$, $\sup(A \cap \beta) \in A$. We say that A is a *club* if it is closed and unbounded if it is a closed subset of κ and $\sup A = \kappa$. We say that $A \subseteq \kappa$ is *stationary* if it has a non-empty intersection with every club.

Theorem 4.18. *Suppose that $S \subseteq \omega_1$ is unbounded. We define* $\mathrm{Club}(S)$ *to be the forcing whose conditions are closed subsets of S ordered by end-extensions. Namely, $q \leq p$ if and only if $q \cap \max p + 1 = p$. Then:*

1. *If G is a V-generic filter, then $\bigcup G$ is a club which is a subset of S. In particular, if S is co-stationary,* $\mathrm{Club}(S)$ *destroys the stationarity of $\omega_1 \setminus S$.*
2. $\mathrm{Club}(S)$ *is σ-closed if and only if S is a club.*
3. $\mathrm{Club}(S)$ *is σ-distributive if and only if S is stationary. In particular,* $\mathrm{Club}(S)$ *does not collapse ω_1 when S is stationary.*

We call the forcing $\mathrm{Club}(S)$ "club shooting". This forcing can be generalised to arbitrary regular cardinal κ and will have similar properties when S is a fat stationary set. Where a stationary set S is *fat* if for all $\alpha < \kappa$ and all clubs $C \subseteq \kappa$, there is a closed subset of $S \cap C$ of order type α. In the case where $\kappa = \omega_1$, all stationary sets are fat.

Proof. The first property is easily proved by the usual density arguments. We will prove the other two. It is also not hard to see that if S is a club and $\{p_n \mid n < \omega\}$ is a descending sequence of conditions, then let $p^- = \bigcup\{p_n \mid n < \omega\}$, if it is a condition in $\mathrm{Club}(S)$, we are done, otherwise by S being closed, $p = p^- \cup \{\sup p^-\}$ is a condition. Similarly, if S is not a club, as it is unbounded, it is not closed. Letting α be the least ordinal not in S for which $\sup(\alpha \cap S) = \alpha$, then there is a sequence $\alpha_n \in S$ such that $\sup \alpha_n = \alpha$, then defining $p_n = S \cap (\alpha_n + 1)$ is a sequence of closed subsets of S, but there is no $p \subseteq S$ such that $p_n \subseteq p$ for all $n < \omega$.

For the third property, if S is non-stationary, let C be a club disjoint from S, as we added a club subset of S, in $V[G]$ we can write ω_1 as the union of two closed and unbounded sets, but this is impossible if ω_1 is regular, so it mus have collapsed and new reals were added.

Finally, suppose that S is stationary and let D_n be dense open sets for $n < \omega$. Given some $p \in \mathrm{Club}(S)$ we find some countable elementary submodel $M \prec H(\theta)$ for a regular cardinal $\theta > 2^{2^{\aleph_1}}$, such that S, p, and the sets D_n are all members of M, and $\delta = M \cap \omega_1 \in S$. The reason we can find such M with $\delta \in S$ is that S is stationary, so we may start with some M_0 satisfying the other requirements, take elementary extensions, M_α, with more and more countable ordinals, and take unions at limit steps. Since this chain is elementary, it is not hard to see that the set of $M_\alpha \cap \omega_1$ forms a club, so we can find M with $\delta \in S$.

Since M is countable we may assume without loss of generality that $\{D_n \mid n < \omega\}$ are all the dense open sets of $\mathrm{Club}(S)$ inside M. We can find a descending sequence in $\mathrm{Club}(S) \cap M$ of conditions such that $p_n \in D_n$ and $p_0 \leq p$. Let $q = \bigcup\{p_n \mid n < \omega\} \cup \{\delta\}$. We claim that $q \in \mathrm{Club}(S)$. For this it is enough to show that $\sup_{n<\omega} \max p_n = \delta$. But since $\mathbb{1} \Vdash \sup \bigcup \dot{G} = \check{\omega}_1$, this must be true in $H(\theta)$ and therefore in M, so for any $\alpha < \delta$, the set of conditions with ordinals above α is a dense open set inside M, so it must be one of the D_ns. Since each D_n was dense and open, it has to be that q lies in their intersection, as wanted. □

Theorem 4.19. *Let κ be a regular cardinal, $\mathbb{P}$ a forcing, and $G \subseteq \mathbb{P}$ be a V-generic filter.*

1. *If $\mathbb{P}$ is κ-.c.c., then every club $C \subseteq \kappa$ in $V[G]$ contains a club from V. Therefore $\mathbb{P}$ preserves stationary subsets of κ.*
2. *If $\mathbb{P}$ is κ-closed, then it preserves stationary subsets of κ.*

In particular, c.c.c. forcings preserve clubs and stationary sets, so $\mathrm{Club}(S)$ *is not c.c.c.*

Proof. Suppose that $\mathbb{P}$ is κ-c.c., and let $p \in \mathbb{P}$ and $\dot{C}$ be such that $p \Vdash$ "$\dot{C} \subseteq \check{\kappa}$ is a club". We define $D = \{\alpha < \kappa \mid p \Vdash \check{\alpha} \in \dot{C}\}$, and we claim that D is a club. If $\sup D \cap \beta = \beta$, then $p \Vdash \sup \dot{C} \cap \check{\beta} = \check{\beta}$, and therefore $\beta \in D$, so D is closed. We need to show that it is unbounded.

Let $\alpha_0 < \kappa$ we define a sequence of ordinals by recursion. Suppose that α_n was defined, since p forces $\dot{C}$ to be unbounded, there is a maximal antichain below p, D_n, such that if $q \in D_n$, then for some $\gamma_q > \alpha_n$, $q \Vdash \check{\gamma}_q \in \dot{C}$, use the mixing lemma to create a name $\dot{\gamma}_n$. By κ-c.c. and the regularity of κ, $\alpha_n = \sup \gamma_q$. Since $p \Vdash \dot{\gamma}_n \in \dot{C}$ for all $n < \omega$, then $p \Vdash \sup \dot{\gamma}_n \in \dot{C}$ as well. However, $p \Vdash \dot{\gamma}_n \leq \check{\alpha}_n < \dot{\gamma}_{n+1}$, so $p \Vdash \sup \check{\alpha}_n = \sup \dot{\gamma}_n$, as wanted.

Suppose that $\mathbb{P}$ is κ-closed, if $S \subseteq \kappa$ was stationary in V, let $\dot{C}$ be a name such that $p \Vdash$ "$\dot{C} \subseteq \check{\kappa}$ is a club". We construct a decreasing sequence of conditions of length κ, p_α, such that $p_0 = p$ and p_α decided the first $\alpha+1$ members of $\dot{C}$. Let D be the club $\{\xi < \kappa \mid \exists \alpha, p_\alpha \Vdash \check{\xi} \in \dot{C}\}$, then $D \cap S \neq \varnothing$. So p has an extension which forces $\dot{C} \cap \check{S} \neq \varnothing$. □

Exercise 4.20. Let $\mathbb{P}$ be the forcing whose conditions are pairs $\langle s, \mathcal{F} \rangle$ where $s \subseteq \omega_1$ is a closed and bounded set and $\mathcal{F}$ is a countable family of clubs. Define $\langle s_q, \mathcal{F}_q \rangle \leq \langle s_p, \mathcal{F}_p \rangle$ when $s_q \cap \max s_p + 1 = s_p$ and $\mathcal{F}_p \subseteq \mathcal{F}_q$, and $s_q \setminus s_p \subseteq \bigcap \mathcal{F}_p$. Show that $\mathbb{P}$ is σ-closed and adds a club. Moreover, show that this club is almost contained in any other club from V. (Compare this to Hechler forcing.)

4.4 Suslin trees

Definition 4.21. We say that a tree T is a *Suslin tree* if it has height ω_1 and no uncountable antichains.[14,15] We will implicitly assume that T is a *normal* tree: every node has an extension to any level of the tree, every node is a splitting node, and any chain in the tree has at most a single upper bound in a given level. When we force with the tree we take its reverse order.

[14] Note that antichains in the "order sense" match with the "forcing sense" when the partial order is a tree.

[15] Since every level is an antichain, all levels are countable. So a Suslin tree is an *Aronszajn tree*.

Theorem 4.22. *Suppose that T is a Suslin tree, then T is σ-distributive and c.c.c., in particular it is not σ-closed.*

Proof. That T is a c.c.c. forcing is by definition. Suppose that $\{D_n \mid n < \omega\}$ is a family of maximal antichains, since each is countable there is α_n such that D_n is contained in the levels below α_n. By regularity of ω_1, $\alpha = \sup_{n<\omega} \alpha_n$ is a countable ordinal, taking the αth level of the tree is now a refinement of $\{D_n \mid n < \omega\}$. □

Theorem 4.23. *Suppose that $\mathbb{P}$ adds a subset to ω_1 and is σ-distributive and c.c.c., then $\mathbb{P}$ projects onto a Suslin tree.*

Proof. Let $\dot{f}$ be a name such that $\mathbb{1} \Vdash \dot{f}\colon \check{\omega}_1 \to \check{2}$ and that it is not in the ground model. Since $\mathbb{P}$ is σ-distributive, it must be that all of its initial segments lie in the ground model. However, since $\mathbb{P}$ is c.c.c. there can only be countably initial segments of any given length. Let $T_\alpha = \{t\colon \alpha \to 2 \mid \exists p \in \mathbb{P}, p \Vdash \check{t} = \dot{f} \restriction \alpha\}$, then $T = \bigcup\{T_\alpha \mid \alpha < \omega_1\}$ is a Suslin tree and the function mapping $p \in \mathbb{P}$ to its maximally decided initial segment is the projection. □

Exercise 4.24 (*). In the previous theorem, show that if $\mathbb{P}$ has (a dense subset of) size $\aleph_1$, then $\mathbb{P}$ is forcing equivalent to a Suslin tree.

One is left to wonder. Are there Suslin trees in the universe? If $V = L$ holds, then the answer is yes; more generally if $\diamondsuit$ holds, then the answer is yes. However, it is consistent that the answer is no, as we will see later. Regardless, one can always add Suslin trees to the universe.

Theorem 4.25. *There is a σ-closed forcing which adds a Suslin tree.*

Proof. Let $\mathbb{P}$ be the forcing whose conditions are countable, ω-splitting, normal trees of countable height, ordered by end-extension. Namely, if T_0, T_1 are two conditions in $\mathbb{P}$, then $T_1 \leq T_0$ when there is some α such that $T_0 = T_1 \restriction \alpha$.

It is not hard to see that $\mathbb{P}$ is σ-closed and that if G is a V-generic filter and $T = \bigcup G$, then T is a normal tree of height ω_1. Suppose that $\dot{A}$ was a name such that $T_0 \Vdash$ "$\dot{A} \subseteq \dot{T}$ is a maximal antichain". We will show that there is some $T_* \leq T_0$ such that T_* decides all the values of $\dot{A} \cap \check{T}_*$ and that it is a maximal antichain there. In this case, any further extension cannot add new elements to $\dot{A}$, since those will be have to extend some node from T_*, and therefore some node in that intersection. Therefore, this would mean that T_* will force that $\dot{A}$ is in fact a subset of T_* and therefore countable.

To find this T_* we first assume that T_0 has a maximal level, α_0, or else we can extend it to one which has. We enumerate this level as $\{t_n \mid n < \omega\}$, and then we define a decreasing sequence of conditions, $T_{0,n}$ such that $T_{0,n}$ contains t for which $T_{0,n} \Vdash \check{t} \in \dot{A} \wedge \check{t}_n \mid\mid \check{t}$. Let T_1 be a lower bound of the $T_{0,n}$ which has a maximal level, α_1. Repeat the process by recursion and let $T_* = \bigcup_{n<\omega} T_n$. We claim that T_* is as wanted.

If $t \in T_*$, then there is some T_n such that $t \in T_n$, therefore in T_{n+1} we have decided a condition in $\dot{A}$ which is compatible with t, so this condition must already be in T_*, and T_* must agree with that decision. Therefore $\{t \in T_* \mid T_* \Vdash \check{t} \in \dot{A}\}$ is a maximal antichain in T_*. So by density argument, any antichain in T, the generic tree, is bounded. □

Remark. The above is quite similar to the proof that L has a $\diamondsuit$ sequence or that $\diamondsuit$ provides us with a Suslin tree. The reason is that we can use $<_L$ or the $\diamondsuit$ in lieu of a generic to provide us with these sort arguments.

Remark. It is a far more difficult and involved construction, but $\mathrm{Add}(\omega, 1)$ also adds a Suslin tree.

Chapter 5

Combining forcing notions in the ground model

5.1 Finite products

Definition 5.1. Suppose that $\mathbb{P}$ and $\mathbb{Q}$ are two notions of forcing. We define the order on $\mathbb{P}\times\mathbb{Q}$ as $\langle p_1, q_1\rangle \leq \langle p_0, q_0\rangle$ if and only if $p_1 \leq p_0$ and $q_1 \leq q_0$.

Exercise 5.2. $\mathbb{P}, \mathbb{Q} \lessdot \mathbb{P}\times\mathbb{Q} \cong \mathbb{Q}\times\mathbb{P}$.

Theorem 5.3. *Suppose that $G\times H$ is a V-generic filter for $\mathbb{P}\times\mathbb{Q}$, then G is in fact $V[H]$-generic and H is $V[G]$-generic.*

Proof. We will show that H is $V[G]$-generic. Suppose that $D \in V[G]$ is a dense open subset of $\mathbb{Q}$, then D has a $\mathbb{P}$-name, $\dot{D}$, and without loss of generality $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}}$ "$\dot{D} \subseteq \check{\mathbb{Q}}$ is dense". Let $D_* = \{\langle p, q\rangle \mid p \Vdash_{\mathbb{P}} \check{q} \in \dot{D}\}$, then since $\dot{D}$ is forced to be dense, for any $\langle p, q\rangle \in \mathbb{P}\times\mathbb{Q}$ there is some $\langle p', q'\rangle$ such that $p' \leq p$, $q' \leq q$ and $p' \Vdash \check{q}' \in \dot{D}$. In other words, D_* is a dense subset of $\mathbb{P}\times\mathbb{Q}$. Since $G\times H$ is V-generic for the product, there is some $\langle p, q\rangle \in D_* \cap G\times H$, but then $p \Vdash_{\mathbb{P}} \check{q} \in \dot{D}$, and since $p \in G$ we have that $D \cap H \neq \varnothing$. $\square$

Corollary 5.4. *If $\mathbb{P}$ is any atomless forcing and G is a V-generic filter, then $G\times G$ is not V-generic for $\mathbb{P}\times\mathbb{P}$.* $\square$

Definition 5.5. Let $G \subseteq \mathbb{P}$ and $H \subseteq \mathbb{Q}$ be two filters. We say that G and H are *mutually V-generic* if G is $V[H]$-generic and H is $V[G]$-generic.

We can now rephrase Theorem 5.3 as stating that if $G\times H$ is a generic filter for the product, then G and H are mutually generic. The following theorem shows that the converse is also true.

Theorem 5.6. *Suppose that $G \subseteq \mathbb{P}$ and $H \subseteq \mathbb{Q}$ are mutually generic, then $G\times H$ is generic.*

Proof. Suppose that $D \subseteq \mathbb{P}\times\mathbb{Q}$ is a dense set in V. Define, in $V[G]$,

$$D_G = \{q \in \mathbb{Q} \mid \exists p \in G, \langle p, q\rangle \in D\},$$

then we claim that D_G is a dense subset of $\mathbb{Q}$. If this is indeed the case, then $D_G \cap H \neq \varnothing$, and so there is some $\langle p, q\rangle \in D \cap G\times H$.

If $q_0 \in \mathbb{Q}$, then the set $D_{q_0} = \{p \in \mathbb{P} \mid \exists q \leq q_0, \langle p, q\rangle \in D\}$ is a dense subset of $\mathbb{P}$. Indeed, if $p_0 \in \mathbb{P}$ is any condition, there is some $\langle p, q\rangle \leq \langle p_0, q_0\rangle$ such that $\langle p, q\rangle \in D$. Let $p \in G \cap D_{q_0}$, then there is some $q \leq q_0$ such that $\langle p, q\rangle \in D$, which means that $q \in D_G$. Therefore D_G is dense, so there is some $q \in D_G \cap H$, as H is $V[G]$-generic. Therefore $\langle p, q\rangle \in D \cap G \times H$. □

Exercise 5.7. Suppose that $\mathbb{P}$ and $\mathbb{Q}$ are κ-closed, then $\mathbb{P} \times \mathbb{Q}$ is κ-closed.

Exercise 5.8. If T is a Suslin tree, then $T \times T$ is not c.c.c.

Remark. It is consistent that there is a Suslin tree T such that $T \times T$ collapses ω_1, which in particular shows that the product of two σ-distributive forcings need not be σ-distributive. It is also provable that if T is Suslin, then there is some c.c.c. forcing $\mathbb{P}$ such that $\mathbb{P} \times T$ collapses ω_1, but $\mathbb{P}$ need not be isomorphic to T.

Theorem 5.9. *Suppose that $\mathbb{P}$ is κ-c.c. and $\mathbb{Q}$ is κ-closed, then $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\check{\mathbb{Q}}$ is $\check{\kappa}$-distributive".*

The proof will follow immediately from the two lemmas below.

Lemma 5.10. *Suppose that $\mathbb{P}$ is κ-c.c. and $\mathbb{Q}$ is κ-distributive. If $\mathbb{1}_{\mathbb{Q}} \Vdash$ "$\check{\mathbb{P}}$ is $\check{\kappa}$-c.c.", then $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\check{\mathbb{Q}}$ is $\check{\kappa}$-distributive".*

Proof. Suppose that $\dot{f}$ is a $\mathbb{P} \times \mathbb{Q}$-name for a sequence of ordinals of length $\gamma < \kappa$. Let H be a V-generic filter for $\mathbb{Q}$, then as per the assumptions we have that in $V[H]$ we still have that $\mathbb{P}$ is κ-c.c., and that $\dot{f}^H$ is a $\mathbb{P}$-name for a sequence of ordinals of length $< \kappa$. For any $\alpha < \gamma$ there is a maximal antichain A_α of conditions in $\mathbb{P}$ which decide the value of $\dot{f}^H(\check{\alpha})$, and by κ-c.c., $|A_\alpha| < \kappa$. We can therefore restrict $\dot{f}^H$ into a name of size $< \kappa$ in $V[H]$, and therefore we get that without loss of generality, $\dot{f}^H \in V$.

It follows that any γ-sequence of ordinals, for $\gamma < \kappa$, in $V[G \times H]$ must be in $V[G]$. Therefore, $V[G] \models$ "$\mathbb{Q}$ is κ-distributive", and as we took an arbitrary V-generic G, this is forced by $\mathbb{1}_{\mathbb{P}}$. □

Lemma 5.11. *Suppose that $\mathbb{P}$ is κ-c.c. and $\mathbb{Q}$ is κ-closed. Then $\mathbb{1}_{\mathbb{Q}} \Vdash$ "$\check{\mathbb{P}}$ is $\check{\kappa}$-c.c."*

Proof. Let $\dot{A}$ be a $\mathbb{Q}$-name for a subset of $\mathbb{P}$ of size κ. Define by recursion a descending sequence q_α deciding the first α elements of $\dot{A}$. Now consider $\{p \in \mathbb{P} \mid \exists \alpha < \kappa, q_\alpha \Vdash \check{p} \in \dot{A}\}$. Since this set in V and must have size κ, so it is not an antichain. In particular, there is some p_α, p_β, witnessed to be in the set by q_α and q_β respectively, which are compatible. Without loss of generality, $\alpha < \beta$, so $q_\beta \Vdash \check{p}_\alpha, \check{p}_\beta \in \dot{A}$, and therefore q_β cannot force that $\dot{A}$ is an antichain. In particular, any condition extends to one which forces this, so $\dot{A}$ cannot be forced to be an antichain by any condition. □

Proposition 5.12. *Suppose that $\mathbb{P}$ is σ-closed and atomless. Then any forcing $\mathbb{Q}$ that adds a real, and in particular* $\operatorname{Add}(\omega, 1)$*, forces that $\mathbb{P}$ is not σ-closed.*

Proof. Since $\mathbb{P}$ is atomless, every condition has at least two incompatible extensions. We construct a copy of $2^{<\omega}$ by recursion starting with $p_0 = \mathbb{1}_{\mathbb{P}}$; if $s \in 2^{<\omega}$ and p_s was defined, let $p_{s\frown 0}$ and $p_{s\frown 1}$ be two incompatible extensions of p_s. Since $\mathbb{P}$ is σ-closed in V, for every $f\colon \omega \to 2$ the sequence $\langle p_{f\restriction n} \mid n < \omega\rangle$ has a lower bound. Moreover, if $p \in \mathbb{P}$ is a lower bound of any infinitely many p_s, then it defines a function $f\colon \omega \to 2$.

If c is the real added by $\mathbb{Q}$, then $\{p_{c\restriction n} \mid n < \omega\}$ is a descending sequence of conditions in $\mathbb{P}$ without a lower bound. □

5.2 General products

Definition 5.13. Suppose that $\{\mathbb{P}_\alpha \mid \alpha < \delta\}$ is a family of forcings. We define $\prod_{\alpha<\delta} \mathbb{P}_\alpha$ as the forcing whose order is given pointwise. That is, $q \leq p$ if and only if for all $\alpha < \delta$, $q(\alpha) \leq_\alpha p(\alpha)$.

Given $p \in \prod_{\alpha<\delta} \mathbb{P}_\alpha$, the *support of* p is $\operatorname{supp}(p) = \{\alpha < \delta \mid p(\alpha) \neq \mathbb{1}_\alpha\}$. We will often restrict to a subfamily of conditions with a specific type of support. Most commonly, if κ is a cardinal, then the *κ-support product* is $\{p \in \prod_{\alpha<\delta} \mathbb{P}_\alpha \mid |\operatorname{supp}(p)| < \kappa\}$. In the case $\kappa = \omega$, we will say "finite support", in the case $\kappa = \omega_1$, we say "countable support", and in the case where $\kappa = \delta^+$, we say "full support".

Exercise 5.14. If κ is an infinite and regular cardinal and $\lambda \geq 2$, then $\operatorname{Add}(\kappa, \lambda)$ is the κ-support product $\prod_{\alpha<\lambda} \operatorname{Add}(\kappa, 1)$.

Exercise 5.15. Suppose that G is V-generic for $\prod_{\alpha<\delta} \mathbb{P}_\alpha$, then set $A \subseteq \delta$, $G \restriction A$ and $G \restriction (\delta \setminus A)$ are mutually generic.

Exercise 5.16. Suppose that $\mathbb{P}_\alpha$ is κ-closed for all $\alpha < \delta$, then the product $\prod_{\alpha<\delta} \mathbb{P}_\alpha$ is closed when taking a κ-support or a full support. (This is true in a much broader generality.)

Exercise 5.17 (*). Show that if c is a Cohen real, then in $V[c]$ there is a family $\{c_\alpha \mid \alpha < 2^{\aleph_0}\}$ of reals such that any two are mutually V-generic. But there is no H which V-generic for $\operatorname{Add}(\omega, \omega_1)$.

Proposition 5.18. *Suppose that $\delta \geq \omega$ and for each $\alpha < \delta$, $\mathbb{P}_\alpha$ contains at least two incompatible conditions. Then the finite support product $\prod_{\alpha<\delta} \mathbb{P}_\alpha$ adds a Cohen real.*

Proof. It is enough to prove this for the case $\delta = \omega$. Let $p_n \in \mathbb{P}_n$ be a condition which is not $\mathbb{1}_n$,[16] then we can project the product $\prod_{n<\omega} \mathbb{P}_n$ onto $\operatorname{Add}(\omega, 1)$ by mapping q to $c(n) = 0$ when $q(n) \perp_n p_n$ and $c(n) = 1$ when $q(n) \leq_n p_n$. $\square$

Exercise 5.19. Suppose that for all $n < \omega$, $\mathbb{P}_n$ contains an antichain of size ω_n. Show that the finite support product $\prod_{n<\omega} \mathbb{P}_n$ collapses $\aleph_\omega$ to be countable.

Theorem 5.20. *Assume* GCH *holds and let $f\colon \omega \to \mathrm{Card}$ be a non-decreasing function such that $\operatorname{cf}(f(n)) > \omega_n$. Then there is a cofinality preserving generic extension $V[G]$ where $2^{\aleph_n} = \aleph_{f(n)}$.*

Proof. Let $\mathbb{P}$ be the full support product of $\operatorname{Add}(\omega_n, \omega_{f(n)})$ and let G be a V-generic filter for $\mathbb{P}$. For every $n < \omega$, write $\mathbb{P}_n = \mathbb{P} \restriction n$ and $\mathbb{P}^n = \mathbb{P} \restriction (n, \omega)$. Then we get that $\mathbb{P}^n$ is ω_n-closed. Moreover, if $D \subseteq \mathbb{P}_n$ is a maximal antichain, we can find a large enough regular cardinal θ and an elementary submodel $M \prec H(\theta)$ such that M is ω_n-closed, and $D, \mathbb{P}_n \in M$. If $p \in D \setminus M$, then $p \cap M \in M$ and must be compatible with some $q \in D \cap M$, which would be impossible since $p \setminus M$ is also compatible with q, and so p was compatible with q to begin with. Note that this proof also shows that $\mathbb{P}$ itself is $\aleph_{\omega+1}$-c.c.

Any cofinality change must occur for an ordinal with cofinality ω_n for some $n < \omega$, and in that case we must have collapsed that ω_n. So it is enough to show that no cardinals were collapsed below $\aleph_\omega$. Since $\mathbb{P} \cong \mathbb{P}_n \times \mathbb{P}^n$ is the product of an ω_n-c.c. forcing with an ω_n-closed forcing, by Theorem 5.9 we get that any subset of ω_k for $k < n$ is added by $\mathbb{P}_n$. But indeed, $\mathbb{P}_n$ does not collapse ω_k for any $k < n$. So cofinalities are preserved.

Moreover, by the chain condition we have that $\mathbb{1}_{\mathbb{P}_{n+1}} \Vdash 2^{\aleph_n} = \aleph_{f(n)}$, as wanted. $\square$

[16] Or rather, not $\mathbb{1}$ of the separative quotient.

Remark. We can in fact extend this to any set, and indeed the whole class, of regular cardinals. Namely, starting from a model of GCH, if $F\colon \mathrm{Ord} \to \mathrm{Card}$ is a definable class function which is non-decreasing and such that whenever $\aleph_\alpha$ is regular, $\mathrm{cf}(F(\alpha)) > \aleph_\alpha$, then there is a cofinality preserving (possibly class-)generic extension where $2^{\aleph_\alpha} = \aleph_{F(\alpha)}$ for any regular cardinal.

For that we need to work a bit harder and defined the *Easton support* on a cardinal κ: a is an Easton support if for all regular cardinals $\gamma < \kappa$, $\sup a \cap \gamma < \gamma$. We can now define the Easton support product, and show that it satisfies all of the wanted properties.

Theorem 5.21. *The full support product $\prod_{n<\omega} \mathrm{Add}(\omega, 1)$ collapses $2^{\aleph_0}$ to be countable. In other words, it is equivalent to $\mathrm{Col}(\omega, 2^{\aleph_0})$.*

Proof. Let G be a generic for $\prod_{n<\omega} \mathrm{Add}(\omega, 1)$, which we can think of as a function $G\colon \omega\times\omega \to 2$. Given $n < \omega$ we define a sequence k_n by recursion: $k_0 = n_0$, $k_{i+1} = \min\{k > k_i \mid G(i+1, k) = 1\}$. Finally, let $r_n(i) = G(i, k_i)$.

We claim that if $r\colon \omega \to 2$ is a binary sequence in V, then there is some $n < \omega$ such that $r_n = r$. To see that, note that if p is any condition and $r \in V$ is a real number, we can extend p by letting $n = \sup \mathrm{dom}\, p(0) + 1$, then setting $p(0, n) = r(0)$, and letting $\ell_1 = \sup \mathrm{dom}\, p(1) + 1$ and adding to $p(0)$ the correct number of 0s such that when defining k_n, we get $k_1 = \ell_1$, then set $p(1, \ell_1) = r(1)$, and continue in that fashion. It is clear that the extended condition must force that $r_n = r$. Now, by density, we get that any ground model real is coded into some r_n. □

5.3 Lottery sums

Definition 5.22. Suppose that $\{\mathbb{P}_i \mid i \in I\}$ is a family of forcing notions. The *lottery sum* of $\{\mathbb{P}_i \mid i \in I\}$ is the partial order $\{\mathbb{1}\} \cup \{\langle i, p\rangle \mid p \in \mathbb{P}_i\}$ with the order $\langle i', p'\rangle \leq \langle i, p\rangle$ if and only if $i = i'$ and $p' \leq_{\mathbb{P}_i} p$, and $\mathbb{1}$ is the maximum element. We denote this sum by $\bigoplus_{i\in I} \mathbb{P}_i$.

The idea is that we do not know which forcing we are going to use. Instead we want to let the generic "decide". This can be used in a myriad of ways to produce odd counterexamples. For example, starting with a model of GCH, consider $\bigoplus_{n<\omega} \mathrm{Add}(\omega_n, \omega_{n+2})$. We violate GCH below $\aleph_\omega$, but we do not know a priori where this violation with occur.

Or, for example, $\mathbb{P} = \bigoplus_{\alpha<\omega_1} \mathrm{Add}(\omega, 1)$ will only add a single Cohen real. However, as a forcing, this partial order is not c.c.c. at all, and therefore not isomorphic to $\mathrm{Add}(\omega, 1)$. Therefore, despite the generic extension being characterised by a single real, there are no dense embeddings from $\mathrm{Add}(\omega, 1)$ into $\mathbb{P}$. However, this lead us to a natural notion of local c.c./closure/distributivity.

Definition 5.23. We say that $\mathbb{P}$ is *locally κ-c.c.* if for every p there is some $q \leq p$ such that $\mathbb{P} \restriction q$ is κ-c.c.

Proposition 5.24. *If $\mathbb{P}$ is locally κ-c.c., then it is isomorphic to a lottery sum of κ-c.c. forcings.*

Proof. Let $D = \{p \in \mathbb{P} \mid \mathbb{P} \restriction p \text{ is } \kappa\text{-c.c. }\}$, then by definition, this is a dense open set. We can therefore find a maximal antichain $D' \subseteq D$. It is not hard to check that $\mathbb{P} \cong \bigoplus_{p\in D'} \mathbb{P} \restriction p$. □

Exercise 5.25. We define locally κ-closed and locally κ-distributive in a similar fashion. Show that these concepts are redundant. Namely, if $\mathbb{P}$ is locally κ-closed/distributive, then it is isomorphic to a forcing notion that is κ-closed/distributive.

Exercise 5.26. Suppose that $\mathbb{P}_i$ is κ_i-c.c./closed/distributive for $i < \kappa$. Analyse the local and global chain condition/closure/distributivity of $\bigoplus_{i<\kappa} \mathbb{P}_i$ in terms of κ_i and κ.

Chapter 6

Iterated forcing

6.1 Two-step iterations

Product forcing is all fun and games, but sometimes the partial order we want to force "in the next step" is not even in the ground model. For example, perhaps we first added a generic Suslin tree, and then we wanted to shoot a branch through it? In this case, we can of course describe this process by first adding a Suslin tree, T, and then a branch b, and simply work with $V \to V[T] \to V[T][b]$. But there must be a simpler way.

Definition 6.1. Suppose that $\mathbb{P}$ is a forcing notion and $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a forcing notion. The iteration, $\mathbb{P} * \dot{\mathbb{Q}}$, is the set of pairs $\langle p, \dot{q}\rangle$ such that $p \in \mathbb{P}$ and $\mathbb{1}_{\mathbb{P}} \Vdash \dot{q} \in \dot{\mathbb{Q}}$. The order is given by $\langle p_1, \dot{q}_1\rangle \leq \langle p_0, \dot{q}_0\rangle \iff p_1 \leq p_0 \wedge p_1 \Vdash \dot{q}_1 \leq \dot{q}_0$.

Remark. In the literature this definition is often given as pairs $\langle p, \dot{q}\rangle$ where $p \Vdash \dot{q} \in \dot{\mathbb{Q}}$. In both of these cases the problem is, of course, that we may end up with a proper class of names for a given $\dot{q}$. There are many ways to restrict this back to a set, for example, we may require that the name $\dot{\mathbb{Q}}$ is replaced by an ordinal, so now $\dot{q}$ can be replaced by $\check{\alpha}$, and the name of the ordering on $\dot{\mathbb{Q}}$ becomes "the important bit". Or, more commonly, we can define an equivalence relation on $\mathbb{P}$-names, namely $\dot{x} \sim \dot{y} \iff \mathbb{1} \Vdash \dot{x} = \dot{y}$, and then use Scott's trick to turn the equivalence classes for those $\dot{q} \in \dot{\mathbb{Q}}$ into only a set of names. We may also skip this option and simply find a rank that is high enough and insist on the cut-off being there. My personal preference is to require that $\dot{q}$ is mixed from names that actually appear inside $\dot{\mathbb{Q}}$.

Exercise 6.2. Show that a lottery sum is a two-step iteration.

Definition 6.3. Let $\mathbb{P} * \dot{\mathbb{Q}}$ be a two step iteration. If $G_0 \subseteq \mathbb{P}$ is a V-generic filter and $G_1 \subseteq \dot{\mathbb{Q}}^{G_0}$ is a $V[G_0]$-generic filter, then $G_0 * G_1$ is $\{\langle p, \dot{q}\rangle \in \mathbb{P} * \dot{\mathbb{Q}} \mid p \in G_0, \dot{q}^{G_0} \in G_1\}$.

Theorem 6.4. *Let $\mathbb{P} * \dot{\mathbb{Q}}$ be a two-step iteration. G is a V-generic filter for $\mathbb{P} * \dot{\mathbb{Q}}$ if and only if there are V-generic $G_0 \subseteq \mathbb{P}$ and a $V[G_0]$-generic, $G_1 \subseteq \dot{\mathbb{Q}}^{G_0}$, such that $G = G_0 * G_1$.*

Proof. Suppose that G is a V-generic filter, we define $G_0 = \{p \in \mathbb{P} \mid \langle p, \dot{\mathbb{1}}_{\mathbb{Q}}\rangle \in G\}$. We first claim that G_0 is V-generic, and indeed, if $D \subseteq \mathbb{P}$ is a dense open set, it is not hard to check that $\{\langle p, \dot{q}\rangle \mid p \in D, \mathbb{1}_P \Vdash \dot{q} \in \dot{\mathbb{Q}}\}$ is a dense subset of $\mathbb{P} * \dot{\mathbb{Q}}$, so there is some $\langle p, \dot{q}\rangle \in G$ such that $p \in D$, and of course, if $\langle p, \dot{q}\rangle \in G$, then $\langle p, \dot{\mathbb{1}}_{\mathbb{Q}}\rangle \in G$ as well.

Next, we define $G_1 = \{q \in \dot{\mathbb{Q}}^{G_0} \mid \exists\langle p, \dot{q}\rangle \in G, \dot{q}^{G_0} = q\}$. We claim that G_1 is $V[G_0]$-generic. If $D \in V[G_0]$ is a dense open subset of $\mathbb{Q}$, let $\dot{D}$ be a $\mathbb{P}$-name for D in V such that $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\dot{D} \subseteq \dot{\mathbb{Q}}$ is dense open". We may assume without loss of generality that if $\dot{q}$ is a name appearing inside $\dot{D}$, then $\mathbb{1}_{\mathbb{P}} \Vdash \dot{q} \in \dot{\mathbb{Q}}$. Then $\dot{D}$ itself is a subset of $\mathbb{P} * \dot{\mathbb{Q}}$, and we claim it is predense.

Pick any $\langle p_0, \dot{q}_0\rangle$, then $p_0 \Vdash \exists q \in \dot{D}, q \leq_{\mathbb{Q}} \dot{q}_0$, and by The Mixing Lemma there is some $\dot{q}_*$ such that $p_0 \Vdash \dot{q}_* \in \dot{D}$. So, by definition there is some $\langle p, \dot{q}\rangle \in \dot{D}$, such that p is compatible with p_0 and $p \Vdash \dot{q} = \dot{q}_*$. Taking any $r \leq p, p_0$, we get that $\langle r, \dot{q}\rangle \leq \langle p_0, \dot{q}_0\rangle$, and therefore $\dot{D}$ is a predense set. It follows, therefore, that there is some $\langle p, \dot{q}\rangle \in G \cap \dot{D}$, so $p \in G_0$, and therefore $\dot{q}^{G_0} \in G_1 \cap \dot{D}^{G_0}$, as wanted. Now, by definition, $G = G_0 * G_1$.

In the other direction, suppose that $G_0 \subseteq \mathbb{P}$ and $G_1 \subseteq \dot{\mathbb{Q}}^{G_0}$ are two suitably generic filters. We want to show that $G_0 * G_1$ is V-generic for the iteration. If $D \subseteq \mathbb{P} * \dot{\mathbb{Q}}$ is a dense open set, we first define in $V[G_0]$ the set $D_1 = \{\dot{q}^{G_0} \mid \exists p \in G_0, \langle p, \dot{q}\rangle \in D\}$. We claim that this is a dense subset of $\dot{\mathbb{Q}}^{G_0}$, as given any $\langle p_0, \dot{q}_0\rangle$, there is some $\langle p, \dot{q}\rangle \in D$ extending it, and in particular $\{p \in \mathbb{P} \mid \exists \dot{q}, \langle p, \dot{q}\rangle \in D, p \Vdash \dot{q} \leq \dot{q}_0\}$ is a dense subset of $\mathbb{P}$, so there is some $p \in G$ and $\dot{q}$ such that $\langle p, \dot{q}\rangle \in D$ and $p \Vdash \dot{q} \leq \dot{q}_0$. So, $D_1 \cap G_1 \neq \varnothing$, so we can take some $\dot{q}^{G_0}$ in the intersection, and some $p \in G$ such that $\langle p, \dot{q}\rangle \in D$, and we get that $\langle p, \dot{q}\rangle \in D \cap G_0 * G_1$, as wanted. □

So we see, quite immediately, that $\mathbb{P} \times \mathbb{Q} \cong \mathbb{P} * \check{\mathbb{Q}}$.

Definition 6.5. Suppose that $\mathbb{P}_0 \lessdot \mathbb{P}_1$, and let $\pi\colon \mathbb{P}_1 \to \mathbb{P}_0$ be a projection witnessing that. We define the quotient $\mathbb{P}_1/\mathbb{P}_0$ to be the $\mathbb{P}_0$-name, $\{\langle \pi(p), \check{p}\rangle \mid p \in \mathbb{P}_1\}$, with the obvious order, $\{\langle \check{q}, \check{p}\rangle \mid q \leq_{\mathbb{P}_1} p\}^\bullet$.

Theorem 6.6. *Suppose that $\mathbb{P}_0 \lessdot \mathbb{P}_1$, then $\mathbb{P}_1 \cong \mathbb{P}_0 * \mathbb{P}_1/\mathbb{P}_0$.*

Proof. Let $\pi\colon \mathbb{P}_1 \to \mathbb{P}_0$ be the projection map defining the quotient. We will show that the map $p \mapsto \langle \pi(p), \mathring{p}\rangle$ is a dense embedding, where $\mathring{p}$ is the name obtained by mixing $\check{p}$ below $\pi(p)$ and $\check{\mathbb{1}}_{\mathbb{P}_1}$ on any incompatible condition.

Assume that $q \leq_{\mathbb{P}_1} p$, then we immediately get that $\pi(q) \leq_{\mathbb{P}_0} \pi(p)$, moreover since $q \leq_{\mathbb{P}_1} p$, then we have by definition that $\pi(q) \Vdash \mathring{q} \leq_{\mathbb{P}_1/\mathbb{P}_0} \mathring{p}$. In the other direction, if $\pi(q) \leq \pi(p)$, then $\pi(q)$ must interpret correctly both $\mathring{p}$ and $\mathring{q}$, so the order is preserved.

To show density, given any $\langle \pi(p), \dot{r}\rangle$ in the iteration, we may extend $\pi(p)$ to some $\pi(q)$ which decides $\dot{r} = \check{p}'$ for some $p' \in \mathbb{P}_1$, this implies that $\pi(q) \leq \pi(p')$. By the definition of a projection there is some $q' \leq_{\mathbb{P}_1} p$ such that $\pi(q') \leq \pi(q)$, and so, $\langle \pi(q'), \mathring{q}'\rangle \leq \langle \pi(p), \dot{r}\rangle$, as wanted. □

Theorem 6.7. *Suppose that $G \subseteq \mathbb{P}$ is a V-generic filter, and let $x \subseteq V$ be an element of $V[G]$. Then there is some $\mathbb{P}_x \lessdot \mathbb{P}$ such that x is the V-generic for $\mathbb{P}_x$ obtained from G. Moreover, there is a forcing $\mathbb{Q} \in V[x]$ such that there is $H \in V[G]$ which is $V[x]$-generic, and $V[G] = V[x][H]$.*

Proof. We define $\mathbb{P}_x$ as a quotient of $\mathbb{P}$, similar to what we did in Theorem 3.3. Fix a $\mathbb{P}$-name $\dot{x}$, and without loss of generality consider it as a function $\dot{x}\colon \check{X} \to \check{2}$. Define an equivalence relation on $\mathbb{P}$, $p \sim q$ if and only if for all $u \in X$ and $\varepsilon < 2$,

$$p \Vdash \dot{x}(\check{u}) = \check{\varepsilon} \iff q \Vdash \dot{x}(\check{u}) = \check{\varepsilon}.$$

This defines a quotient (in the order-theoretic sense) of $\mathbb{P}$. Now apply Theorem 6.6. □

Exercise 6.8. Suppose that $\mathbb{P}$ is κ-closed, and $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\dot{\mathbb{Q}}$ is $\check{\kappa}$-closed", then $\mathbb{P} * \dot{\mathbb{Q}}$ is κ-closed.

Theorem 6.9. *Let κ be an uncountable regular cardinal. Suppose that $\mathbb{P}$ is κ-c.c., and $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\dot{\mathbb{Q}}$ is $\check{\kappa}$-c.c.", then $\mathbb{P} * \dot{\mathbb{Q}}$ is κ-c.c.*

Proof. Suppose that $A = \{\langle p_\alpha, \dot{q}_\alpha\rangle \mid \alpha < \kappa\}$ is an antichain in $\mathbb{P} * \dot{\mathbb{Q}}$. Given any two α, β, either p_α and p_β are incompatible, or (if they are) every common extension must force that $\dot{q}_\alpha \perp \dot{q}_\beta$.

Let $G \subseteq \mathbb{P}$ be V-generic, then in $V[G]$ consider $A^G = \{\dot{q}_\alpha^G \mid p_\alpha \in G\}$. By the observation we made, this set is an antichain in $\dot{\mathbb{Q}}^G$, so it must have fewer than κ elements. In other words, as a $\mathbb{P}$-name, $\mathbb{1}_\mathbb{P} \Vdash |A| < \check{\kappa}$.

Let B be a maximal antichain in $\mathbb{P}$ which decides the upper bound of the indices of elements in A^G in $V[G]$, by regularity and κ-c.c. of $\mathbb{P}$, there is a large enough $\beta < \kappa$ such that $\mathbb{1}_\mathbb{P} \Vdash \dot{q}_\beta \notin A$. This is of course a contradiction, since $p_\beta \Vdash_\mathbb{P} \dot{q}_\beta \in A$. □

Exercise 6.10. Show that the converse is also true. Namely, if $\mathbb{P} * \dot{\mathbb{Q}}$ is κ-c.c., then $\mathbb{P}$ is κ-c.c. and $\mathbb{1}_\mathbb{P} \Vdash$ "$\dot{\mathbb{Q}}$ is $\check{\kappa}$-c.c.", but that this may not be the case for κ-closure.

6.2 General iterations

Definition 6.11. We say that $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \beta < \delta, \alpha \leq \delta \rangle$ is an *iteration system* if:[17]

1. $\mathbb{P}_0 = \{\mathbb{1}\}$.
2. For all $\alpha < \delta$, $\mathbb{1}_{\mathbb{P}_\alpha} \Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha$ is a forcing notion".
3. For all $\alpha \leq \delta$, $p \in \mathbb{P}_\alpha$ is a function with domain α, and for all $\xi < \alpha$, $\mathbb{1}_{\mathbb{P}_\xi} \Vdash_{\mathbb{P}_\xi} p(\xi) \in \dot{\mathbb{Q}}_\xi$ and for all $\beta < \alpha$, $p \restriction \beta \in \mathbb{P}_\beta$.
4. For all $\alpha \leq \delta$, the order of $\mathbb{P}_\alpha$ is given by, $q \leq_{\mathbb{P}_\alpha} p$ if and only if for all $\xi < \alpha$, $q \restriction \xi \leq_{\mathbb{P}_\xi} p \restriction \xi$ and if $\alpha = \xi + 1$, then we require that $q \restriction \xi \Vdash_{\mathbb{P}_\xi} q(\xi) \leq p(\xi)$ as well.

For limit $\alpha \leq \delta$ will usually have a condition on $\operatorname{supp}(p) = \{\xi < \alpha \mid \mathbb{1}_{\mathbb{P}_\xi} \Vdash_{\mathbb{P}_\xi} p(\xi) \neq \dot{\mathbb{1}}_{\mathbb{Q}_\xi}\}$. We say that $\mathbb{P}_\delta$ is the iteration of length δ of the $\dot{\mathbb{Q}}_\alpha$ (with the specified support).

Exercise 6.12. In the definition of an iteration system, if $\delta = \alpha + 1$, then $\mathbb{P}_\delta \cong \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$.

We will refer to the $\dot{\mathbb{Q}}_\alpha$ as "iterands", and we will usually write $\mathbb{1}_\alpha$ and $\Vdash_\alpha$ instead of $\mathbb{P}_\alpha$ subscripts. Moreover, since we can reconstruct the $\mathbb{P}_\alpha$ by knowing the $\dot{\mathbb{Q}}_\alpha$s and the support system at limits, we will often just omit the $\mathbb{P}_\alpha$ from the iteration system, or just write that $\mathbb{P}_\delta$ is the such and such iteration of a given sequence of names, understanding that $\mathbb{P}_\alpha$ will then denote the partial steps in that iteration.

Given an iteration $\mathbb{P}_\delta$ and $\gamma < \delta$, $\mathbb{P}_\gamma$ is "an initial segment" of the iteration. There is a clear projection map $\mathbb{P}_\delta \to \mathbb{P}_\gamma$, given by $p \mapsto p \restriction \gamma$. We will sometimes refer to $\mathbb{P}_\gamma$ as $\mathbb{P}_\delta \restriction \gamma$, which is useful when subscripts are not used to denote the iteration itself.[18]

We will mainly work with finite support iterations, but countable, κ-support, and "full support" will all be mentioned. However, except for the first one, these supports do not play all too well with the natural quotients of the iteration (that is, $\mathbb{P}_\delta / \mathbb{P}_\gamma$). Let us start with an example.

Let $\mathbb{Q}_0 = \operatorname{Add}(\omega, 1)$. If c_0 is the Cohen real, $\mathbb{Q}_1 = \prod_{n \in c_0} \operatorname{Add}(\omega_{n+1}, 1)$ as a full support product, and modifying the forcing slightly to add the generic subset to the interval $[\omega_n, \omega_{n+1})$. Let $c_1 \subseteq \omega_\omega$ be the union of all the generic subsets added by $\mathbb{Q}_1$, then $\mathbb{Q}_2 = \prod_{\alpha \in c_1} \operatorname{Add}(\omega_{\alpha+1}, 1)$ as a full support product with the generic subsets living in the intervals as before.

Let $\mathbb{P}_\omega$ be the full support iteration of this forcing. We want to argue that it does not collapse cardinals, for example. In the case of products, we separated the product into a finite

[17] This is a recursive definition, of course.

[18] We can modify the definition to use partial functions, in which case we simply have $\operatorname{supp}(p) = \operatorname{dom}(p)$ and $\mathbb{P}_\gamma \subseteq \mathbb{P}_\delta$ when $\gamma \leq \delta$.

"initial segment" and a "tail segment"[19] arguing that the tail segment is closed and the initial segment has a nice chain condition.

Ideally, we would like to do this here, so have $\mathbb{P}_n$ and $\mathbb{P}_\omega/\mathbb{P}_n$, and argue that $\mathbb{1}_n \Vdash_n$ "$\mathbb{P}_\omega/\mathbb{P}_n$ is sufficiently closed", or something along those lines. The naive approach is to say that $\mathbb{P}_\omega/\mathbb{P}_n$ is itself a full support iteration of very closed iterands, so it should be. But is the quotient really a full support iteration? After the first step, we have added a new subset to ω, if this is a full support iteration, surely we can find a condition whose support are exactly those coordinates. But if we did, we can "pull" the Cohen real back to the ground model. Of course, in this case, this is not too much of an issue, and we can recover from this sort of scenario. But in other cases, this might be a real problem.

To solve this, in this case, we simply notice that as far as each iterand is concerned, this is not a problem they are defined in the universe where c_0 already appeared, and if we had any condition in the iteration whose support needed to be "exactly" c_0 itself, then we can just strengthen it to a condition with a full support. So, if we can show that even after the whole iteration, every countable set in the generic extension is covered by a countable set in the ground model, then the quotient is dense in the countable support iteration of the iterands. Or, the full support in this case where the length of the iteration is countable.

Note that this problem presents itself also in the case of products, as they are degenerate iterations, but since the product arguments are often done in the ground model, this is not as big a problem as it can get with iterations. Note that in the case $\kappa = \omega$, that is the case of finite support iterations, this is always true.

Definition 6.13. We say that an iteration $\langle \dot{\mathbb{Q}}_\alpha \mid \alpha < \delta \rangle$ is an iteration of "property φ", e.g. c.c.c., if for all $\alpha < \delta$, $\mathbb{1}_\alpha \Vdash_\alpha$ "$\dot{\mathbb{Q}}_\alpha$ has property φ".

Theorem 6.14. *Finite support iteration of κ-c.c. forcings is itself κ-c.c.*

Proof. We can assume that κ is an uncountable and regular, since the first failure of chain conditions must occur at an uncountable regular cardinal. We prove this by induction on the length of the iteration, δ. For $\delta = 0$ this is vacuously true. For a successor ordinal δ this is a consequence of Theorem 6.9. So, we may assume that δ is a limit ordinal. Let $A = \{p_\alpha \mid \alpha < \kappa\}$ be a subset of $\mathbb{P}_\delta$.

We divide it into two cases, $\operatorname{cf}(\delta) \neq \kappa$ and $\operatorname{cf}(\delta) = \kappa$. In the first case, there is some $\gamma < \delta$ such that $A^* = \{\alpha < \kappa \mid \operatorname{supp}(p_\alpha) \subseteq \gamma\}$ has size κ, then A^* is a subset of $\mathbb{P}_\gamma$ of size κ, and by the induction hypothesis it contains two compatible conditions, say p_α and p_β. It is not hard to verify that p_α and p_β are also compatible in $\mathbb{P}_\delta$, so A is not an antichain.

In the second case, fix $\{\delta_\xi \mid \xi < \kappa\}$ to be a continuous and cofinal sequence in δ, and let $C \subseteq \kappa$ be a club such that when $\eta \in C$, for all $\xi < \eta$, $\operatorname{supp}(p_\xi) \subseteq \delta_\eta$.

For any limit point of C, η, there is some $\xi(\eta) < \eta$ such that $\operatorname{supp}(p_\eta) \cap \delta_\eta \subseteq \delta_{\xi(\eta)}$, so by Fodor's lemma there is an unbounded[20] subset $D \subseteq C$, such that $\xi(\eta) = \xi$ for all $\eta \in D$. Considering $D^* = \{p_\alpha \restriction \delta_\xi \mid \alpha \in D\}$ as a subset of $\mathbb{P}_{\delta_\xi}$. We want to find $\alpha < \beta$, both from D and $q \in \mathbb{P}_\xi$ such that $q \leq_\xi p_\alpha \restriction \delta_\xi, p_\beta \restriction \delta_\xi$ such that we can extend q to $\bar{q} \in \mathbb{P}_\delta$ which will extend both p_α and p_β.

If D^* has fewer than κ distinct elements, then there are $\alpha < \beta$ whose restrictions are equal and we can take $q = p_\alpha \restriction \delta_\xi$. Otherwise this is a subset of size κ, so is not an antichain in $\mathbb{P}_{\delta_\xi}$, so there are some $\alpha < \beta$, both in D, such that $p_\alpha \restriction \delta_\xi$ and $p_\beta \restriction \delta_\xi$ are compatible in $\mathbb{P}_{\delta_\xi}$, so we can find some $q \leq_{\delta_\xi} p_\alpha \restriction \delta_\xi, p_\beta \restriction \delta_\xi$.

[19]Being a product means that the indexation is more about the set than its order, hence the quotation marks.
[20]Stationary, in fact!

Finally, consider the following extension of q into $\mathbb{P}_\delta$: $\bar{q} = q^\frown p_\alpha \restriction [\delta_\xi, \delta_\beta)^\frown p_\beta \restriction [\delta_\beta, \delta)$. Namely, below δ_ξ we are exactly q; until δ_α we are exactly what p_α did; and for the remainder we are exactly p_β. Note that the key point here is that if $\alpha < \beta \in D$, then $\operatorname{supp}(p_\alpha) \cap \operatorname{supp}(p_\beta) \subseteq \delta_\xi$. It is not hard to verify that $\bar{q} \leq_\delta p_\alpha, p_\beta$, so A is not an antichain. □

Exercise 6.15. The full support iteration of $\mathrm{Add}(\omega, 1)$ is not a c.c.c. forcing.

Exercise 6.16. κ-support iteration of κ-closed forcings is itself κ-closed.

Exercise 6.17. If $\mathbb{1}_n \Vdash_n$ "$\dot{\mathbb{Q}}_n$ has two incompatible conditions", then the finite support iteration adds a Cohen real.

Exercise 6.18. Finite support iterations of $\dot{\mathbb{Q}}_n$, such that $\mathbb{1}_n \Vdash_n$ "$\dot{\mathbb{Q}}_n$ does not have the $\aleph_{n+1}$-c.c." collapses $\aleph_\omega$.

Exercise 6.19. What goes wrong when trying to iterate with full support $\dot{\mathrm{Add}}(\dot{\omega}_n, \dot{\omega}_{n+2})$, or any other value compatible with the continuum function there?

Chapter 7

Martin's Axiom

7.1 What is... a Forcing Axiom?

Given a class of forcing notions (e.g., all c.c.c. forcings; all σ-closed forcings; $\{\mathbb{P}\}$ for some partial order), we want to understand the amount of genericity we can obtain "inside the universe" for members of the class. Namely, given some collection of dense open sets, is there a filter that meets all of them?

Clearly, if the collection of dense open sets is just "all of them", then the answer is no, except for trivial cases. We can always meet countably many dense open sets, this is just the Rasiowa–Sikorski lemma,[21] or its topological equivalent, the Baire Category Theorem. But maybe we can meet more?

Suppose that $\mathbb{P}$ is σ-closed, if $\{D_\alpha \mid \alpha < \omega_1\}$ are dense open sets, then we can construction by recursion a descending sequence, $p_{\alpha+1} \leq p_\alpha$ such that $p_\alpha \in D_\alpha$, using the σ-closed to get through the limit steps. Or, if $\mathbb{P}$ was $\aleph_2$-distributive, then the intersection $\bigcap_{\alpha<\omega_1} D_\alpha$ is just a single dense open set, which we can most certainly meet.

Definition 7.1. Let Γ be a class of forcings and let κ be an infinite cardinal. We say that $\mathsf{FA}_\kappa(\Gamma)$ holds if for every $\mathbb{P} \in \Gamma$ and family of dense open subsets of $\mathbb{P}$, $\mathcal{D} = \{D_\alpha \mid \alpha < \kappa\}$, there is a $\mathcal{D}$-generic filter G. Namely, $G \cap D_\alpha \neq \varnothing$ for all $\alpha < \kappa$.

Exercise 7.2. If $\lambda < \kappa$, then $\mathsf{FA}_\kappa(\Gamma) \implies \mathsf{FA}_\lambda(\Gamma)$, and $\mathsf{FA}_{\aleph_0}(\Gamma)$ always holds.

Exercise 7.3. If $\mathrm{Add}(\omega,1) \in \Gamma$, then $\mathsf{FA}_{2^{\aleph_0}}(\Gamma)$ is false. If $\kappa > \omega$ and $\mathrm{Col}(\omega,\kappa) \in \Gamma$, then $\mathsf{FA}_\kappa(\Gamma)$ is false.

Martin's Axiom is the particular case where Γ is a subclass of "c.c.c.", in which case we denote it by $\mathsf{MA}_\kappa(\Gamma)$, and we omit Γ when it is exactly the class of c.c.c. forcings. We also use MA to denote $\mathsf{MA}_{<2^{\aleph_0}}$, namely, $\forall\kappa < 2^{\aleph_0}, \mathsf{MA}_\kappa$.

7.2 Some consequences of Martin's Axiom

Proposition 7.4. *Assume* $\mathsf{MA}_{\aleph_1}$ *holds, then there are no Suslin trees.*

Proof. Let T be a Suslin tree, then as a forcing notion, T is a c.c.c. forcing notion. Let D_α be the dense open set of all nodes of height α or higher, then a $\{D_\alpha \mid \alpha < \omega_1\}$-generic filter must define a cofinal branch. □

[21] Essentially, Theorem 1.14.

Proposition 7.5. *Assume* MA_κ *holds, then* $2^{\aleph_0} > \kappa$.

Proof. Suppose that $\lambda \leq \kappa$, and $\{r_\alpha \colon \omega \to 2 \mid \alpha < \lambda\}$ is a set of reals, let $D_\alpha \subseteq \operatorname{Add}(\omega, 1)$ be the dense open set $\{p \mid p \nsubseteq r_\alpha\}$ and let $E_n = \{p \mid n \in \operatorname{dom} p\}$. If G is a $\{D_\alpha, E_n \mid \alpha < \lambda, n < \omega\}$-generic filter, then G defines a real, g, and by its genericity, $g \neq r_\alpha$ for all $\alpha < \lambda$. In particular, $2^{\aleph_0} > \lambda$. □

Proposition 7.6. *Assume* MA_κ *holds, then given any* $\{r_\alpha \colon \omega \to \omega \mid \alpha < \kappa\}$, *there is* r *such that* $r_\alpha \leq^* r$ *for all* $\alpha < \kappa$.

Proof. Recall the Hechler forcing is c.c.c., and for each α let $D_\alpha = \{\langle s, F\rangle \mid r_\alpha \in F\}$ and let $E_n = \{\langle s, F\rangle \mid n \in \operatorname{dom} s\}$, then if G is a $\{D_\alpha, E_n \mid \alpha < \kappa, n < \omega\}$-generic filter, then $\bigcup_{\langle s,F\rangle \in G} s = r$ is a real which dominates all the r_αs simultaneously. □

Theorem 7.7. *Assume* MA *holds, then* $2^\lambda = 2^{\aleph_0} = \mathfrak{c}$ *for all* $\lambda < \mathfrak{c}$. *Consequently,* $\mathfrak{c}^{<\mathfrak{c}} = \mathfrak{c}$ *and therefore* $\mathfrak{c}$ *is a regular cardinal..*

Proof. By König's lemma, $\operatorname{cf}(2^\kappa) > \kappa$, so if $\mathfrak{c} = 2^\lambda$, $\operatorname{cf}(\mathfrak{c}) > \lambda$, and if we show $2^\lambda = \mathfrak{c}$ for all $\lambda < \mathfrak{c}$, then we must have that $\mathfrak{c}$ is regular. Note, moreover, that $(2^\lambda)^\lambda = 2^\lambda$, so proving this equality will also imply that $\mathfrak{c}^{<\mathfrak{c}} = \mathfrak{c}$.

Fix an almost disjoint family of subsets of ω of size $\mathfrak{c}$, say $\{A_\alpha \mid \alpha < \mathfrak{c}\}$. Namely, each A_α is an infinite subset of ω, and if $\alpha \neq \beta$, then $A_\alpha \cap A_\beta$ is finite.[22] We will use this almost disjoint family of sets to show that given any $\lambda < \mathfrak{c}$ and any $X \subseteq \lambda$, there is a real, r_X, which "codes" X, in the sense that $X = \{\alpha < \lambda \mid |r_X \cap A_\alpha| = \aleph_0\}$.

For $X \subseteq \lambda < \mathfrak{c}$ we define $\mathbb{P}_X$ whose conditions are pairs, $\langle s, A\rangle$, where $s \colon \omega \to 2$ is a finite partial function and $A \in [\lambda]^{<\omega}$; with the ordering given by $\langle t, B\rangle \leq \langle s, A\rangle$ when:

1. $s \subseteq t$ and $A \subseteq B$; and
2. for any $n \in \operatorname{dom}(t \setminus s)$, if $t(n) = 1$ then for some $\alpha \in A \cap X$, $n \in A_\alpha$.

We first notice that $\mathbb{P}_X$ is c.c.c., since any two conditions p, q such that $s_p = s_q$ are compatible. Next, consider for all $\alpha < \lambda$ the dense open set

$$D_{\alpha,n} = \{\langle s, A\rangle \mid \alpha \in A, \alpha \in X \to \exists m(m > n, m \in A_\alpha, s(m) = 1)\}.$$

We claim that if G is $\{D_{\alpha,n} \mid \alpha < \lambda, n < \omega\}$-generic, then $r_X = \bigcup_{\langle s,A\rangle} s^{-1}(1)$ is a code for X.

First we need to check that each $D_{\alpha,n}$ is a dense open set. Indeed, if $\langle s, A\rangle$ is any condition, let $m = \min\{k \in A_\alpha \mid k > n, k \notin \operatorname{dom} s\}$, then $\langle s \cup \{\langle m, 1\rangle\}, A \cup \{\alpha\}\rangle$ is a condition in $D_{\alpha,n}$, so it is a dense set. It is not hard to see that it is also open.

Next, if $\alpha \notin X$, then it is clear from the definition of the conditions, that $r_X \cap A_\alpha$ is finite. On the other hand, if $\alpha \in X$, then by meeting all the sets $D_{\alpha,n}$ we are guaranteed to have an infinite intersection with A_α. □

Theorem 7.8. *Assume* $\mathsf{MA}_{\aleph_1}$ *holds, if* $\mathbb{P}$ *and* $\mathbb{Q}$ *are c.c.c., then* $\mathbb{P} \times \mathbb{Q}$ *is c.c.c.*

Lemma 7.9. *Assume* $\mathsf{MA}_{\aleph_1}$, *then every c.c.c. forcing is Knaster. That is, if* $\mathbb{P}$ *is a c.c.c. forcing, and* W *is an uncountable subset, then there is an uncountable* $W' \subseteq W$ *such that any two elements in* W' *are compatible.*

[22] Enumerate the binary tree, and consider the set of branches, for example.

Proof. We want to show that if $\mathbb{P}$ is c.c.c. and $W \subseteq \mathbb{P}$ is uncountable, say $\{p_\alpha \mid \alpha < \omega_1\}$, then there is an uncountable $W' \subseteq W$ such that any two points in W' are compatible. First we argue that there is some $p \in W$ such that any extension of p is compatible with uncountably many elements of W, otherwise for all $\alpha < \omega_1$ there is some $\beta > \alpha$, and $q_\alpha \leq p_\alpha$, such that $q_\alpha \perp \{p_\gamma \mid \gamma \geq \beta\}$. In this case we can recursively construct an uncountable antichain from $\{q_\alpha \mid \alpha < \omega_1\}$.

Say p_0 is compatible with uncountably many p_α, then $D_\alpha = \{p \leq p_0 \mid \exists\gamma \geq \alpha, p \leq p_\gamma\}$ is dense open below p_0, so by MA there is some G which is $\{D_\alpha \mid \alpha < \omega_1\}$-generic. So $G \cap W = W'$ must be uncountable, and any two elements there are compatible. □

Proof of Theorem 7.8. Suppose that $W \subseteq \mathbb{P} \times \mathbb{Q}$ is uncountable. If there is some $p \in \mathbb{P}$ such that $\{q \in \mathbb{Q} \mid \langle p, q\rangle \in W\}$ is uncountable, then we are done, since $\mathbb{Q}$ is c.c.c., that means that there are $q, q' \in \mathbb{Q}$ which are compatible and $\langle p, q\rangle, \langle p, q'\rangle \in W$. Otherwise, for each $p \in \mathbb{P}$ there are at most countably many $q \in \mathbb{Q}$ such that $\langle p, q\rangle \in W$.

Let W_0 be $\{p \in \mathbb{P} \mid \exists q \in \mathbb{Q}, \langle p, q\rangle \in W\}$, in that case, as an uncountable set, by Lemma 7.9, there is $W_0' \subseteq W_0$ of pairwise compatible elements. Let $W_1 = \{q \in \mathbb{Q} \mid \exists p \in W_0', \langle p, q\rangle \in W\}$, then there is some $q, q' \in W_1$ which are compatible and $p, p' \in W_0'$ such that $\langle p, q\rangle, \langle p', q'\rangle \in W$. To see this, either W_1 is uncountable, and we use the fact that $\mathbb{Q}$ is c.c.c., or it is countable, so we can find $q = q'$ by cardinality argument. □

Remark. The above theorem can be extended to any finite support product of c.c.c. forcings!

7.3 The consistency of Martin's Axiom

This section will be devoted for the proof of the following theorem.

Theorem 7.10. *Assume* GCH *holds and let* $\kappa > \omega_1$ *be an uncountable regular cardinal. There is a c.c.c. forcing* $\mathbb{P}$ *such that* $\mathbb{1}_\mathbb{P} \Vdash \mathsf{MA} + 2^{\aleph_0} = \check{\kappa}$.

The idea is going to be simple, we will iterate "enough" c.c.c. forcing notions to guarantee that MA holds. Since the finite support iterations of c.c.c. forcings is itself a c.c.c. forcing, we will preserve cofinalities and cardinals. But what does it mean "enough"? After all, we want to obtain MA for *all* c.c.c. forcings, of which is there a proper class.

Lemma 7.11. MA_λ *holds if and only if it holds for c.c.c. forcing notions of size* $\leq \lambda$.

Proof. One direction is trivial. In the other direction, suppose that $\mathbb{P}$ is a c.c.c. notion and that $\{D_\alpha \mid \alpha < \lambda\}$ are dense open sets. Refine each to a countable antichain $E_\alpha \subseteq D_\alpha$, and let $\mathbb{Q}$ be an elementary submodel of $\mathbb{P}$ generated by $\bigcup_{\alpha<\lambda} E_\alpha$ of size λ. Each E_α generates a dense open set, D^*_α inside $\mathbb{Q}$, so by our assumption there is G which meets all of the D^*_α. However, in meeting D^*_α, we must have picked one of the members of E_α, and so our filter generates in $\mathbb{P}$ a filter which is $\{D_\alpha \mid \alpha < \lambda\}$-generic. □

So, if we want to force that MA holds and $2^{\aleph_0} = \kappa$, it is enough to make sure that all c.c.c. forcings of size $< \kappa$ will have satisfied MA. We will define a finite support iteration of c.c.c. forcings using a bookkeeping device. The idea is to iterate, up to κ all the small partial orders, repeatedly. Then, given a small c.c.c. forcing and a collection of dense open sets, we will argue that there is a bounded stage in the iteration where the forcing and all the sets have appeared, so the generic at that stage is indeed meeting all the sets in our collection.

We begin by fixing a function $f\colon \kappa \to [\kappa \times \kappa]^{<\kappa}$ such that for each $a \in [\kappa \times \kappa]^{<\kappa}$, $f^{-1}(a)$ is unbounded in κ. We construct our finite support iteration by recursion. Suppose that $\mathbb{P}_\alpha$ was defined, if $f(\alpha)$ codes a $\mathbb{P}_\alpha$-name, take a maximal antichain which decides whether or not it is a c.c.c. forcing, and over this antichain mix either $f(\alpha)$ below the conditions which force to the positive and the trivial forcing on the others. This is our $\dot{\mathbb{Q}}_\alpha$, and plainly, $\mathbb{1}_\alpha \Vdash_\alpha$ "$\dot{\mathbb{Q}}_\alpha$ is c.c.c."

For this to make sense, we need to understand in what sense $f(\alpha)$ is coding a name for a partial order. If we know that $\mathbb{P}_\alpha$ can be seen as a subset of κ, and we can think of the next iterand (or a candidate for one), $\mathbb{Q}$, as a set of ordinals, then its name, $\dot{\mathbb{Q}}$, can be seen as a set of pairs of ordinals, $\langle \xi, \zeta \rangle$ where ξ is a condition in $\mathbb{P}_\alpha$ which forces $\check{\zeta} \in \dot{\mathbb{Q}}$. Since $\dot{\mathbb{Q}}$ is forced to be small and $\mathbb{P}_\alpha$ is c.c.c. we get that $\dot{\mathbb{Q}}$ can be coded as some $f(\alpha)$.

Lemma 7.12. *Let μ be a cardinal such that $\mu^{\aleph_0} = \mu$. Suppose that $\langle \dot{\mathbb{Q}}_\alpha \mid \alpha < \mu \rangle$ is a finite support iteration of c.c.c. forcings such that $\mathbb{1}_\alpha \Vdash_\alpha |\dot{\mathbb{Q}}_\alpha| < \check{\mu}$, then $|\mathbb{P}_\alpha| \leq \mu$ for all $\alpha < \mu$.*

Proof. We prove this by recursion on α. For $\alpha = 0$ this is trivial; if α is a limit ordinal, then $|\mathbb{P}_\alpha| = |(\bigcup_{\beta<\alpha} \mathbb{P}_\beta)^{<\omega}| \leq \mu$. Assume that $|\mathbb{P}_\alpha| \leq \mu$, then since $\mathbb{1}_\alpha \Vdash_\alpha |\dot{\mathbb{Q}}_\alpha| < \check{\mu}$, since there are only countably many possible values for $|\mathbb{Q}_\alpha|$, these are bounded below μ by some γ. So we can think of $\dot{\mathbb{Q}}_\alpha$ as a function from γ into the ordinals. For each $\xi < \gamma$ there is a maximal antichain of size $\aleph_0$ which decides its value, A_ξ, so the name $\{\langle p, \check{\xi} \rangle \mid p \in A_\xi, p \Vdash_\alpha \check{\xi} \in \dot{\mathbb{Q}}_\alpha\}$ is forced to be equal to $\dot{\mathbb{Q}}_\alpha$, and as an object in the ground model it has size $|\gamma| \cdot \aleph_0$.

Finally, the elements of $\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$ can be thought of as mixing over names which appear inside $\dot{\mathbb{Q}}_\alpha$. Since these are determined by a maximal antichain in $\mathbb{P}_\alpha$ and $|\dot{\mathbb{Q}}_\alpha| < \mu$, we have that

$$|\mathbb{P}_{\alpha+1}| \leq (|\mathbb{P}_\alpha| \cdot |\dot{\mathbb{Q}}_\alpha|)^{\aleph_0} \leq \mu^{\aleph_0} = \mu. \qquad \square$$

So, each $\mathbb{P}_\alpha$ in our iteration has size $\leq \kappa$ and we can code it as a subset of κ. But we will do it in the following way. Split κ into $\{C_\alpha \mid \alpha < \kappa\}$ such that $|C_\alpha| = \kappa$ for all $\alpha < \kappa$. Then we take an injection of $c_\alpha\colon \mathbb{P}_\alpha \to \bigcup_{\beta \leq \alpha} C_\beta$ such that whenever $\beta < \alpha$, $c_\beta = c_\alpha \upharpoonright \mathbb{P}_\beta$.

The reason we do this is to ensure that if $a \in [\kappa \times \kappa]^{<\omega}$ is a $\mathbb{P}_\alpha$-name, then it will remain a $\mathbb{P}_\alpha$-name, even if interpreted as a $\mathbb{P}_\beta$-name.

Lemma 7.13. *Suppose that $\mathbb{P}_\delta$ is a finite support iteration of c.c.c. forcings of length δ, and let G be a V-generic filter. If $A \in V[G]$ is a set of ordinals such that $|A| < \operatorname{cf}(\delta)$, then there is some $\alpha < \delta$ such that $A \in V[G \upharpoonright \alpha]$.*

Proof. If $\operatorname{cf}(\delta) \leq \omega$, this is trivial, since the assumption on A implies it is finite, so we can assume that $\operatorname{cf}(\delta) > \omega$. Suppose that $p \in \mathbb{P}_\delta$ and $p \Vdash \operatorname{otp}(\dot{A}) = \check{\gamma}$, where $\dot{A}$ is a name for A. For every $\xi < \gamma$ there is a maximal antichain, A_ξ, deciding the ξth member of A, since the iteration is c.c.c. and $\operatorname{cf}(\delta) > \omega$, there is some α_ξ such that $\operatorname{supp}(p) \subseteq \alpha_\xi$ for all $p \in A_\xi$. Let $\alpha = \sup_{\xi<\gamma} \alpha_\xi$, then the name $\dot{A}_* = \{\langle p, \check{\beta} \rangle \mid p \in A_\xi, p \Vdash \check{\beta} \in \dot{A}\}$ is a $\mathbb{P}_\alpha$-name, and it is not hard to see that $\mathbb{1}_\delta \Vdash \dot{A} = \dot{A}_*$. $\square$

Finally, suppose that $G \subseteq \mathbb{P}_\kappa$ is a V-generic filter. Since $|\mathbb{P}_\kappa| = \kappa$ and it is a c.c.c. forcing, $V[G] \models 2^{\aleph_0} \leq (\kappa^{\aleph_0})^V = \kappa$. We also added Cohen reals in every limit stage of countable cofinality, and therefore $V[G] \models 2^{\aleph_0} = \kappa$. Suppose that $\mathbb{Q} \in V[G]$ is a c.c.c. forcing of size $< \kappa$, and suppose that for some $\gamma < \kappa$, $\mathcal{D} = \{D_\alpha \mid \alpha < \gamma\}$ is a family of dense open subsets of $\mathbb{Q}$.

By Lemma 7.13 there is some α such that $\mathbb{Q}$ and $\mathcal{D}$ are in $V[G \upharpoonright \alpha]$. So, there is some $\mathbb{P}_\alpha$-name for a c.c.c. forcing which will be interpreted as $\mathbb{Q}$. In particular, in V this name is some $a \in [\kappa \times \kappa]^{<\kappa}$, so it must appear as $f(\beta)$ for some $\beta \geq \alpha$, since $V[G] \models$ "$\mathbb{Q}$ is c.c.c.", we are below a condition which forced $\dot{\mathbb{Q}}_\beta = f(\beta)$, so we forced with $\mathbb{Q}$ at that stage. This means that $G(\beta)$ is a $V[G \upharpoonright \beta]$-generic filter for $\mathbb{Q}$, and must meet all the sets in $\mathcal{D}$ as wanted.

Chapter 8

Proper forcing

8.1 Basics

Definition 8.1. Let $\mathbb{P}$ be a forcing notion, let θ be a large enough regular cardinal, and let $M \prec H(\theta)$ be a countable elementary submodel such that $\mathbb{P} \in M$. We say that $q \in \mathbb{P}$ is an *M-generic condition* if whenever $D \in M$ is predense, $D \cap M$ is predense below q.

Exercise 8.2. Show that we can limit ourselves only to dense open sets/maximal antichains in M.

Exercise 8.3. Show that if q is M-generic and $r \leq q$, then r is M-generic.

What is "large enough" in this context, then? Well. Forcing is about the dense and predense subsets of the forcing notion, so large enough is simply a cardinal which can capture the power set of $\mathbb{P}$. Namely, $\theta > |2^{\mathbb{P}}|$. We will adopt the terminology that M is *suitable* if it is a countable elementary submodel of a large enough $H(\theta)$ and $\mathbb{P} \in M$.

Theorem 8.4. *Let $\mathbb{P}$ be a forcing notion and $M \prec H(\theta)$ a suitable model. The following are equivalent for $q \in \mathbb{P}$:*

1. *q is M-generic.*
2. *If $\dot\alpha \in M$ is a name for an ordinal, then $q \Vdash \dot\alpha \in \check M$.*
3. *$q \Vdash \dot G \cap M$ is M-generic for $\mathbb{P} \cap M$.*

Proof. We will prove that (2) and (3) are equivalent to (1).

Assume (1) holds, and suppose that q is M-generic and let $\dot\alpha \in M$ be a name for an ordinal. Then the set $D = \{p \in \mathbb{P} \mid \exists\beta, p \Vdash \dot\alpha = \check\beta\}$ is a dense open set, and by elementarity it lies in M. It follows, from elementarity, that if $p \in D \cap M$, then for some $\beta \in M$, $p \Vdash \dot\alpha = \check\beta$. Since $D \cap M$ is predense below q, it must be that $q \Vdash \dot\alpha \in \check M$. Given any $\bar q \leq q$ such that $\bar q \in D$, it is compatible with a condition in $D \cap M$ and therefore must force that $\dot\alpha$ equals to some $\beta \in M$.

Assume now that (2) holds. Let $D \in M$ be a maximal antichain. Pick, in M, a well-ordering of $D = \{p_\xi \mid \xi < \delta\}$. Note that $p_\xi \in M$ if and only if $\xi \in M$. Consider the name $\dot\alpha$ obtained by mixing such that $p_\xi \Vdash \dot\alpha = \check\xi$. Since this name is definable from D and the well-ordering, so by elementarity $\dot\alpha \in M$. Therefore $q \Vdash \dot\alpha \in \check M$, so $D \cap M$ must be predense below q, since it means that any $\bar q \leq q$ which extends some $p_\xi \in D$ must have that $p_\xi \in D \cap M$.

Next, assume that (1) holds, and we will show that (3) holds. Suppose that G is a V-generic filter for $\mathbb{P}$ and $q \in G$, then by being M-generic, given any $D \in M$ predense, $D \cap M$ is predense below q. Since $q \in G$, there is some condition $p \in D \cap M \cap G$ as well. Therefore $G \cap M$ meets every predense set $D \in M$. Note that $M \models$ "D is predense" if and only if $D \cap M$ is predense in $\mathbb{P} \cap M$ by the elementarity of M, so indeed $G \cap M$ is M-generic for $\mathbb{P} \cap M$.

Finally, assume (3), then given any $D \in M$ which is predense, as we noted this is equivalent to $D \cap M$ being predense in $\mathbb{P} \cap M$. Since $q \Vdash \dot{G} \cap \check{M} \cap \check{D} \neq \check{\varnothing}$, we have that $D \cap M$ must be predense below q. $\square$

Definition 8.5. We say that a forcing notion $\mathbb{P}$ is *proper* if for all large enough regular θ, if $M \prec H(\theta)$ is suitable, then for any $p \in \mathbb{P} \cap M$ there is $q \leq p$ such that q is M-generic.

Exercise 8.6. Show that "for all large enough regular θ" is equivalent to "some regular cardinal $\theta \geq |2^{\mathbb{P}}|^+$" and to "for $\theta = |2^{\mathbb{P}}|^+$".

8.2 Proper properties of forcing

In a sense, properness is a generalisation of c.c.c., since what we are saying is that q being M-generic means that "relevant maximal antichains below q are countable". But as the following theorem shows, this is a generalisation in an even stronger sense.

Theorem 8.7. *If $\mathbb{P}$ is c.c.c., then $\mathbb{P}$ is proper. In fact, $\mathbb{P}$ is c.c.c. if and only if $\mathbb{1}_{\mathbb{P}}$ is M-generic for any suitable M.*

Proof. Let M be a suitable model, taking any maximal antichain $D \in M$. Since $\mathbb{P}$ is a c.c.c. forcing, D is countable, and therefore $D \subseteq M$.[23] And so we get that $D \cap M = D$ is predense below $\mathbb{1}_{\mathbb{P}}$.

In the other direction of the equivalence, assume that $\mathbb{1}_{\mathbb{P}}$ is M-generic for any suitable M, and let D be a maximal antichain, taking a suitable M such that $D \in M$, we get that $D \cap M$ is predense. However, since $D \cap M \subseteq D$ is itself an antichain, it has to be that $D \cap M = D$, and therefore D is countable. $\square$

Equivalently, of course, we can say that every condition is M-generic for any suitable M. Being a generalisation of c.c.c., we want to know if it is an actual generalisation, or if this is just a recasting of c.c.c. in fancy terms.

Theorem 8.8. *If $\mathbb{P}$ is σ-closed, then $\mathbb{P}$ is proper.*

Proof. Let $\mathbb{P}$ be a σ-closed and let M be suitable. Let $\{D_n \mid n < \omega\}$ be an enumeration of all dense open subsets of $\mathbb{P}$. Let $p \in \mathbb{P} \cap M$, we define a descending sequence of conditions, $p_0 \leq p$ such that $p_0 \in D_0 \cap M$, and $p_{n+1} \leq p_n$ is a condition such that $p_{n+1} \in D_{n+1} \cap M$. Since $\mathbb{P}$ is σ-closed, let q be a lower bound for $\{p_n \mid n < \omega\}$. We claim that q is M-generic.

Let $D \in M$ be a dense open set, then there is some $n < \omega$ such that $D = D_n$, therefore $p_n \in D \cap M$. Since $q \leq p_n$ we get that $\{p_n\}$ is predense below q, and therefore $D \cap M$ must also be predense below q. $\square$

Since we know that σ-closed forcing is only c.c.c. if it is trivial, this shows that we have a lot more than just c.c.c. forcings in the class of all proper forcings. But maybe it is the entire class of all forcings?

[23]Since $\omega + 1 \subseteq M$ and there is a bijection $f \colon \omega \to D$ in M, $f(n) \in M$ for all $n < \omega$.

Theorem 8.9. *Let $\mathbb{P}$ be a proper forcing, and let $S \subseteq \omega_1$ be a stationary set. Then $\mathbb{P}$ preserves the stationarity of S. In particular, ω_1 is not collapsed.*

We saw that $\mathrm{Club}(S)$ is σ-distributive, but if S is stationary and co-stationary, then $\mathrm{Club}(S)$ destroys the stationarity of $\omega_1 \setminus S$ and therefore cannot be proper.[24] This also shows that $\mathrm{Col}(\omega, \kappa)$ is not proper for any $\kappa > \omega$.

Proof. Let S be a stationary set and let $\mathbb{P}$ be a proper forcing. Suppose that $\dot{f}$ is a name such that $p \Vdash$ "$\dot{f}\colon \check{\omega}_1 \to \check{\omega}_1$ is a normal function",[25] we will show that there is $q \leq p$ such that $q \Vdash \exists\delta \in \check{S}, \dot{f}``\delta \subseteq \delta$. Let M be a suitable model such that $p, S, \dot{f} \in M$ and $M \cap \omega_1 = \delta \in S$.[26] Since $p \Vdash \dot{f}\colon \check{\omega}_1 \to \check{\omega}_1$, if $\alpha \in M$ is a countable ordinal, then there is a name $\dot{\beta} \in M$ such that $\mathbb{1}$ forces $\dot{\beta}$ to be a name for an ordinal and $p \Vdash \dot{f}(\check{\alpha}) = \dot{\beta}$. Let $q \leq p$ be an M-generic condition, then $q \Vdash \dot{\beta} \in \check{M}$, but since $q \leq p$, it not hard to verify that $q \Vdash \dot{\beta} < \check{\delta}$. It follows that q forces that if $\alpha < \delta$, then $f(\alpha) < \delta$, as wanted. □

Corollary 8.10. *Not every σ-distributive forcing is proper.* □

Remark. We can define the notions of club and stationary sets in $[\kappa]^\omega$ when $\kappa > \omega$. It turns out that $\mathbb{P}$ is proper if and only if it preserves stationarity in that sense for all $\kappa > \omega$.

Exercise 8.11. Suppose that $\mathbb{P}$ is proper, $G \subseteq \mathbb{P}$ is V-generic, and $A \in V[G]$ is a countable set of ordinals. Then there is some $B \in V$ such that B is countable and $A \subseteq B$.

Proposition 8.12. *The lottery sum of any number of proper forcings is proper.*

Proof. Suppose $\mathbb{P} = \bigoplus_{i \in I} \mathbb{P}_i$ where each $\mathbb{P}_i$ is proper, and let M be a suitable model for $\mathbb{P}$. Given any $p \in \mathbb{P} \cap M$, there is some $i \in I \cap M$ such that $p \in \mathbb{P}_i \cap M$. Since M is also suitable for $\mathbb{P}_i$, there is some $q \leq p$ which is M-generic. □

8.3 Baumgartner clubs

Theorem 8.13. *Let $\mathbb{B}$ denote the partial order of finite partial functions $p\colon \omega_1 \to \omega_1$, such that there is a normal function $f\colon \omega_1 \to \omega_1$ with $p \subseteq f$. We order $\mathbb{B}$ by reverse inclusion, namely $q \leq p$ if and only if $p \subseteq q$. Then $\mathbb{B}$ adds a new club to ω_1 and $\mathbb{B}$ is proper.*

Proof. Let $F = \bigcup G$, where G is a V-generic filter for $\mathbb{B}$. It is not hard to see that $F\colon \omega_1 \to \omega_1$ and that $\alpha < \beta$ implies $F(\alpha) < F(\beta)$. To see that F is continuous, let δ be a limit ordinal, and let $\sup F``\delta = \gamma$, then $\gamma \leq F(\delta)$. Towards a contradiction, assume that $\gamma < F(\delta)$, then there is some $p \in G$ such that $p \Vdash \forall\alpha < \check{\delta}, \dot{F}(\alpha) \leq \check{\gamma}$. However, since p is finite, we can find some large enough $\alpha < \delta$ and set $q = p \cup \{\langle \alpha+1, \gamma+1 \rangle\}$. This means that $q \Vdash \dot{F}(\check{\alpha}+1) > \check{\gamma}$, which would be impossible. Therefore $F\colon \omega_1 \to \omega_1$ is continuous. It remains to verify that $\mathbb{B}$ is proper.

Suppose that M is a suitable model and $p \in \mathbb{B} \cap M$. Let $\delta = \omega_1 \cap M$, then we claim that $q = p \cup \{\langle \delta, \delta \rangle\}$ is M-generic. Suppose that $D \in M$ is dense and let $r \leq q$ be any condition. Consider $r \cap M$, since for all $\alpha \in \operatorname{dom} r$, $\alpha < \delta$ if and only if $r(\alpha) < \delta$, $r \cap M \in \mathbb{B} \cap M$. Therefore, there is some $r' \in D$ such that $r' \leq r \cap M$. Easily, r' is compatible with r, so $D \cap M$ is predense below q. □

Exercise 8.14 (*). Let B be the generic club added by $\mathbb{B}$, then B contains no infinite subsets from V. In other words, if $B \cap A \in V$, then $B \cap A$ is finite.

[24] There is a generalisation of properness to *S-proper* when S is a stationary set and $\mathrm{Club}(S)$ is indeed S-proper.

[25] Recall that a function is normal if it is continuous and strictly increasing, or equivalently, if it is the enumeration of a club.

[26] Recall the proof of Theorem 4.18.

8.4 General facts about properness

Fact 8.15. *If $\mathbb{P}$ is proper and $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\dot{\mathbb{Q}}$ is proper", then $\mathbb{P} * \dot{\mathbb{Q}}$ is proper. In fact, the countable support iteration of proper forcings is proper.*

Exercise 8.16. If $\mathbb{P} * \dot{\mathbb{Q}}$ is proper, then $\mathbb{P}$ is proper and $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\dot{\mathbb{Q}}$ is proper". In other words, if $\mathbb{P} \lessdot \mathbb{R}$ and $\mathbb{R}$ is proper, then $\mathbb{P}$ is proper and $\mathbb{R}/\mathbb{P}$ is proper.

As a consequence of the fact above, the countable support iteration of $\mathrm{Add}(\omega, 1)$ is proper and therefore does not collapse ω_1. This is in stark contrast to Theorem 5.21.

Definition 8.17. The *Proper Forcing Axiom*, or PFA, is $\mathsf{FA}_{<2^{\aleph_0}}(\mathbf{Proper})$ where **Proper** is the class of all proper forcings.

Since every c.c.c. forcing is proper, PFA implies MA.

Fact 8.18. PFA *proves that $2^{\aleph_0} = \aleph_2$, and so it is equivalent to* $\mathsf{FA}_{\omega_1}(\boldsymbol{Proper})$.

Fact 8.19. *We cannot prove* PFA *is consistent starting from "just"* ZFC *as we did with* MA*. It is known that* PFA *implies the consistency of fairly strong large cardinal axioms (e.g., it implies the Axiom of Determinacy holds in an inner model, which already implies the consistency of infinitely many Woodin cardinals). We can prove* PFA *is consistent by starting from a supercompact cardinal which is a fairly large large cardinal axiom.*

We can weaken the definition of properness in the following way, p is an *M-semigeneric condition* if whenever $\dot{\alpha} \in M$ is a name for a *countable* ordinal, then $p \Vdash \dot{\alpha} \in \check{M}$. With this, $\mathbb{P}$ is *semiproper* if for every suitable M, every $p \in \mathbb{P} \cap M$ extends to an M-semigeneric condition. The above proof shows that semiproper forcing must also preserve stationary sets. Indeed, we can focus on the even-larger class of forcings which preserve stationary subsets of ω_1.

Both of these have their own forcing axioms, SPFA is the semiproper forcing axiom, and Martin's Maximum (MM) is the forcing axiom for stationary set preserving forcings. These can be augmented further, into MM^+ and MM^{++}, where we require that the $\{D_\alpha \mid \alpha < \omega_1\}$-generic will also interpret a given name for a stationary set correctly, or in the case of MM^{++}, we have a family of size $\aleph_1$ of names for stationary sets.

Chapter 9

Coda for Forcing: Prikry forcing

9.1 Measurable cardinals

Definition 9.1. We say that a cardinal κ is *measurable* if there is a free κ-complete ultrafilter on κ. Namely, there is an ultrafilter $\mathcal{U} \subseteq \mathcal{P}(\kappa)$ which contains all the cofinite sets and given any $\{A_\alpha \mid \alpha < \gamma\} \subseteq \mathcal{U}$ with $\gamma < \kappa$, $\bigcap_{\alpha<\gamma} A_\alpha \in \mathcal{U}$.

We will always assume that our κ is uncountable, although in some instances it is useful to allow ω to be a measurable cardinal. In any case, if κ is a measurable cardinal, a measure would always mean a free κ-complete ultrafilter.

Exercise 9.2. If κ is measurable, then κ is regular and a strong limit.

Exercise 9.3 (*). If κ is the least cardinal with a free and $\aleph_1$-complete ultrafilter, then κ is measurable.

Theorem 9.4. *κ is measurable if and only if there is an elementary embedding $j\colon V \to M$, where M is a transitive class, and $\operatorname{crit}(j) = \kappa$. Namely, $\kappa = \min\{\alpha \in \mathrm{Ord} \mid \alpha < j(\alpha)\}$.*

Proof. Suppose that κ is measurable, let $\mathcal{U}$ be a measure and consider the ultrapower $V^\kappa/\mathcal{U}$. Since $\mathcal{U}$ is κ-complete, the ultrapower is well-founded. To see this, suppose that $f_n\colon \kappa \to V$ is a sequence of functions such that $[f_{n+1}]_\mathcal{U} \in_\mathcal{U} [f_n]_\mathcal{U}$, then $A_n = \{\alpha < \kappa \mid f_{n+1}(\alpha) \in f_n(\alpha)\} \in \mathcal{U}$ for all n, then by κ-completeness, $\bigcap_{n<\omega} A_n \in \mathcal{U}$ and so non-empty. Let α be an element of the intersection, then $\{f_n(\alpha) \mid n < \omega\}$ is ill-founded, which is impossible.

Let M be the transitive collapse of $V^\kappa/\mathcal{U}$,[27] and let j be the ultrapower embedding, namely $j(x) = [c_x]_\mathcal{U}$ where $c_x\colon \kappa \to \{x\}$. We claim that $\kappa = \operatorname{crit}(j)$. For $\alpha < \kappa$, if $M \models \beta \in j(\alpha)$, then there is a function f such that $[f]_\mathcal{U} = \beta$, then $F = \{\xi < \kappa \mid f(\xi) < \alpha\} \in \mathcal{U}$. Consider for all $\gamma < \alpha$ the set $\{\xi < \kappa \mid f(\xi) = \gamma\}$, then this is a partition of F into $|\alpha|$ parts, so by κ-completeness, exactly one of them must be in $\mathcal{U}$, so $[f]_\mathcal{U} = [c_\beta]_\mathcal{U}$ for some $\beta < \alpha$. So the ordinal $j(\alpha)$ must be equal to α itself. On the other hand, for all $\alpha < \kappa$, $[c_\alpha]_\mathcal{U} < [\mathrm{id}]_\mathcal{U} < [c_\kappa]_\mathcal{U}$, where id is the identity function, so $\kappa < j(\kappa)$.

In the other direction, if κ is $\operatorname{crit}(j)$, let $\mathcal{U} = \{A \subseteq \kappa \mid \kappa \in j(A)\}$. We claim that $\mathcal{U}$ is a measure on κ. Easily, this is a filter. It clearly does not contain any finite subset, since if $A \subseteq \kappa$ is finite, then $j(A) = A$, so it is free. Since for all $A \subseteq \kappa$ either $\kappa \in j(A)$ or $\kappa \notin j(A)$, $\mathcal{U}$ is also an ultrafilter. Finally, if $\gamma < \kappa$ and $\{A_\alpha \mid \alpha < \gamma\} \subseteq \mathcal{U}$, then $j(\{A_\alpha \mid \alpha < \gamma\}) = \{j(A_\alpha) \mid \alpha < \gamma\}$ since $j(\gamma) = \gamma$, since κ belongs to each $j(A_\alpha)$, it must be in the intersection. □

[27] We will judicially confuse the elements of M with their ultrapower representations.

The measure we defined from j is called "the derived measure".

Exercise 9.5. If κ is a measurable cardinal, show that there exists a measure $\mathcal{U}$ on κ such that $\{\alpha < \kappa \mid \alpha \text{ is strongly inaccessible}\} \in \mathcal{U}$. (Hint: use the measure derived from j in the above proof and show that κ is strongly inaccessible in M.)

Definition 9.6. We say that a measure $\mathcal{U}$ on κ is normal if whenever $f\colon \kappa \to \kappa$ is regressive, namely $f(\alpha) < \alpha$ for all $\alpha > 0$, there is some $A \in \mathcal{U}$ such that f is constant on A.

Exercise 9.7. Show that the measure derived from an elementary embedding is always normal.

Exercise 9.8. $\mathcal{U}$ is a normal measure on κ if and only if $[\mathrm{id}]_{\mathcal{U}} = \kappa$ if and only if it is closed under diagonal intersections.[28]

9.2 Prikry forcing

Definition 9.9. Let κ be a measurable cardinal and let $\mathcal{U}$ be a normal measure on κ. $\mathbb{P}_{\mathcal{U}}$ is the forcing notions whose conditions are $\langle s, A\rangle$ where $s \in [\kappa]^{<\omega}$ is an increasing finite sequence and $A \in \mathcal{U}$ such that $\max \operatorname{rng} s < \min A$. We order $\mathbb{P}_{\mathcal{U}}$ by $\langle s_q, A_q\rangle \leq \langle s_p, A_p\rangle$ if and only if s_q is an end-extension of s_p, $A_q \subseteq A_p$ and $s_q \setminus s_p \subseteq A_p$.

This forcing is also known as *Prikry forcing*. It has many generalisations, from tree-type for measures that are not normal, to larger and more complicated large cardinals which have an embedding or an ultrafilter associated with them. We will refer to s as "the stem" and to A as "the upper part".

Theorem 9.10. *Let $\mathbb{P} = \mathbb{P}_{\mathcal{U}}$ for some normal measure on κ, then the following properties hold:*

1. $\mathbb{1}_{\mathbb{P}} \Vdash \operatorname{cf}(\check{\kappa}) = \check{\omega}$.
2. *$\mathbb{P}$ does not add bounded subsets to κ, so in particular $\mathbb{P}$ preserves cardinals up to κ.*
3. *$\mathbb{P}$ has κ^+-c.c., so $\mathbb{P}$ preserves all cardinals above κ (and so, all cardinals).*

Proof. The first property is easy to verify, let $c = \bigcup_{\langle s,A\rangle \in G} s$, where G is a V-generic, then by genericity, it is easy to see that c is a cofinal sequence of order type ω. The third property is also easy to show, since clearly if $\langle s, A\rangle$ and $\langle s, B\rangle$ are compatible, and there are only κ different stems.

For the second property we define an auxiliary order, we say that $q = \langle s_q, A_q\rangle \leq^* \langle s_p, A_p\rangle = p$ (or that q is a *direct extension* of p) if $s_q = s_p$. It is immediate, from the κ-completeness of $\mathcal{U}$, that $\leq^*$ is a κ-closed order. To prove that no bounded subsets of κ are added we will need the following lemma: if p is any condition and φ is any statement in the language of forcing, then there is a direct extension $q \leq^* p$ such that q decides the truth value of φ.

Let us assume the truth of this lemma first. Suppose that $\dot{a}$ is a name such that $p \Vdash \dot{a} \subseteq \check{\gamma}$ for some $\gamma < \kappa$. We can now find a decreasing sequence of direct extensions p_α such that p_α decides $\check{\alpha} \in \dot{a}$, for $\alpha < \gamma$. Since $\leq^*$ is κ-closed, there is some $q \leq^* p$ such that q has decided all the information about $\dot{a}$. Namely, letting $u = \{\alpha < \gamma \mid q \Vdash \check{\alpha} \in \dot{a}\}$ we have that $q \Vdash \dot{a} = \check{u}$. Therefore no new sets are added below κ and therefore all cardinals are preserved. ☐

The cofinal sequence added by Prikry forcing is called a "Prikry sequence".

[28] Recall that $\triangle_{\alpha<\kappa} A_\alpha = \{\xi < \kappa \mid \forall \beta < \xi, \xi \in A_\beta\}$.

Lemma 9.11 (The Prikry Lemma). *Let $\mathbb{P} = \mathbb{P}_{\mathcal{U}}$ be a Prikry forcing, where $\mathcal{U}$ is a normal measure on κ. Then given any $p \in \mathbb{P}$ and φ, there is $q \leq^* p$ such that q decides φ.*

Proof. Let $p = \langle s, A\rangle$ be any condition and φ a formula. Consider the function on $[A]^{<\omega}$:

$$F(t) = \begin{cases} 0, & \exists B \subseteq A, \langle s \cup t, B\rangle \Vdash \varphi \\ 1, & \exists B \subseteq A, \langle s \cup t, B\rangle \Vdash \neg\varphi \\ 2, & \text{Otherwise.} \end{cases}$$

Note that if $F(t) < 2$, this does not depend on B, since if $\langle t, B\rangle \Vdash \varphi$ and $C \subseteq A$ is any other set, then $\langle t, C\rangle$ is compatible with $\langle t, B\rangle$, so no extension of $\langle t, C\rangle$ can force $\neg\varphi$, and therefore $\langle t, C\rangle \Vdash \varphi$ (and similarly for the negation of φ).

Let us assume that there is some $H \subseteq A$ such that $H \in \mathcal{U}$ and for all $n < \omega$, $F``[H]^n$ is a singleton. We claim that $q = \langle s, H\rangle$ decides the value of φ. Assume otherwise, then there are $t, t' \subseteq H$ such that $\langle s \cup t, B\rangle$ and $\langle s \cup t', B'\rangle$ force opposite values of φ. Without loss of generality we can assume that $|t| = |t'| = n$, but in that case $F(t) \neq F(t')$ which is impossible since F is constant on $[H]^n$. Of course, it could very well be that $[H]^n$ takes the value of 0 everywhere, but this is impossible since there will be eventual $t \subseteq H$ such that $\langle s \cup t, B\rangle$ decides φ, so it follows that any extension of that length must have value 0 or 1. □

Lemma 9.12. *Suppose that $\mathcal{U}$ is a normal measure on κ and let $F\colon [\kappa]^{<\omega} \to \gamma$ be a function such that $\gamma < \kappa$, then there is some $H \in \mathcal{U}$ such that for all $n < \omega$, F is constant on $[H]^n$.*

Proof. It is enough to show that given any function defined on $[\kappa]^n \to \gamma$, for any γ, there is such a homogeneous set. If this is the case, let $H_n \in \mathcal{U}$ be the homogeneous set for $[\kappa]^n$, then $H = \bigcap_{n<\omega} H_n$ is our desired set. We prove this by induction on n.

For $n = 1$ this is just the κ-completeness of $\mathcal{U}$. Assume this is true for $[\kappa]^n$, and let $F\colon [\kappa]^{n+1} \to \gamma$ be a function with $\gamma < \kappa$. We define for each $\alpha < \kappa$, $F_\alpha(t) = F(t \cup \{\alpha\})$, defined on $[\kappa \setminus \{\alpha\}]^n$. By the induction hypothesis, for each α there is some $A_\alpha \in \mathcal{U}$ which is homogeneous for F_α. We take $A = \triangle_{\alpha<\kappa} A_\alpha$, then by the normality we have that $A \in \mathcal{U}$ as well. By definition, if $y, x \in [A]^{n+1}$ and $\alpha = \min x = \min y$, then $F(y) = F_\alpha(y \setminus \{\alpha\}) = F_\alpha(x \setminus \{\alpha\}) = F(x) = \xi < \gamma$. Since $\gamma < \kappa$, there is $H \subseteq A$ such that F is constant on $[H]^{n+1}$. □

Proposition 9.13. *Prikry forcing is semiproper but not proper.*

Proof. The Prikry sequence is a countable set of ordinals, and it is easy to see it is not covered by any countable set in the ground model, since any countable subset of κ is bounded in κ. So it is not proper. Let us see that it is semiproper.

First note that if $\dot\alpha$ is a name for a countable ordinal and p is a condition, then there is exactly one $\beta < \omega_1$ for which there is a direct extension $q \leq^* p$ such that $q \Vdash \dot\alpha = \check\beta$, since any two direct extensions are compatible, and by the Prikry Lemma such q must exist. Moreover, if $p, \dot\alpha \in M$ a suitable model, then by elementarity such $q \in M$ as well, and therefore $\beta \in M$.

Let $\dot\alpha_n$, for $n < \omega$, be an enumeration of all the names for countable ordinals in M, and let $p \in \mathbb{P} \cap M$ be any condition. Then there is a descending sequence of direct extensions, $q_n \in M$, such that q_n decides the value of $\beta_n \in M \cap \omega_1$. Let $q \leq^* q_n$ for all $n < \omega$, then if $\dot\alpha \in M$ is a name for a countable ordinal, there is some n such that $\dot\alpha = \dot\alpha_n$, and therefore $q \Vdash \dot\alpha = \check\beta_n \in \check M$. □

Exercise 9.14 (*). Suppose that κ is measurable in V and $\mathcal{U}$ is a normal measure and in $W \supseteq V$ there is a sequence α_n such that $\sup_{n<\omega} \alpha_n = \kappa$. Suppose that for any $A \in \mathcal{U}$ there is some n_0 such that $\{\alpha_n \mid n > n_0\} \subseteq A$. Then $\{\alpha_n \mid n < \omega\}$ is a Prikry sequence over V for $\mathbb{P}_{\mathcal{U}}$.

Part II

Choiceless Constructions: Symmetric and Generic Alike

Chapter 10

The basics of symmetric extensions: automorphisms and whatnot

10.1 Introduction

The problem with generic extensions, as you may have noticed, at least in the context of choiceless results, is that generic extensions preserve the Axiom of Choice. This is not a bad thing when studying models of ZFC, of course, but if one wants to prove that AC is not provable from ZF, then forcing, on its own, will require us to start with a ground model of $\mathsf{ZF} + \neg\mathsf{AC}$, which defeats the purpose.

However, we can observe that two pairwise generic objects (for the same forcing) will, in a sense, be indiscernible to the ground model. That is, from the forcing-perspective of the ground model, it is impossible to tell a priori which object will have which property. For example, adding two Cohen reals, we cannot decide in advance which one appears first in the linear ordering of the reals. Of course, there will be some condition that decides this information, but it is not $\mathbb{1}$.

Utilising this very observation we can implement, as Cohen did, the ideas that began with Fraenkel, Mostowski, and later Specker, for constructing models of ZF with atoms where the Axiom of Choice fails[29] and using automorphisms of the forcing notion to find an inner model of the generic extension where the Axiom of Choice fails.

10.2 Automorphisms

For the rest of the discussion, $\mathbb{P}$ is an arbitrary, but fixed, notion of forcing in V.

Definition 10.1. $\pi\colon \mathbb{P} \to \mathbb{P}$ is an *automorphism* if it is an bijection satisfying $\pi q \leq \pi p \iff q \leq p$.

Exercise 10.2. If π is an automorphism of $\mathbb{P}$, then $\pi(\mathbb{1}) = \mathbb{1}$.

Exercise 10.3. If π is an automorphism of $\mathbb{P}$ and $D \subseteq \mathbb{P}$ is predense/dense/dense open (below p), then $\pi``D = \{\pi q \mid q \in D\}$ is predense/dense/dense open (below πp).

Exercise 10.4. If π is an automorphism of $\mathbb{P}$, then it extends (uniquely) to its Boolean completion. Indeed, if π is an automorphism defined on a dense subset of $\mathbb{P}$, then π extends (uniquely) to the Boolean completion of $\mathbb{P}$.

[29] Also known as *permutation models*, or FM models, or FMS models.

Definition 10.5. Let π be an automorphism of $\mathbb{P}$, then π acts on the $\mathbb{P}$-names with the following recursive definition:

$$\pi\dot{x} = \{\langle \pi p, \pi\dot{y}\rangle \mid \langle p, \dot{y}\rangle \in \dot{x}\}.$$

Exercise 10.6. If $x \in V$, then $\pi\check{x} = \check{x}$ for any automorphism π.

Exercise 10.7. If π is an automorphism of $\mathbb{P}$, then π preserves the $\mathbb{P}$-rank of $\mathbb{P}$-names.

Lemma 10.8 (The Symmetry Lemma). *Suppose that π is an automorphism of $\mathbb{P}$, then for any $\mathbb{P}$-names $\dot{x}_0, \ldots, \dot{x}_{n-1}$ and formula φ*

$$p \Vdash \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1}) \iff \pi p \Vdash \varphi(\pi\dot{x}_0, \ldots, \pi\dot{x}_{n-1}).$$

Proof. We prove this by induction on the complexity of φ, and by induction on the ranks of $\dot{x}_0, \dot{x}_1$ for $\dot{x}_0 \in \dot{x}_1$ and $\dot{x}_0 = \dot{x}_1$.

$$\begin{aligned}
&p \Vdash \dot{x}_0 \in \dot{x}_1 \\
&\qquad\iff \{q \in \mathbb{P} \mid \exists\langle r, \dot{y}\rangle \in \dot{x}_1, q \leq r \wedge q \Vdash \dot{x}_0 = \dot{y}\} \text{ is dense below } p \\
&\qquad\iff \{\pi q \in \mathbb{P} \mid \exists\langle r, \dot{y}\rangle \in \dot{x}_1, q \leq r \wedge q \Vdash \dot{x}_0 = \dot{y}\} \text{ is dense below } \pi p \\
&\qquad\iff \{\pi q \in \mathbb{P} \mid \exists\langle \pi r, \pi\dot{y}\rangle \in \pi\dot{x}_1, \pi q \leq \pi r \wedge \pi q \Vdash \pi\dot{x}_0 = \pi\dot{y}\} \text{ is dense below } \pi p \\
&\qquad\iff \{q \in \mathbb{P} \mid \exists\langle \dot{r}, \dot{y}\rangle \in \pi\dot{x}_1, q \leq r \wedge q \Vdash \pi\dot{x}_0 = \dot{y}\} \text{ is dense below } \pi p \\
&\qquad\iff \pi p \Vdash \pi\dot{x}_0 \in \pi\dot{x}_1.
\end{aligned}$$

The proof for $\dot{x}_0 = \dot{x}_1$ is similar, as are the proofs for more complicated φ. □

Corollary 10.9. *For any automorphism π, $\mathbb{1} \Vdash \pi\dot{G}$ is a V-generic filter.* □

Exercise 10.10. $(\pi^{-1}\dot{G})^G = \pi``G$.

Proposition 10.11. *Suppose that $\dot{x}$ is a $\mathbb{P}$-name, π is an automorphism of $\mathbb{P}$, and G is a V-generic filter. Then $\dot{x}^G = (\pi\dot{x})^{\pi``G}$.*

Proof. We prove this by induction on the $\mathbb{P}$-rank of $\dot{x}$.

$$\begin{aligned}
y \in \dot{x}^G &\iff \exists p \in G, \dot{y} : \langle p, \dot{y}\rangle \in \dot{x} \wedge \dot{y}^G = y \\
&\iff \exists \pi p \in \pi``G, \pi\dot{y}, \langle \pi p, \pi\dot{y}\rangle \in \pi\dot{x} \wedge (\pi\dot{y})^{\pi``G} = y \\
&\iff y \in (\pi\dot{x})^{\pi``G}.
\end{aligned}$$

□

10.3 ...and whatnot

Definition 10.12. Let $\mathscr{G}$ be a group. We say that $\mathscr{F}$ is a *filter of subgroups (on $\mathscr{G}$)* if it is a non-empty family of subgroups of $\mathscr{G}$ which is closed under finite intersections and supergroups. We say that $\mathscr{F}$ is a *normal filter* if whenever $H \in \mathscr{F}$ and $\pi \in \mathscr{G}$, $\pi H \pi^{-1} \in \mathscr{F}$.

Definition 10.13. We say that $\langle \mathbb{P}, \mathscr{G}, \mathscr{F}\rangle$ is a *symmetric system* if $\mathbb{P}$ is a notion of forcing, $\mathscr{G}$ is a group of automorphisms of $\mathbb{P}$, and $\mathscr{F}$ is a normal filter of subgroups on $\mathscr{G}$.

Having fixed $\mathbb{P}$, let us now fix a symmetric system around it, $\langle \mathbb{P}, \mathscr{G}, \mathscr{F}\rangle$.

Definition 10.14. Let $\dot{x}$ be a $\mathbb{P}$-name. We say that $\dot{x}$ is *$\mathscr{F}$-symmetric* if the stabiliser of $\dot{x}$, $\mathrm{sym}_{\mathscr{G}}(\dot{x}) = \{\pi \in \mathscr{G} \mid \pi\dot{x} = \dot{x}\}$, is in $\mathscr{F}$. If this property holds hereditarily for all the names which appear in $\dot{x}$, we say that $\dot{x}$ is *hereditarily $\mathscr{F}$-symmetric*. The class $\mathsf{HS}_{\mathscr{F}}$ is the class of all hereditarily $\mathscr{F}$-symmetric names.

Exercise 10.15. Show that $\mathrm{sym}_{\mathscr{G}}(\pi\dot{x}) = \pi\,\mathrm{sym}_{\mathscr{G}}(\dot{x})\pi^{-1}$, and therefore if $\mathscr{F}$ is normal, the property of being $\mathscr{F}$-symmetric is preserved under the automorphisms from $\mathscr{G}$.

Definition 10.16. Let G be a V-generic filter for $\mathbb{P}$. The class $M = \mathsf{HS}_{\mathscr{F}}^G = \{\dot{x}^G \mid \dot{x} \in \mathsf{HS}_{\mathscr{F}}\}$ is called a *symmetric extension.*

Theorem 10.17. *Let M be the symmetric extension $\mathsf{HS}_{\mathscr{F}}^G$. Then the following properties hold:*

1. *$V \subseteq M \subseteq V[G]$.*
2. *M is a transitive class in $V[G]$.*
3. *$M \models \mathsf{ZF}$.*

Proof. To see that $V \subseteq M$, note that if $x \in V$, then $\check{x} \in \mathsf{HS}_{\mathscr{F}}$. The hereditary nature of $\mathsf{HS}_{\mathscr{F}}$ ensures that M is a transitive class in $V[G]$. So it remains to see that M is a model of ZF. We will use the following fact: If M is a transitive class that is almost universal and satisfies Δ_0-Separation, then M is an inner model of ZF.

First we check that M is almost universal. Let $x \in V[G]$ be such that $x \subseteq M$. We need to find $y \in M$ such that $x \subseteq y$. Let $\dot{x}$ be a name for x, and without loss of generality, we can assume that any name which appears in $\dot{x}$ is in $\mathsf{HS}_{\mathscr{F}}$. Let α be the $\mathbb{P}$-name rank of $\dot{x}$, and let $\dot{y} = \{\dot{u} \in \mathsf{HS}_{\mathscr{F}} \mid \mathrm{rank}(\dot{u}) < \alpha\}^\bullet$. It is not hard to see that $\mathbb{1} \Vdash \dot{x} \subseteq \dot{y}$, as any name appearing in $\dot{x}$ has rank smaller than α and is in $\mathsf{HS}_{\mathscr{F}}$. Since $\mathscr{F}$ is normal, $\pi\dot{y} = \dot{y}$ for all $\pi \in \mathscr{G}$, so $\dot{y} \in \mathsf{HS}_{\mathscr{F}}$ as well. Therefore, M is almost universal.

We prove that M satisfies Δ_0-Separation. Namely, if $x, w \in M$ and $\varphi(u)$ is a Δ_0 formula, then the subset it defines, $y = \{u \in x \mid \varphi(u, w)\}$, is in M.[30] Let $\dot{x}, \dot{w} \in \mathsf{HS}_{\mathscr{F}}$ be names for x and w. We define $\dot{y} = \{\langle p, \dot{u}\rangle \mid \mathrm{rank}(\dot{u}) < \mathrm{rank}(\dot{x}), \dot{u} \in \mathsf{HS}_{\mathscr{F}}, p \Vdash \dot{u} \in \dot{x} \wedge \varphi(\dot{u}, \dot{w})\}$.

We claim, to begin with, that $\dot{y}^G = y$. If $u \in y$, then there are some $p, q \in G$ and a name $\dot{u}$ for u, such that $\langle p, \dot{u}\rangle \in \dot{x}$, $q \leq p$, and $q \Vdash \varphi(\dot{u}, \dot{w})$. Therefore $\langle q, \dot{u}\rangle \in \dot{y}$, so $u \in \dot{y}^G$. In the other direction, if $u \in \dot{y}^G$, then there is some $p \in G$ and a name $\dot{u}$ for u such that $\langle p, \dot{u}\rangle \in \dot{y}$, and therefore $p \Vdash \dot{u} \in \dot{x} \wedge \varphi(\dot{u}, \dot{w})$, so indeed $u \in y$ and therefore $\dot{y}$ is indeed a name for y.

It remains to show that $\dot{y} \in \mathsf{HS}_{\mathscr{F}}$. Let $\pi \in \mathrm{sym}_{\mathscr{G}}(\dot{x}) \cap \mathrm{sym}_{\mathscr{G}}(\dot{w})$. Then by the Symmetry Lemma we have

$$p \Vdash \dot{u} \in \dot{x} \wedge \varphi(\dot{u}, \dot{w}) \iff \pi p \Vdash \pi\dot{u} \in \pi\dot{x} \wedge \varphi(\pi\dot{u}, \pi\dot{w}).$$

However, since $\pi\dot{x} = \dot{x}$ and $\pi\dot{w} = \dot{w}$, as well as π preserving being in $\mathsf{HS}_{\mathscr{F}}$ and ranks, we get that $\pi\dot{y} = \dot{y}$, so $\mathrm{sym}_{\mathscr{G}}(\dot{x}) \cap \mathrm{sym}_{\mathscr{G}}(\dot{w}) \subseteq \mathrm{sym}_{\mathscr{G}}(\dot{y})$. Moreover, as every name in $\dot{y}$ is by definition in $\mathsf{HS}_{\mathscr{F}}$ it means that $\dot{y} \in \mathsf{HS}_{\mathscr{F}}$ as wanted. □

Definition 10.18. We have a relativised forcing relation, $\Vdash^{\mathsf{HS}}$, defined by requiring that all names and quantifiers are restricted to HS.

Exercise 10.19. If $\pi \in \mathscr{G}$ and $\dot{x} \in \mathsf{HS}_{\mathscr{F}}$, then $p \Vdash^{\mathsf{HS}} \varphi(\dot{x}) \iff \pi p \Vdash^{\mathsf{HS}} \varphi(\pi\dot{x})$. That is, the Symmetry Lemma holds for $\Vdash^{\mathsf{HS}}$ if we restrict to the group $\mathscr{G}$.

Theorem 10.20. *The following are equivalent:*

1. *$p \Vdash^{\mathsf{HS}} \varphi$.*
2. *For any V-generic, G, such that $p \in G$, $\mathsf{HS}_{\mathscr{F}}^G \models \varphi$.* □

[30] Since M is transitive, it agrees with V on Δ_0-definitions where the parameters are in M. If we wanted to verify Separation in its generality, this would require more finesse.

10.4 Cohen's first model

We finish with one of the most important examples. Cohen's [first] model was the first model of $\mathsf{ZF} + \neg\mathsf{AC}$. It was constructed from $V = L$, but we can use any model of ZFC as a ground model. We will omit the subscripts from $\mathrm{sym}_{\mathscr{G}}$ and $\mathsf{HS}_{\mathscr{F}}$ to improve the readability.

Let $\mathbb{P} = \mathrm{Add}(\omega, \omega)$. Our group $\mathscr{G}$ is the group of finitary permutations of ω, with the action on $\mathbb{P}$ defined by $\pi p(\pi n, m) = p(n, m)$. The filter $\mathscr{F}$ is generated by sets of the form $\mathrm{fix}(E) = \{\pi \in \mathscr{G} \mid \pi \restriction E = \mathrm{id}\}$ where $E \in [\omega]^{<\omega}$.

Exercise 10.21. Show that $\mathscr{F}$ is normal.

We say that E is a *support* for a name $\dot{x}$ if $\mathrm{fix}(E) \subseteq \mathrm{sym}(\dot{x})$. Similarly, E is a support for a condition p if $\mathrm{dom}\, p \subseteq E \times \omega$. Note that if E is a support for p, then for all $\pi \in \mathrm{fix}(E)$, $\pi p = p$. We will write $\mathrm{supp}(p)$ to denote the smallest E which is a support for p.

We define $\dot{a}_n = \{\langle p, \check{m}\rangle \mid p(n, m) = 1\}$ and $\dot{A} = \{\dot{a}_n \mid n < \omega\}^\bullet$.

Proposition 10.22. *$\pi \dot{a}_n = \dot{a}_{\pi n}$ and consequently, $\pi \dot{A} = \dot{A}$.* □

Corollary 10.23. $\dot{a}_n, \dot{A} \in \mathsf{HS}$.

Proof. $\{n\}$ is a support for $\dot{a}_n$, and $\varnothing$ is a support for $\dot{A}$. □

Proposition 10.24. *If $n \neq m$, then $\mathbb{1} \Vdash^{\mathsf{HS}} \dot{a}_n \neq \dot{a}_m$. Therefore $\mathbb{1} \Vdash^{\mathsf{HS}}$ "$\dot{A}$ is infinite".* □

Theorem 10.25. $\mathbb{1} \Vdash^{\mathsf{HS}}$ *$\dot{A}$ cannot be well-ordered.*

Proof. Suppose that $\dot{f} \in \mathsf{HS}$ and $p \Vdash^{\mathsf{HS}} \dot{f}\colon \check{\omega} \to \dot{A}$. We will show that $p \Vdash$ "$\dot{f}$ is not injective". Let E be a support for $\dot{f}$ and p. Let $q \leq p$ be a condition such that for some $n \notin E$ and $i < \omega$, $q \Vdash^{\mathsf{HS}} \dot{f}(\check{i}) = \dot{a}_n$. If no such q exists, then $p \Vdash \mathrm{rng}(\dot{f}) \subseteq \{\dot{a}_n \mid n \in E\}^\bullet$, which is a finite set, and therefore p already forces that $\dot{f}$ is not injective.

Next, pick $m \notin E \cup \{n\} \cup \mathrm{supp}(q)$, and consider π to be the 2-cycle $(n\ m)$. Easily, $\pi \in \mathrm{fix}(E)$ and therefore $\pi \dot{f} = \dot{f}$ and $\pi p = p$. By the proposition above, $\pi \dot{a}_n = \dot{a}_m$. So we have that $\pi q \Vdash^{\mathsf{HS}} \dot{f}(\check{i}) = \dot{a}_m$.

Next, we claim that πq is compatible with q. To see that, note that if $\langle i, j\rangle \in \mathrm{dom}\, q \cup \pi q$, then either $i \in \{n, m\}$ in which case $\langle i, j\rangle$ is in exactly one of the conditions, or else $\pi i = i$ and so $q(i, j) = \pi q(\pi i, j) = \pi q(i, j)$.

Therefore, $q \cup \pi q \Vdash^{\mathsf{HS}} \dot{f}(\check{i}) = \dot{a}_n \neq \dot{a}_m = \dot{f}(\check{i})$. This is of course impossible. Therefore, there cannot be such q, as wanted. □

What we actually see here is that $\dot{A}$ is going to be an infinite set without a countably infinite subset. In other words, an infinite Dedekind-finite set. This implies that not only that AC fails, in fact AC_ω already fails. We will get back to the Cohen model later, as well as see its generalisations and extensions.

Proposition 10.26. *In the Cohen model, $\mathbb{1} \Vdash^{\mathsf{HS}} \exists a \in \dot{A}, \check{0} \in a$, but there is no $\dot{a} \in \mathsf{HS}$ such that $\mathbb{1} \Vdash \check{0} \in \dot{a} \in \dot{A}$. Therefore $\Vdash^{\mathsf{HS}}$ does not have an analogue of the Mixing Lemma.*

Proof. It is easy to check that $\mathbb{1} \Vdash^{\mathsf{HS}} \exists a \in \dot{A}, \check{0} \in a$. Suppose that $\dot{a} \in \mathsf{HS}$ was a name witnessing this and let E be its support. Let p be any condition extending $\{\langle\langle n, 0\rangle, 0\rangle \mid n \in E\}$ which decides for some k that $\dot{a} = \dot{a}_k$. Note that $p \Vdash^{\mathsf{HS}} \dot{a} \neq \dot{a}_n$ for all $n \in E$, so $k \notin E$. Find some $m \notin E \cup \mathrm{supp}(p) \cup \{k\}$ and take π to be the 2-cycle $(m\ k)$. Then, as before, πp is compatible with p, $\pi \dot{a} = \dot{a}$, since $\pi \in \mathrm{fix}(E)$, and $\pi p \Vdash \dot{a} = \dot{a}_m \neq \dot{a}_k$. This is, of course, a contradiction, since then $p \cup \pi p \Vdash^{\mathsf{HS}} \dot{a}_m = \dot{a} = \dot{a}_k$. □

10.5 Tenacity

One thing that was helpful in clarifying the argument in the Cohen model was the fact that given a condition p, we could find a group in the filter which does not move p. Namely, $\operatorname{fix}(p) = \{\pi \in \mathscr{G} \mid \pi p = p\}$ was a large group.

Definition 10.27. Given a symmetric system $\langle \mathbb{P}, \mathscr{G}, \mathscr{F} \rangle$ we say that $p \in \mathbb{P}$ is $\mathscr{F}$-tenacious if $\operatorname{fix}(p) \in \mathscr{F}$. We say that the system itself is tenacious if there is a dense set of tenacious conditions.

We saw that the symmetric system for the Cohen model is tenacious, indeed all the conditions were tenacious. On the other hand, if we had taken $\mathscr{F} = \{\mathscr{G}\}$ instead, then the only condition that is tenacious would be $\mathbb{1}$. Incidentally, in this case the only names in HS would have the form $\check{x}$.

Definition 10.28. We say that two symmetric systems, $\langle \mathbb{P}, \mathscr{G}, \mathscr{F} \rangle$ and $\langle \mathbb{P}', \mathscr{G}', \mathscr{F}' \rangle$, are *equivalent* if for every V-generic filter $G \subseteq \mathbb{P}$ there is a filter $G' \subseteq \mathbb{P}'$ such that $\mathsf{HS}^G_{\mathscr{F}} = \mathsf{HS}^{G'}_{\mathscr{F}'}$ and vice versa.

Note that we do not require that G' is V-generic, just that the interpretation of $\mathsf{HS}_{\mathscr{F}'}$ is "correct enough".

Theorem 10.29. *Every symmetric system is equivalent to one where all conditions are tenacious.*

Definition 10.30. Let $\langle \mathbb{P}, \mathscr{G}, \mathscr{F} \rangle$ be a symmetric system. We say that $A \subseteq \mathbb{P}$ is *symmetric* if $\{\pi \in \mathscr{G} \mid \pi``A = A\} \in \mathscr{F}$. We say that $G \subseteq \mathbb{P}$ is a *symmetrically V-generic* if for every symmetrically dense open set $D \in V$, $D \cap G \neq \varnothing$.

Theorem 10.31. *The following are equivalent:*

1. $p \Vdash^{\mathsf{HS}} \varphi$.
2. *For every V-generic, G, such that $p \in G$, $\mathsf{HS}^G \models \varphi$.*
3. *For every symmetrically V-generic, G, such that $p \in G$, $\mathsf{HS}^G \models \varphi$.*

Proof. We already know that (1) and (2) are equivalent. We also know that every generic filter is symmetrically generic, so (3) implies (2) trivially. It remains to prove that (1) implies (3). The complete proof is done by induction on the formula and the rank of the names, but let us only consider the crux of the point induction here. For this, let us be explicit and write $\varphi(\dot{x})$. Note that $\{p \in \mathbb{P} \mid p \Vdash^{\mathsf{HS}} \varphi(\dot{x}) \vee p \Vdash^{\mathsf{HS}} \neg\varphi(\dot{x})\}$ is a dense open set, but it is also symmetric. Indeed, if $\pi \in \operatorname{sym}(\dot{x})$, then $p \Vdash^{\mathsf{HS}} \varphi(\dot{x})$ if and only if $\pi p \Vdash^{\mathsf{HS}} \varphi(\dot{x})$, and likewise for the negation. □

There is a non-trivial part that we have neglected to deal with. Why is HS^G even a model of ZF if G is not V-generic, but only symmetrically V-generic? Well, we do know that $\mathbb{1} \Vdash^{\mathsf{HS}} \mathsf{ZF}$. We can recast the $\Vdash^{\mathsf{HS}}$ relation into explicit terms by using symmetrically dense open sets, and we will see, in doing so, that whereas the truth in a generic extension depends on the dense sets, the truth in a symmetric extension depends only on the symmetrically dense sets.

Before we can finally prove the theorem, we will make one more observation.

Proposition 10.32. *Suppose that $\mathbb{P}$ is a complete Boolean algebra, then* $\sup\{p \in \mathbb{P} \mid p \Vdash^{\mathsf{HS}} \varphi\}$ *is the weakest condition which symmetrically forces φ, and it is tenacious.* □

The proof here is essentially the same idea as the previous proof. Armed with this knowledge, we can now prove Theorem 10.29.

Proof of Theorem 10.29. We may assume without loss of generality that $\mathbb{P}$ is a complete Boolean algebra. Let $\mathbb{B}$ be the subset of $\mathbb{P}$ which contains all the tenacious conditions. We claim that $\langle\mathbb{B},\mathscr{G},\mathscr{F}\rangle$ is equivalent to $\langle\mathbb{P},\mathscr{G},\mathscr{F}\rangle$, in the sense that every symmetrically V-generic filter on the one, is symmetrically V-generic filter on the other.

First, we must argue that $\mathscr{G}$ still acts on $\mathbb{B}$. But, indeed, if $p\in\mathbb{P}$ is tenacious and $\pi\in\mathscr{G}$, then $\operatorname{fix}(\pi p)=\pi\operatorname{fix}(p)\pi^{-1}\in\mathscr{F}$. So $\mathbb{B}$ itself is closed under the action of $\mathscr{G}$, and so $\langle\mathbb{B},\mathscr{G},\mathscr{F}\rangle$ is a symmetric system. As we are using the same filter of groups, we will denote $\mathsf{HS}_\mathbb{P}$ and $\mathsf{HS}_\mathbb{B}$ the two classes of hereditarily $\mathscr{F}$-symmetric names.

Next, since $\mathsf{HS}_\mathbb{B}\subseteq\mathsf{HS}_\mathbb{P}$, we will show that if $\dot x\in\mathsf{HS}_\mathbb{P}$, then there is some $\dot x'\in\mathsf{HS}_\mathbb{B}$ such that $\mathbb{1}_\mathbb{P}\Vdash_\mathbb{P}\dot x=\dot x'$. We prove this by induction on the name rank. Suppose that $\dot x\in\mathsf{HS}_\mathbb{P}$, by the induction hypothesis we may assume that if $\langle p,\dot y\rangle\in\dot x$, then $\dot y\in\mathsf{HS}_\mathbb{B}$.

For each $\langle p,\dot y\rangle\in\dot x$, let $\bar p_y=\sup\{\pi p\mid\pi\in\operatorname{sym}(\dot x)\cap\operatorname{sym}(\dot y)\}$. This is a well-defined condition, as $\mathbb{P}$ is a complete Boolean algebra. Note that $\bar p_y$ is tenacious and $\bar p_y\Vdash\dot y\in\dot x$. We define $\dot x'=\{\langle\bar p_y,\dot y\rangle\mid\langle p,\dot y\rangle\in\dot x\}$, then $\dot x'$ is the wanted name.

Finally, if $G\subseteq\mathbb{P}$ is a V-generic filter, then it is clear that $\mathsf{HS}_\mathbb{P}^G=\mathsf{HS}_\mathbb{B}^G$. It remains to argue in the other direction. Suppose that $G\subseteq\mathbb{B}$ is a V-generic filter, and let G' be the filter on $\mathbb{P}$ generated by G. We claim that $\mathsf{HS}_\mathbb{B}^G=\mathsf{HS}_\mathbb{P}^{G'}$. Indeed, if $\dot x\in\mathsf{HS}_\mathbb{P}$ and $p\Vdash_\mathbb{P}^{\mathsf{HS}}\dot y\in\dot x$, we know that we may assume that $\dot y\in\mathsf{HS}_\mathbb{B}$ and that $p\in\mathbb{B}$, which is enough to argue that the interpretation of the names is the same. ☐

Chapter 11

Homogeneity is not just for milk

11.1 Moving around

Definition 11.1. We say that a forcing $\mathbb{P}$ is *weakly homogeneous* if for every $p, q \in \mathbb{P}$ there is an automorphism π such that πp is compatible with q.

Proposition 11.2. *If $\mathbb{P}$ is weakly homogeneous, then its Boolean completion is weakly homogeneous.*

Proof. Let $\mathbb{B}$ denote the Boolean completion of $\mathbb{P}$. Let $p, q \in \mathbb{B}$, then there are $\bar{p}, \bar{q} \in \mathbb{P}$ such that $\bar{p} \leq p$ and $\bar{q} \leq q$. By weak homogeneity, there is some π such that $\pi\bar{p}$ is compatible with $\bar{q}$, say by some $r \leq \pi\bar{p}, \bar{q}$. However, since $\bar{p} \leq p$, it follows that $\pi\bar{p} \leq \pi p$. Therefore $r \leq \pi p$ and on the other hand, $r \leq \bar{q} \leq q$, so πp and q are compatible. □

The other direction is not true, though, as we can see in the following proposition.

Proposition 11.3. $\mathrm{Add}(\omega, 1)$ *is weakly homogeneous, but it is forcing equivalent to a rigid partial order (that is without non-trivial automorphisms).*

Proof. Given $p, q \in \mathrm{Add}(\omega, 1)$, that is two finite partial functions $\omega \to 2$, consider a permutation of ω, π, which moves $\operatorname{dom} p$ to be disjoint from $\operatorname{dom} q$ acting on $\mathrm{Add}(\omega, 1)$ by $\pi p(\pi n) = p(n)$. Then $\operatorname{dom}(\pi p) \cap \operatorname{dom} q = \varnothing$, so the two conditions are compatible.

To produce a rigid partial order, we construct a finitely splitting tree by recursion. Start with a root t_2 that splits into two nodes, t_3, t_4. Then by recursion, t_i splits into i successors which are indexed by next available integers. It is not hard to see that this tree has no maximal elements, and therefore is forcing equivalent to $\mathrm{Add}(\omega, 1)$. However, every node in the tree is determined by the number of its successors, which, being finite, must be preserved by automorphisms, and therefore the only automorphism is the identity. □

We will say that $\mathbb{P}$ is a *rigid forcing* if its Boolean completion is rigid. So the tree we constructed, despite being a rigid partial order, is not a rigid forcing.

Proposition 11.4. *Suppose that $\mathbb{P}$ is a weakly homogeneous forcing, then for every $\varphi(x)$ in the language of forcing and every $x \in V$, $\mathbb{1} \Vdash \varphi(\check{x})$ or $\mathbb{1} \Vdash \neg\varphi(\check{x})$.*

Proof. Suppose that $p \Vdash \varphi(\check{x})$. Then for any q there is some π such that πp is compatible with q. In particular, it means that $q \nVdash \neg\varphi(\check{x})$. Therefore, there is no condition which forces $\neg\varphi(\check{x})$, and so $\mathbb{1} \Vdash \varphi(\check{x})$. □

Fact 11.5. *Let $\mathbb{P}$ be a complete Boolean algebra, $p, q \in \mathbb{P}$. There is an automorphism π such that πp is compatible with q if and only if there is a V-generic $G \subseteq \mathbb{P}$ such that $p \in G$ and $H \in V[G]$ which is V-generic, $q \in H$, and $V[G] = V[H]$.*

Corollary 11.6. *$\mathbb{P}$ has a rigid Boolean completion if and only if it admits a unique generic which generates any given generic extension.* $\square$

Exercise 11.7. Let $\mathbb{P}$ be a forcing notion, then the following are equivalent:

1. The Boolean completion of $\mathbb{P}$ is weakly homogeneous.
2. If G is V-generic, then for every $p \in \mathbb{P}$ there is some V-generic, G_p, such that $p \in G_p$ and $V[G] = V[G_p]$.
3. For every $\varphi(\dot{x}_1, \dots, \dot{x}_n)$ in the language of forcing and $x_1, \dots, x_n \in V$, $\mathbb{1}$ decides the truth value of $\varphi(\check{x}_1, \dots, \check{x}_n)$.

Definition 11.8. We say that $\mathbb{P}$ is *cone homogeneous* if for every $p, q \in \mathbb{P}$ there are extensions $\bar{p} \leq p$ and $\bar{q} \leq q$ such that $\mathbb{P} \upharpoonright \bar{p} \cong \mathbb{P} \upharpoonright \bar{q}$

Exercise 11.9. Use the fact to prove that cone homogeneous complete Boolean algebras are weakly homogeneous.

Definition 11.10. We say that $\mathscr{G}$ *witnesses the homogeneity of* $\mathbb{P}$ if whenever $p, q \in \mathbb{P}$ there is some $\pi \in \mathscr{G}$ such that πp is compatible with q. We say that a symmetric system $\langle \mathbb{P}, \mathscr{G}, \mathscr{F} \rangle$ is *homogeneous* if $\mathscr{G}$ witnesses the homogeneity of $\mathbb{P}$.

Theorem 11.11. *If $\mathbb{Q}$ is any forcing notion, then there is a homogeneous system $\langle \mathbb{P}, \mathscr{G}, \mathscr{F} \rangle$ such that $\mathbb{Q} \lessdot \mathbb{P}$ and there is a name $\dot{H} \in \mathsf{HS}$ such that $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\dot{H}$ is a V-generic filter for $\check{\mathbb{Q}}$".*

Proof. Let $\mathbb{P}$ be the finite support product $\prod_{n<\omega} \mathbb{Q}$, with $\mathscr{G}$ the group of finitary permutations of ω acting on the product in the natural way: $\pi \langle q_n \mid n < \omega \rangle = \langle q_{\pi n} \mid n < \omega \rangle$, and $\mathscr{F}$ generated by $\operatorname{fix}(E)$ for $E \in [\omega]^{<\omega}$. It is not hard to verify this is symmetric system works. $\square$

Assuming ZFC, however, every forcing also embeds into a rigid forcing. Let us see how this works for the case of $\operatorname{Add}(\omega, 1)$.

Theorem 11.12. *There is a rigid forcing $\mathbb{P}$ such that $\operatorname{Add}(\omega, 1)$ embeds into $\mathbb{P}$.*

Proof. For simplicity, we will assume GCH holds in the ground model, although this is not strictly necessary. We define an iteration of length ω with full support.

$\mathbb{Q}_0 = \operatorname{Add}(\omega, 1)$, and let G_0 be a V-generic filter, with $c_0 = \bigcup G_0$ as the "generic set". Suppose that $V[G_n]$ was defined, with G_n generic for $\mathbb{P}_n$, the iteration of the first n steps, and c_n is the generic set, we let $\mathbb{Q}_n$ be the full support product of $\prod_{\alpha \in c_n} \operatorname{Add}^+(\omega_{\alpha+1}, 1)$.

Here $\operatorname{Add}^+(\kappa^+, 1)$ simply means that the domains of the conditions are restricted to $[\kappa, \kappa^+)$.

If $H_n \subseteq \mathbb{Q}_n$ is $V[G_n]$-generic, then $c_{n+1} = \bigcup H_n$ is the generic set, in that it determines H_n uniquely. We now set $G_{n+1} = G_n * H_n$, and proceed.

Let $\mathbb{P}$ be the iteration of length ω. We argue that $\mathbb{P}$ is a rigid forcing, so it is enough to show that if G is V-generic, then G is unique. But indeed, H was V-generic and $V[G] = V[H]$, then by induction, we get that $\dot{c}_n^G = \dot{c}_n^H$, since it is the set

$$\{\alpha < \sup c_n \mid \omega_{\alpha+1} \text{ has a generic subset over the previous model}\},$$

but since those sets determine the generics of every step it follows that $G_n = H_n$ for all $n < \omega$, so $G = H$. $\square$

11.2 The Feferman–Levy model

Theorem 11.13. *It is consistent with* ZF *that 2^ω is a countable union of countable sets.*

Before proving the theorem, let us see a consequence of this fact.

Proposition 11.14. *Suppose that 2^ω is a countable union of countable sets, then ω_1 is singular.*

Proof. Recall that there is a map from 2^ω onto ω_1. Given $x \in 2^\omega$, decode from x a relation on ω. If this relation is a well-ordering of its domain, map it to its order type. Otherwise map it to 0. Call this surjection $O\colon 2^\omega \to \omega_1$.

Suppose that $2^\omega = \bigcup\{A_n \mid n < \omega\}$, where each A_n is countable. Since the image of a countable set is always finite or countable itself, $O``A_n = B_n$ is countable. Since O is onto, $\omega_1 = \bigcup\{B_n \mid n < \omega\}$, so it is the countable union of countable sets. If there is some B_n which is unbounded, we can recursively construct a cofinal sequence of length ω. Otherwise, set $\alpha_n = \sup B_n$, then $\sup\{\alpha_n \mid n < \omega\} = \omega_1$, and so again we can recursively find a cofinal sequence of length ω. □

Exercise 11.15. ω_2 is never the countable union of countable sets.

Proof of Theorem 11.13. For simplicity, assume $V = L$, or at least GCH. Our forcing $\mathbb{P}$ is going to be the finite support product of $\mathrm{Col}(\omega, \omega_n)$. Namely, $p \in \mathbb{P}$ is a finite function whose domain is a subset of $\omega \times \omega$ with the property that $p(n, \cdot)\colon \omega \to \omega_n$ is a finite partial function. We will write $\mathrm{supp}(p)$ to denote $\{n < \omega \mid \exists m, \langle n, m\rangle \in \operatorname{dom} p\}$. If $E \subseteq \omega$, we will write $p \upharpoonright E$ to denote $p \upharpoonright E \times \omega$, and we will write $\mathbb{P} \upharpoonright E$ to denote $\{p \upharpoonright E \mid p \in \mathbb{P}\}$. Note that for $n < \omega$, $\mathbb{P} \upharpoonright n$ is simply the finite product of collapses.

Our group of automorphisms, $\mathscr{G}$, will be permutations of $\omega \times \omega$ which do not move the left coordinate. Namely, $\pi(n, m) = \langle n, m'\rangle$ for some m'.

For $E \subseteq \omega$, define $\mathrm{fix}(E) = \{\pi \in \mathscr{G} \mid \pi \upharpoonright E \times \omega = \mathrm{id}\}$. We let $\mathscr{F}$ to be the filter generated by $\{\mathrm{fix}(E) \mid E \in [\omega]^{<\omega}\}$. We will say that E is a support for a name $\dot{x}$ if $\mathrm{fix}(E) \subseteq \mathrm{sym}(\dot{x})$, as we did before.

Claim. *$\langle \mathbb{P}, \mathscr{G}, \mathscr{F}\rangle$ is a homogeneous system.*

Proof of Claim. It is not hard to verify that $\mathscr{F}$ is normal, since $\pi\,\mathrm{fix}(E)\pi^{-1} = \mathrm{fix}(E)$. To verify the homogeneity, suppose that $p, q \in \mathbb{P}$, then for every $n \in \mathrm{supp}(p)$ there is some $\pi_n\colon \omega \to \omega$ which makes the domain of $p(n, \cdot)$ disjoint from the domain of $q(n, \cdot)$, for $n \notin \mathrm{supp}(p)$ set $\pi_n = \mathrm{id}$. Let $\pi(n, m) = \langle n, \pi_n m\rangle$, then πp and q have disjoint domains and are therefore compatible. □

Claim. *If $\dot{x} \in \mathsf{HS}$ and $\mathbb{1} \Vdash^{\mathsf{HS}} \dot{x} \subseteq \check{\eta}$, for some ordinal η, then if $p \Vdash \check{\xi} \in \dot{x}$, then $p \upharpoonright E \Vdash \check{\xi} \in \dot{x}$, where E is a support for $\dot{x}$.*

Proof of Claim. If $q \leq p \upharpoonright E$, then there is an automorphism, π, which makes $p \upharpoonright (\mathrm{supp}(p) \setminus E)$ and q compatible. Moreover, this π can be taken such that $\pi \in \mathrm{fix}(E)$, as constructed in the proof of the previous claim. Therefore πp is compatible with q, so q cannot force $\check{\xi} \notin \dot{x}$. Therefore $p \upharpoonright E \Vdash \check{\xi} \in \dot{x}$. □

Indeed, the proof of the claim is a proof for the following lemma for this symmetric system.

Lemma 11.16. *Suppose that $\dot{x} \in \mathsf{HS}$ with support E and $p \Vdash^{\mathsf{HS}} \varphi(\dot{x}, \check{a}_1, \ldots, \check{a}_n)$ for some $a_1, \ldots, a_n \in V$. Then $p \upharpoonright E \Vdash \varphi(\dot{x}, \check{a}_1, \ldots, \check{a}_n)$.* □

It follows that if $\dot{x} \in \mathsf{HS}$ is a name for a real,[31] then there is some $n < \omega$ and some $\mathbb{P}\restriction n$-name, $\dot{x}_*$, such that $\mathbb{1} \Vdash \dot{x} = \dot{x}_*$. Note that any $\mathbb{P} \restriction n$-name is always in HS.

Let $\dot{R}_n = \{\dot{x} \in \mathsf{HS} \mid \dot{x}$ is a nice $\mathbb{P} \restriction n$-name for a real$\}^\bullet$, then $\dot{R}_n \in \mathsf{HS}$ for all $n < \omega$, and indeed, $\langle \dot{R}_n \mid n < \omega\rangle^\bullet \in \mathsf{HS}$. Here a "nice name for a real" can be interpreted simply as requiring any name appearing inside $\dot{x}$ to be $\check{0}$ or $\check{1}$ or any other property that is preserved under all automorphisms.

Claim. *For all $n < \omega$, $\mathbb{1} \Vdash^{\mathsf{HS}} |\dot{R}_n| = \aleph_0$.*

Proof of Claim. Note that $\dot{R}_n$ is the name for 2^ω of the model $V[G\restriction n]$, where G is a V-generic filter for $\mathbb{P}$. Since we assume GCH, $\mathbb{1}_n \Vdash |\dot{R}_n| = \aleph_1 = |\check{\omega}_{n+1}|$. However, since the nth factor of the iteration is $\mathrm{Col}(\omega, \omega_{n+1})$, $\mathbb{1}_{n+1} \Vdash |\dot{R}_n| = \aleph_0$. Since $\mathbb{P} \restriction n + 1$-names are all in HS, it is the case that $\mathbb{1} \Vdash^{\mathsf{HS}} |\dot{R}_n| = \aleph_0$. □

Finally, by the second claim, every name for a real number in HS is equivalent to a name in some $\dot{R}_n$. Therefore $\mathbb{1} \Vdash^{\mathsf{HS}} \bigcup\{\dot{R}_n \mid n < \omega\} = 2^\omega$. So, indeed, $\mathbb{1} \Vdash^{\mathsf{HS}}$ "2^ω is a countable union of countable sets" as wanted. □

Exercise 11.17 (*). Working in the Feferman–Levy model, let $E \subseteq \omega$ be in the ground model and let $R_E = \bigcup\{R_{n+1} \setminus R_n \mid n \in E\}$. Show that $|R_E| \leq |R_{E'}|$ if and only if $E \setminus E'$ is finite.

Exercise 11.18 (*). Show that in the previous exercise we can remove the requirement that E was in the ground model.

As a consequence of this exercise, we can show that CH fails in fairly substantial ways in the Feferman–Levy model, in that there are many intermediate cardinals between ω and 2^ω.

Exercise 11.19. Show that it is consistent with ZF that $\mathrm{cf}(\omega_2) = \omega$ or that $\mathrm{cf}(\omega_2) = \omega_1$.

Remark. One might be tempted to try and construct a model of ZF in which $\mathrm{cf}(\omega_1) = \mathrm{cf}(\omega_2) = \omega$. This, however, requires very large cardinals to be present. Interestingly, we do not know the exact consistency strength of this assumption.

[31] That is, $\mathbb{1} \Vdash \dot{x} \subseteq \check{\omega}$.

Chapter 12

Pseudo-atoms make pseudo-matter

12.1 Generically indiscernible sets

The problem with Cohen reals, or any reals, for that matter is that their generic theory relative to the ground model, so to speak, will determine them completely. In other words, we can split any set of reals into those that contain 0 or not; those that have one pattern or another.

Looking at the Cohen model, we can think of this as having tried to create a "copy" of a structure. Naively, at least, it seems that we tried to preserve the trivial structure on ω, without any relations or functions. Indeed, if R is any n-ary relation on ω that is stable under all finitary permutations which fix a finite set, then R is either finite or co-finite. But as we said above, we can still split the Dedekind-finite set of Cohen reals, A into $\{a \in A \mid 0 \in a\}$ and its complement, and it is not hard to see that both have to be infinite.

What if we wanted to have a better control?

Definition 12.1. Suppose that X, Y are two sets, $\mathscr{G}$ is a group of permutations of X and $\mathscr{H}$ is a group of permutations of Y. The *wreath product*, $\mathscr{G} \wr \mathscr{H}$ is the groups of permutations, π, of $X \times Y$ which have the following properties:

1. There is a sequence $\langle \pi_x \in \mathscr{H} \mid x \in X \rangle$ and a permutation $\pi^* \in \mathscr{G}$.
2. $\pi(x, y) = \langle \pi^*(x), \pi_x(y) \rangle$.

In other words, we assign each Y-section a permutation, and after applying each one of these, we permute the X coordinates. We will refer to the π_x as an *inner* part of π and π^* is called the *outer* part of π.

Remark. The exact definition in terms of notation may slightly differ from the standard one in group theory. However, it is the one that is most clear and useful for us in this context.

In order to find two "generically indiscernible" sets, it seems that sets of reals, or indeed any sets of ordinals, are not going to do the trick. They can canonically be linearly ordered, and if classification theory had taught us anything is that when an order is definable, all hell breaks loose.

12.2 Example: amorphous sets

Definition 12.2. We say that an infinite set is *amorphous* if it cannot be written as a disjoint union of two infinite sets.

Proposition 12.3. *If A is an amorphous set, then A cannot be mapped onto ω.* $\square$

Proposition 12.4. *If A is an amorphous set, then A cannot be linearly ordered.*

Proof. Suppose that $<$ is a linear ordering on an amorphous set. Consider the set A_0 given by $\{a \in A \mid \{b \in A \mid b < a\}$ is finite$\}$, it is either finite or co-finite. Supposing it is co-finite, the size of these finite initial segments is uniquely determined, since $<$ is a linear ordering, which is a function from A onto ω. So it must be a finite set, in which case taking the reversed ordered (or considering tail segments) gives the same map onto ω. $\square$

Theorem 12.5. *It is consistent with* ZF *that there is an amorphous set.*

Proof. Let $\mathbb{P}$ be $\mathrm{Add}(\omega, \omega \times \omega)$, and let $\mathscr{G}$ be the group $S_{<\omega} \wr S_{<\omega}$ acting in the usual way on conditions, we will write π^* to denote the outer part of π and π^*_n to denote the nth inner part. Namely, $\pi p(\pi(n, m), k) = p(n, m, k)$. Finally, the filter $\mathscr{F}$ is generated by fixing finite subsets of $\omega \times \omega$ pointwise, that is, $\mathrm{fix}(E) = \{\pi \in \mathscr{G} \mid \pi \upharpoonright E = \mathrm{id}\}$ for $E \in [\omega \times \omega]^{<\omega}$.[32]

For $n, m < \omega$ we let $\dot{x}_{n,m} = \{\langle p, \check{k}\rangle \mid p(n, m, k) = 1\}$. We then let $\dot{a}_n = \{\dot{x}_{n,m} \mid m < \omega\}^\bullet$ and $\dot{A} = \{\dot{a}_n \mid n < \omega\}^\bullet$.

Claim. *The following hold:*

1. *$\langle \mathbb{P}, \mathscr{G}, \mathscr{F}\rangle$ is a homogeneous system.*
2. *$\pi\dot{x}_{n,m} = \dot{x}_{\pi(n,m)}$.*
3. *$\pi\dot{a}_n = \dot{a}_{\pi^* n}$.*
4. *$\pi\dot{A} = \dot{A}$.* $\square$

Claim. *Given any $p, q \in \mathbb{P}$, there is $\pi \in \mathscr{G}$ such that $\pi^* = \mathrm{id}$ and πp is compatible with q.*

Proof of Claim. For all n, if there is some m for which $\langle n, m\rangle \in \mathrm{supp}(p)$, find π_n such that $\pi_n``\{m \mid \langle n, m\rangle \in \mathrm{supp}(p)\} \cap \{m \mid \langle n, m\rangle \in \mathrm{supp}(q)\} = \varnothing$. Otherwise, set $\pi_n = \mathrm{id}$. Then π for which $\pi^* = \mathrm{id}$ and π_n defined as above satisfies that $\mathrm{dom}(\pi p) \cap \mathrm{dom}(q) = \varnothing$ and the two conditions are therefore compatible. $\square$

Suppose, first, that $B \subseteq \omega$ is such that $\{\dot{a}_n \mid n \in B\}^\bullet \in \mathsf{HS}$, then B is finite or co-finite. Indeed, if B is finite this is trivial. Suppose that B is infinite and let E be a finite set such that $\mathrm{fix}(E) \subseteq \mathrm{sym}(\{\dot{a}_n \mid n \in B\}^\bullet)$. If B is co-infinite, let $n \in B$ and $m \notin B$ such that $\{n, m\} \times \omega \cap E = \varnothing$. We can now take any $\pi \in \mathrm{fix}(E)$ such that $\pi^* = (n\ m)$, which shows that $\pi\dot{B} \neq \dot{B}$. This is impossible, so if B is infinite, it must be that $\omega \setminus B \subseteq E$.

Remark. So far, this argument is not too dissimilar from the structure of the Cohen model. Yet, the set we defined in the Cohen model is a set of real numbers, and is most certainly linearly ordered. Indeed, it is quite easy to see that in Cohen's model, mapping each real to its minimal element (as a subset of ω) will give us a surjection onto ω. What the above argument shows, however, is that even in the Cohen model, any infinite-co-infinite subset of A, when considered in the full generic extension, is not given by a ground model set.

[32]It is worth noting that there are *a lot* of ways to combine filters that will give us the wanted result here. We simply pick the simplest one at this point.

To actually show that $\mathbb{1} \Vdash^{\mathsf{HS}}$ "$\dot{A}$ is amorphous" suppose that $\dot{B} \in \mathsf{HS}$ with some finite support E, and $p \Vdash^{\mathsf{HS}} \dot{B} \subseteq \dot{A}$. It is enough to verify that if $p \Vdash$ "$\dot{B}$ is infinite", then p must force that $\dot{B}$ is co-finite. For simplicity, we may assume that $E = N^2$ for some $N < \omega$ and that $\operatorname{supp}(p) = N^2$ as well.

Suppose that $n > N$ and $q \leq p$ are such that $q \Vdash^{\mathsf{HS}} \dot{a}_n \in \dot{B}$. For any $m > N$, there is some $\pi \in \operatorname{fix}(E)$ such that $\pi^* = (n\ m)$ and πq is compatible with q. Therefore, q cannot force that $\dot{a}_m \notin \dot{B}$. But since this applies to any extension of q as well, it follows that q must therefore force that $\dot{a}_m \in \dot{B}$ as well. Therefore, if $q \leq p$ and $q \Vdash \dot{a}_n \in \dot{B}$ for some $n > N$, then q forces that $\{\dot{a}_n \mid n > N\}^\bullet \subseteq \dot{B}$. But since p forced that $\dot{B}$ is infinite, it must be that any $p' \leq p$ can be extended to such q, and therefore p already must force that inclusion holds and that $\dot{B}$ is co-finite. □

Exercise 12.6. Show that if $\dot{P} \in \mathsf{HS}$ is a name for a partition of $\dot{A}$, then $\mathbb{1} \Vdash^{\mathsf{HS}}$ "$\dot{P}$ has only finitely many non-singleton elements".[33]

The key point in the proof, in contrast to the construction of the Cohen model, is that we can move the $\dot{a}_n$ almost independently of the conditions that force their properties. This is because any two infinite sets of Cohen reals over the same ground models will have "the same generic properties". Anything that can be forced, in terms of the properties of the Cohen reals, must be witnessed by a condition, which is finite, and so must happen repeatedly in both infinite sets.

Exercise 12.7. What happens if we replace the filter $\mathscr{F}$ by the "one-dimensional analogue"? Namely, for $E \subseteq \omega$, let $\operatorname{fix}(E) = \{\pi \in \mathscr{G} \mid \pi \restriction E \times \omega = \mathrm{id}\}$.

Theorem 12.8. *It is consistent with* ZF *that there is a vector space over* $\mathbb{F}_2$ *which is not finitely generated but every proper subspace is finitely generated. In particular, there can be a vector space without a basis.*

Proof. Let W denote the countably generated vector space over $\mathbb{F}_2$, and let $\mathbb{P} = \operatorname{Add}(\omega, W \times \omega)$. We let $\mathscr{G} = \operatorname{Aut}(W) \wr S_{<\omega}$, and $\mathscr{F}$ be the filter generated by $\operatorname{fix}(E) = \{\pi \in \mathscr{G} \mid \pi \restriction E = \mathrm{id}\}$ for $E \in [W \times \omega]^{<\omega}$.

As before, we denote by $\dot{x}_{w,n} = \{\langle p, \check{m}\rangle \mid p(w,n,m) = 1\}$, $\dot{a}_w = \{\dot{x}_{w,n} \mid n < \omega\}^\bullet$, and $\dot{A} = \{\dot{a}_w \mid w \in W\}^\bullet$. Much like in the previous case, etc.

Claim. *Let* $\dot{+} = \{\langle \dot{a}_w, \dot{a}_v, \dot{a}_{w+v}\rangle^\bullet \mid w, v \in W\}^\bullet$*, then* $\dot{+} \in \mathsf{HS}$ *and* $\mathbb{1} \Vdash^{\mathsf{HS}}$ *"*$\langle \dot{A}, \dot{+}\rangle^\bullet$ *is a vector space over* $\mathbb{F}_2$*".* □

Suppose that $\dot{X} \in \mathsf{HS}$ and $p \Vdash^{\mathsf{HS}}$ "$\dot{X}$ is a subspace of $\dot{A}$". Let E be a support for $\dot{X}$, since W is infinite dimensional, there is some $w \notin \operatorname{span}(E^*)$, where E^* is the projection of E to W. Moreover, for any $v \notin \operatorname{span}(E^* \cup \{w\})$ there is an automorphism of W, π^*, such that $\pi^* \restriction E^* = \mathrm{id}$ and $\pi^* w = v$. If $q \leq p$ is such that $\dot{a}_w \in \dot{X}$, by repeating the same arguments as before, we get that $q \Vdash^{\mathsf{HS}} \dot{a}_v \in \dot{X}$ as well. Now, simply note that if $v_0 \in \operatorname{span}(E^*)$ and $v_1 \notin \operatorname{span}(E^* \cup \{w\})$, then both $v = v_0 + v_1$ and v_1 satisfy the conditions, and therefore $q \Vdash^{\mathsf{HS}} \dot{a}_v, \dot{a}_{v_1} \in \dot{X}$ and therefore $q \Vdash^{\mathsf{HS}} \dot{a}_v + \dot{a}_{v_1} \in \dot{X}$, and since we are working over $\mathbb{F}_2$, this is the same as $q \Vdash^{\mathsf{HS}} \dot{a}_{v_0} \in \dot{X}$. Therefore, $q \Vdash^{\mathsf{HS}} \dot{X} = \dot{A}$. Of course, the alternative is that no $w \notin \operatorname{span}(E^*)$ has a condition $q \leq p$ such that $q \Vdash^{\mathsf{HS}} \dot{a}_w \in \dot{X}$, in which case $\dot{X} \subseteq \{\dot{a}_w \mid w \in \operatorname{span}(E^*)\}^\bullet$. □

Exercise 12.9. Show that $\dot{A}$ is amorphous, but it is not strongly amorphous. In fact, for any $n < \omega$, there is some $m > n$, such that there is a partition of A into sets of size m.

[33]This property makes A a *strongly* amorphous set.

Exercise 12.10. Show that any linear operator on A in the symmetric extension must be scalar multiplication.

Remark. We did not use the fact that we are working over $\mathbb{F}_2$, except for two minor places where we did not have to worry about scalar multiplication. Of course we can repeat the argument over any field $\mathbb{F}$ and get the same result. The difference, of course, is that the space will not be amorphous if it is over an infinite field.

12.3 Basic transfer theorem

The essence of the edict "groups preserve structure, filters preserve subsets" can be seen in full-force in the previous example. Ultimate, a countably generated vector space over a finite field is just a countable set. The filters of subgroups were ultimately the same, but the groups themselves differed enough to preserve significantly different structures.

We can try and formalise this idea and create a basic transfer theorem.

Definition 12.11. Let $M \in V$ be a structure in a first-order language $\mathcal{L}$, and let $W \subseteq V[G]$ be a symmetric extension of V. We say that $N \in W$ is a symmetric copy of M if $V[G] \models M \cong_{\mathcal{L}} N$.

We saw, in the previous examples, that A was a symmetric copy of ω in the empty language in the first one, and a symmetric copy of $\mathbb{F}_2^{(\omega)}$ in the second example.

Let $\mathcal{L}$ be a first-order language and let M be an $\mathcal{L}$-structure. Given a group $\mathscr{G} \subseteq \operatorname{Aut}(M)$ and an ideal of subsets of M, $\mathcal{I}$,[34] we have a natural filter of subgroups associated with $\mathcal{I}$, generated by $\operatorname{fix}(A) = \{\pi \in \mathscr{G} \mid \pi \restriction A = \operatorname{id}\}$ for some $A \in \mathcal{I}$. Fixing such $\mathcal{L}, M, \mathscr{G}$, and $\mathcal{I}$, say that $X \subseteq M$ is *stable* under $\pi \in \mathscr{G}$ if $\pi``X = X$, and it is stable if it is stable under all $\pi \in \operatorname{fix}(A)$ for some $A \in \mathcal{I}$. This notion, of course, extends to n-ary relations as well. Note that the stable sets form an algebra of sets which is closed under unions, intersections, and complements.

Exercise 12.12. Suppose that $\mathcal{I}$ is closed under $\mathscr{G}$, that is $\pi``A \in \mathcal{I}$ whenever $A \in \mathcal{I}$, then the natural filter of subgroups is normal.

Theorem 12.13 (Basic transfer theorem). *Let $\mathcal{L}$ be a first-order language, M an $\mathcal{L}$-structure, $\mathscr{G}_M$ an automorphism group of M, and $\mathcal{I}$ an ideal of subsets of M which contains all singletons and is closed under $\mathscr{G}_M$. Then there is a symmetric extension in which there is a symmetric copy of M, N, such that every subset of N^k is a symmetric copy of a stable relation on M.*

Proof. Take $\lambda = |M|^+$ and let $\mathbb{P}$ be $\operatorname{Add}(\lambda, M \times \lambda)$ with $\mathscr{G} = \mathscr{G}_M \wr S_{<\lambda}$. We let the filter of subgroups, $\mathscr{F}$, be generated by $\operatorname{fix}(E) = \{\pi \in \mathscr{G} \mid \pi \restriction E = \operatorname{id}\}$ where E has the following properties:

1. $E^M = \{m \in M \mid \exists\alpha, \langle m, \alpha\rangle \in E\} \in \mathcal{I}$ and
2. for any $m \in M$, $E_m = \{\alpha < \lambda \mid \langle m, \alpha\rangle \in E\} \in [\lambda]^{<\lambda}$.

We let $\dot{x}_{m,\alpha} = \{\langle p, \check{\beta}\rangle \mid p(m, \alpha, \beta) = 1\}$, $\dot{a}_m = \{\dot{x}_{m,\alpha} \mid \alpha < \lambda\}^\bullet$, and $\dot{N} = \{\dot{a}_m \mid m \in M\}^\bullet$.

[34] Recall that ideals of sets are closed under finite unions and subsets.

Claim. *The following hold.*

1. $\langle \mathbb{P}, \mathscr{G}, \mathscr{F} \rangle$ *is a homogeneous system.*
2. $\pi \dot{a}_m = \dot{a}_{\pi^* m}$.
3. *If* p, q *are two conditions, then there is some* π *such that* πp *is compatible with* q *and* $\pi^* = \mathrm{id}$.

Proof of Claim. We have essentially seen all three in action before, so we will simply prove (3). For every $m \in \operatorname{supp}(p)^M$, let π_m be a permutation of λ such that $\operatorname{supp}(p)_m \cap \operatorname{supp}(q)_m = \varnothing$. Finally, let π be such that $\pi^* = \mathrm{id}$ and π_m is the selected permutation for $m \in \operatorname{supp}(p)^M$ or id otherwise. It is not hard to verify that πp is indeed compatible with q, as they now have disjoint domains by design. □

For each n-relation symbol R, we define $\dot{R}^N = \{\langle \dot{a}_{m_0}, \dots, \dot{a}_{m_{n-1}} \rangle^\bullet \mid \langle m_0, \dots, m_{n-1} \rangle \in R^M\}^\bullet$. Since $\mathscr{G}_M$ is a group of $\mathcal{L}$-automorphisms, it is not hard to see that $\dot{R}^N$ are preserved by any $\pi \in \mathscr{G}$. It is also not hard to see that with this definition, $\mathbb{1} \Vdash$ "$\dot{N}$ is a symmetric copy of M".

Finally, if $\dot{X} \in \mathsf{HS}$ is such that $p \Vdash^{\mathsf{HS}} \dot{X} \subseteq \dot{N}$, let E be a support for $\dot{X}$. If $p \Vdash^{\mathsf{HS}} \dot{a}_m \in \dot{X}$, then for all $\pi \in \operatorname{fix}(E)$, $\pi p \Vdash \pi \dot{a}_m = \dot{a}_{\pi^* m} \in \dot{X}$. It is enough to show that p and πp can be made compatible by some σ for which $\sigma^* = \mathrm{id}$. In that case, no extension of p can force to the contrary, and so p must have forced $\dot{a}_{\pi^* m} \in \dot{X}$ as well.

Since $\pi \in \operatorname{fix}(E)$, it must be the case that $\pi p \upharpoonright E = p \upharpoonright E$. But, we can now repeat the same proof as in the claim on the coordinates outside of E. So, if $p \Vdash^{\mathsf{HS}} \dot{a}_m \in \dot{X}$, then $p \Vdash^{\mathsf{HS}} \{\dot{a}_{\pi^* m} \mid \pi^* \in \operatorname{fix}(E^M)\}^\bullet \subseteq \dot{X}$. In particular, p must force that $\dot{X}$ contains a copy of a stable set.

However, if $p \Vdash \dot{a}_m \notin \dot{X}$, the same argument shows that the orbit of $\dot{a}_m$ is disjoint from $\dot{X}$. And so we get the wanted conclusion. Finally, since $\mathbb{P}$ is λ^+-closed, it is the case that any subset of M in the generic extension lies in the ground model, so we only need to worry about $\bullet$-names. □

Exercise 12.14. Show that $\mathbb{1} \Vdash^{\mathsf{HS}}$ "$\dot{N}$ is isomorphic to $\check{M}$" if and only if the filter of subgroups is improper (i.e., contains the trivial group).

We can now apply this theorem for a myriad of situations. Taking the algebraic closure of $\mathbb{Q}$ with its automorphism group will produce an algebraic closure of $\mathbb{Q}$ which cannot be well-ordered.

Chapter 13

Preserving mild choice principles

13.1 Dependent Choice

Definition 13.1. Recall that DC_κ is the statement "If T is a κ-closed tree, then T has a maximal node or a chain of order type κ". We use DC to denote DC_ω and $\mathsf{DC}_{<\kappa}$ to denote $\forall\lambda < \kappa, \mathsf{DC}_\lambda$.

Exercise 13.2. If κ is singular, then $\mathsf{DC}_{<\kappa}$ implies DC_κ. Consequently, when discussion about $\mathsf{DC}_{<\kappa}$ we may always assume that κ is regular. Otherwise, $\mathsf{DC}_{<\kappa}$ implies $\mathsf{DC}_{<\kappa^+}$.

Proposition 13.3. $\mathsf{DC}_{<\kappa}$ *is equivalent to "Every κ-closed forcing is κ-distributive".*

Proof. Assume $\mathsf{DC}_{<\kappa}$, and let $\mathbb{P}$ be a notion of forcing which is κ-closed and for some $\gamma < \kappa$, let $\{D_\alpha \mid \alpha < \gamma\}$ be a family of dense open subsets of $\mathbb{P}$ and let p be a condition. Since DC_γ holds, consider the tree of all sequences $\vec{p} = \langle p_\alpha \mid \alpha < \beta\rangle$ for some $\beta < \gamma$ such that $p_\alpha \in D_\alpha$ and $p_{\alpha+1} \leq_\mathbb{P} p_\alpha \leq_\mathbb{P} p$. Since $\mathbb{P}$ is κ-closed, this tree is γ-closed, so by DC_γ it has a branch, some $\langle p_\alpha \mid \alpha < \gamma\rangle$. But since $\mathbb{P}$ is κ-closed, there is some $q \leq_\mathbb{P} p_\alpha$ for all $\alpha < \gamma$, so the intersection of the D_α is dense.

In the other direction, assuming $\mathsf{DC}_{<\kappa}$ fails, then there is a least $\lambda < \kappa$ and a tree T which is λ-closed, without maximal nodes, and no branches of order type λ, note that this implies vacuously that T must have height at most λ and that it is κ-closed as well. Taking D_α to be all the nodes of rank above α, we have that $\{D_\alpha \mid \alpha < \lambda\}$ is a family of dense open sets with an empty intersection. □

Proposition 13.4. *Assume* ZFC *holds in the ground model. Let $\langle\mathbb{P}, \mathscr{G}, \mathscr{F}\rangle$ be a symmetric system and suppose that $\mathbb{P}$ is κ-closed and $\mathscr{F}$ is κ-complete. Then $\mathbb{1} \Vdash^{\mathsf{HS}} \mathsf{DC}_{<\kappa}$.*

Proof. Suppose that $\dot{T} \in \mathsf{HS}$ and $p \Vdash^{\mathsf{HS}}$ "$\dot{T}$ is a λ-closed tree without maximal nodes" for some $\lambda < \kappa$. We will find some $q \leq p$ and $\dot{f} \in \mathsf{HS}$ such that $q \Vdash^{\mathsf{HS}}$ "$\dot{f}\colon \check{\lambda} \to \dot{T}$ is a branch". Set $p_0 = p$ and $\dot{t}_0 \in \mathsf{HS}$ is a name such that $p_0 \Vdash^{\mathsf{HS}}$ "$\dot{t}_0$ is the root of $\dot{T}$". Continue by recursion, using the fact that $\mathbb{P}$ is λ-closed and that $\dot{T}$ is forced to be λ-closed. Finally, let $q \leq p_\alpha$ for all $\alpha < \lambda$ and let $\dot{f} = \{\langle\check{\alpha}, \dot{t}_\alpha\rangle^\bullet \mid \alpha < \lambda\}^\bullet$. Since $\mathscr{F}$ is κ-complete, the intersection of all the $\operatorname{sym}(\dot{t}_\alpha)$ is in the filter and so $\dot{f} \in \mathsf{HS}$ as wanted. □

We can make the following general statement which will simplify our lives. Say that V is κ-closed in W, if for any $f \in W$ such that $f\colon \gamma \to V$ for $\gamma < \kappa$, it holds that $f \in V$.

Theorem 13.5. *Suppose that $V \subseteq W$ are models of* ZF. *If V is κ-closed in W and $W \models \mathsf{DC}_{<\kappa}$, then $V \models \mathsf{DC}_{<\kappa}$.*

Proof. Let $\lambda < \kappa$ be some cardinal. If T is a λ-closed tree in V, then by closure of V, T is λ-closed in W. Since $W \models \mathsf{DC}_\lambda$, T has a branch in W. It is not hard to see that this branch must be in V. □

Proposition 13.6. *We can replace "$\mathbb{P}$ is κ-closed" by "$\mathbb{P}$ is κ-c.c." in the statement of Proposition 13.4.*

Proof. It is enough to show that HS^G is κ-closed in $V[G]$. We may also assume that $\langle \mathbb{P}, \mathscr{G}, \mathscr{F} \rangle$ is tenacious. Suppose that $\dot{f}$ is a $\mathbb{P}$-name such that some p forces that $\operatorname{dom} \dot{f} = \check{\lambda} < \check{\kappa}$ and that $\operatorname{rng} \dot{f}$ is included in HS.

Since $\mathbb{P}$ is κ-c.c., for every $\alpha < \lambda$, let D_α be a maximal antichain such that if $p_\alpha \in D_\alpha$, then there is some $\dot{x}_{p_\alpha} \in \mathsf{HS}$ such that $p_\alpha \Vdash \dot{f}(\check{\alpha}) = \dot{x}_{p_\alpha}$. Let $\dot{f}_* = \{\langle p_\alpha, \dot{x}_{p_\alpha} \rangle \mid p_\alpha \in D_\alpha, \alpha < \lambda\}$. Since $\lambda < \kappa$ and $|D_\alpha| < \kappa$, we have that $\bigcap_{\alpha<\lambda} \bigcap_{p \in D_\alpha} \operatorname{fix}(p) \cap \operatorname{sym}(\dot{x}_p) \in \mathscr{F}$, so $\dot{f}_* \in \mathsf{HS}$. It is not hard to see that $p \Vdash \dot{f} = \dot{f}_*$, and therefore HS^G must be κ-closed, as wanted. □

When we have a condition which holds for both c.c.c. and σ-closed forcings, the natural question is to check if it holds for proper forcing. Namely, will assuming $\mathbb{P}$ is proper suffice to preserve DC?

Proposition 13.7. *Assume that* ZFC *holds in V and suppose that $\langle \mathbb{P}, \mathscr{G}, \mathscr{F} \rangle$ is a symmetric system such that $\mathbb{P}$ is proper and $\mathscr{F}$ is σ-complete. Then $\mathbb{1} \Vdash^{\mathsf{HS}} \mathsf{DC}$.*

Proof. Let $\dot{f}$ be a name such that $p \Vdash \dot{f} \colon \check{\omega} \to \mathsf{HS}$. We let θ be a large enough regular cardinal, and let $M \prec H(\theta)$ be a countable elementary submodel such that $\langle \mathbb{P}, \mathscr{G}, \mathscr{F} \rangle, \dot{f}, p \in M$. Then for every $n < \omega$, $D_n = \{\bar{p} \mid \bar{p} \perp p \vee \exists \dot{x} \in \mathsf{HS}, \bar{p} \Vdash \dot{f}(\check{n}) = \dot{x}\}$ must also be in M, by elementarity, and therefore if $q \leq p$ is M-generic, it follows that $D_n \cap M$ is predense below q. For $\bar{p} \in D_n \cap M$, let $\dot{x}_n^{\bar{p}}$ be the name that $\bar{p}$ decided to be the value of $\dot{f}(\check{n})$, then $\dot{f}_* = \{\langle \bar{p}, \dot{x}_n^{\bar{p}} \rangle \mid n < \omega, \bar{p} \in D_n\}$ is a name in HS, since we can intersect $\operatorname{sym}(\dot{x}_n^{\bar{p}})$ and $\operatorname{fix}(\bar{p})$ over all the countably many values we have.

Clearly, $q \Vdash \dot{f} = \dot{f}_*$. This holds for any p and $\dot{f}$, so it must be that HS^G is σ-closed in $V[G]$, and so DC holds there. □

Remark. It is worth noting that there is a generalisation of properness to κ which is uncountable. It is simply not a very interesting notion, as it fails to have good iteration properties and things can get very disastrous very quickly. It is true, however, that the above proposition will generalise in the same sense.

Theorem 13.8 (Improved transfer theorem). *In the conditions of Theorem 12.13, if the ideal of sets is κ-closed we can require that the symmetric extension satisfies $\mathsf{DC}_{<\kappa}$.* □

13.2 Hartogs, Lindenbaum, and weird partitions

Recall that for a set X, the *Hartogs number* of X, denoted by $\aleph(X)$, is $\sup\{\alpha \in \mathrm{Ord} \mid \alpha \leq X\}$, and the *Lindenbaum number* of X, denoted by $\aleph^*(X)$, is $\sup\{\alpha \in \mathrm{Ord} \mid \alpha \leq^* X\}$.[35]

[35] Here $A \leq^* B$ if there is a partial surjection from B onto A. This relation is transitive and reflexive, but ZF does not prove it to be antisymmetric.

Theorem 13.9. *Given any $\lambda \leq \kappa$, it is consistent with* ZF *that there is a set X such that $\aleph(X) = \lambda$ and $\aleph^*(X) = \kappa$.*

Proof. Let $\mathbb{P}$ be $\mathrm{Add}(\omega, \kappa \times \lambda)$, with the permutation group $\mathscr{G} \subseteq S_{\kappa\times\lambda}$ of those permutations which move $< \lambda$ points, with the action on the index set defined in the usual way. The filter $\mathscr{F}$ is given by the following groups. For $E \in [\kappa]^{<\kappa}$, $E' \in [E \times \lambda]^{<\lambda}$, let $H_{E,E'}$ be the group of π such that:

1. $\pi \restriction E \times \lambda \in \{\mathrm{id}\} \wr S_\lambda$.

2. $\pi \restriction E' = \mathrm{id}$.

In other words, π fixes pointwise a set of size $< \lambda$, and also for $< \kappa$ coordinates on the κ-axis, π "freezes" these coordinates. We will refer to E as the "weak support" and E' as the "strong support". We write $p \restriction E$ to denote $p \restriction E \times \lambda$ if E is a weak support, and $p \restriction E'$ is well-defined in the case of a strong support. We first prove the following homogeneity lemma.

Lemma 13.10. *Let $H_{E,E'} \in \mathscr{F}$ and suppose q, q' are two conditions such that $q \restriction E'$ is compatible with $q' \restriction E'$, then there is some $\pi \in H_{E,E'}$ such that πq is compatible with q'.*

Proof of Lemma 13.10. For any $\langle\alpha, \beta\rangle \in \operatorname{supp} q$ where $\alpha \notin E$, we have free reign in moving $\langle\alpha, \beta\rangle$ outside the support of q', so we may find some α' such that $\{\alpha'\} \times \lambda \cap \operatorname{supp} q' = \varnothing$ and a permutation $\pi_E \in H_{E,E'}$ such that $\pi_E``\operatorname{supp}(q \setminus q \restriction E) \subseteq \{\alpha'\} \times \lambda$. For any $\alpha \in E$ we can find some $\pi_\alpha \in S_\lambda$ such that π_α moves the αth coordinate in $\operatorname{supp}(q \setminus q \restriction E')$ to be disjoint of the support of q' at α. Now, let π be the permutation which is π_α for $\alpha \in E$ and π_E otherwise. □

We let $\dot{x}_{\alpha,\beta} = \{\langle p, \check{n}\rangle \mid p(\alpha, \beta, n) = 1\}$, $\dot{X}_\alpha = \{\dot{x}_{\alpha,\beta} \mid \beta < \lambda\}^\bullet$ and $\dot{X} = \bigcup_{\alpha<\kappa} \dot{X}_\alpha$.

It is not hard to see, as we did previously, that $\pi\dot{x}_{\alpha,\beta} = \dot{x}_{\pi(\alpha,\beta)}$. Therefore all of the names above are in HS. Moreover, if $\mu < \kappa$, then $\{\langle\check{\alpha}, \dot{X}_\alpha\rangle^\bullet \mid \alpha < \mu\}^\bullet \in \mathsf{HS}$, so we immediately get that $\mathbb{1} \Vdash^{\mathsf{HS}} \aleph^*(\dot{X}) \geq \check{\kappa}$, by considering the function $\{\langle\dot{x}_{\alpha,\beta}, \check{\xi}_\alpha\rangle^\bullet \mid \alpha < \kappa\}^\bullet$ where $\xi_\alpha = \alpha$ for $\alpha < \mu$ or 0 otherwise.

and similarly for any $\mu < \lambda$ and a fixed $\alpha < \kappa$, $\{\langle\check{\beta}, \dot{x}_{\alpha,\beta}\rangle^\bullet \mid \beta < \mu\}^\bullet \in \mathsf{HS}$ as well, so $\mathbb{1} \Vdash^{\mathsf{HS}} \aleph(\dot{X}) \geq \check{\lambda}$.

It remains to show that both of these inequalities are in fact equalities. We start with the $\aleph^*$ case. Suppose that $\dot{f} \in \mathsf{HS}$ and $p \Vdash^{\mathsf{HS}} \dot{f} \colon \dot{X} \to \check{\kappa}$. Let $H = H_{E,E'} \subseteq \operatorname{sym}(\dot{f})$ and we may assume without loss of generality that $H \subseteq \operatorname{fix}(p)$ as well.

It follows that if $\langle\alpha, \beta\rangle \notin E'$ and $q \leq p$ such that $q \Vdash^{\mathsf{HS}} \dot{f}(\dot{x}_{\alpha,\beta}) = \check{\xi}$, then $q \restriction E' \cup \{\langle\alpha, \beta\rangle\}$ already forces that. Simply apply the above lemma to $H_{E\cup\{\alpha\},E'\cup\{\langle\alpha,\beta\rangle\}}$.

Next, since $\mathbb{P}$ is a c.c.c. forcing, for each $\langle\alpha, \beta\rangle \notin E'$ there is a maximal antichain $D_{\alpha,\beta}$ below p which decides the possible values of $\dot{f}(\dot{x}_{\alpha,\beta})$. If $\pi \in H$, then $\pi``D_{\alpha,\beta}$ is a maximal antichain below p which decides the values of $\dot{f}(\dot{x}_{\pi(\alpha,\beta)})$, and so must agree with the same set of values that $D_{\pi(\alpha,\beta)}$ have produced.

In particular, it follows that the image of $\dot{f}$ outside the strong support is fully determined by the orbits under H. Namely, if $\pi \in H$ and $\pi\langle\alpha, \beta\rangle = \langle\alpha', \beta'\rangle$, the two antichains $D_{\alpha,\beta}$ and $D_{\alpha',\beta'}$ must have the same countable set of possible values for $\dot{f}(\dot{x}_{\alpha,\beta})$ and $\dot{f}(\dot{x}_{\alpha',\beta'})$ respectively. Therefore, if $\langle\alpha, \beta\rangle$ is not in the weak support, the orbit is any other point not in the weak support, and therefore there can only be countably many values attained by any point there, and within the weak support (but outside the strong support) this value only depends on α, so there can be at most $< \kappa$ different values to $\dot{f}$, so p must force that it is not surjective. Therefore $\mathbb{1} \Vdash^{\mathsf{HS}} \aleph^*(\dot{X}) = \check{\kappa}$.

To prove that $\mathbb{1} \Vdash^{\mathsf{HS}} \aleph(\dot{X}) = \check{\lambda}$, suppose that $\dot{f} \in \mathsf{HS}$ and $p \Vdash^{\mathsf{HS}} \dot{f} \colon \check{\lambda} \to \dot{X}$. Let $H = H_{E,E'}$ be such that $H \subseteq \operatorname{sym}(\dot{f}) \cap \operatorname{fix}(p)$. Suppose that $q \leq p$ was such that for some $\langle \alpha, \beta \rangle \notin E'$, $q \Vdash^{\mathsf{HS}} \dot{f}(\check{\xi}) = \dot{x}_{\alpha,\beta}$. We can find some $\langle \gamma, \delta \rangle \notin E' \cup \{\langle \alpha, \beta \rangle\}$ such that $\{\alpha, \gamma\} \cap E \neq \varnothing$ if and only if $\alpha = \gamma$, such that $\langle \gamma, \delta \rangle \notin \operatorname{supp} q$, and therefore the 2-cycle, π, switching $\langle \alpha, \beta \rangle$ and $\langle \gamma, \delta \rangle$ is in H and q is compatible with πq. But that means that $q \cup \pi q \Vdash^{\mathsf{HS}} \dot{x}_{\alpha,\beta} = \dot{f}(\check{\xi}) = \dot{x}_{\gamma,\delta}$, which is impossible, since $\dot{f}$ is supposed to be a function. Therefore, no such q exists to begin with, and therefore p must force that the image of $\dot{f}$ is included in $\{\dot{x}_{\alpha,\beta} \mid \langle \alpha, \beta \rangle \in E'\}$, so in particular, it is not injective. □

Exercise 13.11. See how much Dependent Choice we can preserve or fail by varying the different parameters of this symmetric system.

Theorem 13.12. *It is consistent with* $\mathsf{ZF} + \mathsf{DC}_{<\kappa}$ *that there is a partition of the real numbers with cardinality strictly greater. Namely, there is a set X such that 2^ω maps onto X, into X, but not at the same time.*

Proof. Let $\kappa > 2^{\aleph_0}$ be any regular cardinal, we force with $\operatorname{Add}(\omega, \kappa \times \omega)$. Our permutation group is going to be $\mathscr{G} = \{\mathrm{id}\} \wr S_\omega$, which means that if we think about the forcing as adding a sequence of length κ of countable sets of reals, then the action of the permutation group on the forcing is permuting each of the blocks independently. For $E \subseteq \kappa$, we let $\operatorname{fix}(E) = \{\pi \in \mathscr{G} \mid \pi \restriction E \times \omega = \mathrm{id}\}$, and $\mathscr{F}$ is the filter generated by $\{\operatorname{fix}(E) \mid E \in [\kappa]^{<\kappa}\}$. So, in other words, we fix the blocks indexed by E pointwise.

Since this filter of groups is κ-complete and Cohen forcing is c.c.c., we get that $\mathsf{DC}_{<\kappa}$ holds in the model. Now, let $\dot{x}_{\alpha,n} = \{\langle p, \check{m} \rangle \mid p(\alpha, n, m) = 1\}$ and let $\dot{X}_\alpha = \{\dot{x}_{\alpha,n} \mid n < \omega\}^\bullet$. Easily, $\pi \dot{x}_{\alpha,n} = \dot{x}_{\pi(\alpha,n)}$ and therefore $\pi \dot{X}_\alpha = \dot{X}_\alpha$. Therefore, these names as well as the name $\langle \dot{X}_\alpha \mid \alpha < \kappa \rangle^\bullet$ are all in HS. Note that $\mathbb{1} \Vdash^{\mathsf{HS}} |\dot{X}_\alpha| = \aleph_0$ for all $\alpha < \kappa$, as $\operatorname{fix}(\{\alpha\})$ fixes pointwise the obvious enumeration of $\dot{X}_\alpha$, and therefore $\mathbb{1} \Vdash^{\mathsf{HS}} \kappa < \aleph([2^\omega]^\omega)$.

Since 2^ω always maps onto $[2^\omega]^\omega$, and always injects into $[2^\omega]^\omega$, if we can show that $\kappa \geq \aleph(2^\omega)$, then the map from 2^ω onto $[2^\omega]^\omega$ will induce a partition of the reals which has the wanted properties.

Suppose that $\dot{f} \in \mathsf{HS}$ and $p \Vdash^{\mathsf{HS}} \dot{f} \colon \check{\kappa} \to \dot{2}^\omega$. Let E be such that $\operatorname{fix}(E) \subseteq \operatorname{sym}(\dot{f}) \cap \operatorname{fix}(p)$, and let $q \leq p$ be such that for some $\alpha \notin E$ and some $n < \omega$, $q \Vdash^{\mathsf{HS}} \dot{f}(\check{\xi}) = \dot{x}_{\alpha,n}$. Easily, we can find large enough m such that $\langle \alpha, m \rangle \notin \operatorname{supp} q$ and consider the permutation π which is the 2-cycle $(\langle \alpha, n \rangle\ \langle \alpha, m \rangle)$. As usual, πq and q are compatible, but this is impossible, since that implies $\dot{f}$ is not a function. Therefore, no such q exists, so $p \Vdash^{\mathsf{HS}} \operatorname{rng} \dot{f} \subseteq \{\dot{x}_{\alpha,n} \mid \alpha \in E, n < \omega\}^\bullet$, and in particular, this set cardinality $< \kappa$ so $\dot{f}$ cannot be injective. □

Exercise 13.13. Show that in the construction above, every generic real is symmetric.

Exercise 13.14. Show that we can partition the reals in the construction given above such that the partition is incomparable in cardinality with 2^ω.

Exercise 13.15. What happens in the construction above if we simply require $\kappa > 2^{\aleph_0}$?

Exercise 13.16. Analyse $\aleph(2^\omega)$ and $\aleph^*(2^\omega)$ in the Feferman–Levy model. Conclude that there are strange partitions of 2^ω in that model as well.

13.3 Restricted choice

Theorem 13.17. *Let κ be a regular cardinal, then it is consistent with* ZF *that* $\mathsf{AC}_{\mathsf{WO}}$ *holds and* DC_κ *holds, but* DC_{κ^+} *fails.*

Proof. We force with $\mathrm{Add}(\omega,\kappa^+)$, $\mathscr{G}$ is S_{κ^+}, and $\mathscr{F}$ is generated by $\mathrm{fix}(E)=\{\pi\in\mathscr{G}\mid \pi\restriction E=\mathrm{id}\}$ for $E\in[\kappa^+]^{<\kappa^+}$. The resulting model satisfies DC_κ, as we have seen before. As usual, we let $\dot{x}_\alpha=\{\langle p,\check{n}\rangle\mid p(\alpha,n)=1\}$ and $\dot{X}=\{\dot{x}_\alpha\mid\alpha<\kappa^+\}^\bullet$.

It is as we have seen before, $\dot{x}_\alpha$ and $\dot{X}$ are all in HS, and indeed $\dot{X}^{<\kappa^+}$ is a tree witnessing the failure of DC_{κ^+}. It remains to show that if λ is an ordinal and $\dot{F}\in\mathsf{HS}$ such that $p\Vdash^{\mathsf{HS}}$ "$\dot{F}$ is a function with domain $\check{\lambda}$ and for all $\check{\alpha}<\check{\lambda}$, $\dot{F}(\check{\alpha})\neq\check{\varnothing}$", then there is some $q\leq p$ and $\dot{f}\in\mathsf{HS}$ such that for all $\alpha<\lambda$, $q\Vdash\dot{f}(\check{\alpha})\in\dot{F}(\check{\alpha})$.

If $\lambda\leq\kappa$, there is not much to verify, since DC_κ implies AC_κ, and therefore AC_λ. So we may assume that $\lambda>\kappa$. Let E and E' be two disjoint sets such that $\mathrm{fix}(E)\subseteq\mathrm{sym}(\dot{F})\cap\mathrm{fix}(p)$ and $|E'|=\kappa$, we will define a name $\dot{f}$ as above such that $\mathrm{fix}(E\cup E')\subseteq\mathrm{sym}(\dot{f})$.

First, for a fixed α, let D_α be a maximal antichain below p such that if $q\in D_\alpha$, then there is some $\dot{a}^q_\alpha\in\mathsf{HS}$ such that $q\Vdash\dot{a}^q_\alpha\in\dot{F}(\check{\alpha})$. We may assume that $\dot{a}^q_\alpha$ is such that any condition appearing inside of it is below q, otherwise replace $\dot{a}^q_\alpha$ by

$$\{\langle r,\dot{y}\rangle\mid r\leq q, r\Vdash\dot{y}\in\dot{a}^q_\alpha, \dot{y}\text{ appears in }\dot{a}^q_\alpha\},$$

note that this name is symmetric as witnessed by $\mathrm{sym}(\dot{a}^q_\alpha)\cap\mathrm{fix}(q)$. Next, define $\dot{a}_\alpha=\bigcup_{q\in D_\alpha}\dot{a}^q_\alpha$. This name is again in HS, since $|D_\alpha|\leq\kappa$ and $\mathscr{F}$ is κ^+-complete, it follows that $\dot{a}_\alpha\in\mathsf{HS}$, and clearly $p\Vdash\dot{a}_\alpha\in\dot{F}(\check{\alpha})$.

We want to say that this completes the proof, since we can take $\{\langle\check{\alpha},\dot{a}_\alpha\rangle^\bullet\mid\alpha<\lambda\}^\bullet$ to be $\dot{f}$. But unfortunately, λ is too big for us to argue so simply that this name is in HS. If, however, we can show that $\dot{a}_\alpha$ can be replaced by some name stable under $\mathrm{fix}(E\cup E')$, this will then complete the proof.

For each $\dot{a}_\alpha$, let E_α be such that $\mathrm{fix}(E_\alpha)\subseteq\mathrm{sym}(\dot{a}_\alpha)$. We can find some $\pi\in\mathscr{G}$ such that:

1. $\pi_\alpha\in\mathrm{fix}(E)$.
2. $\pi_\alpha``E_\alpha\subseteq E\cup E'$.

The reason for that is that $|E_\alpha\setminus E|\leq|E'|$. Let $\dot{c}_\alpha=\pi_\alpha\dot{a}_\alpha$. Then $\pi_\alpha p\Vdash\pi_\alpha\dot{a}_\alpha\in\pi_\alpha\dot{F}(\pi_\alpha\check{\alpha})$, but since $\pi_\alpha\in\mathrm{fix}(E)$ we get that $p\Vdash\dot{c}_\alpha\in\dot{F}(\check{\alpha})$. Finally, we let $\dot{f}=\{\langle\check{\alpha},\dot{c}_\alpha\rangle^\bullet\mid\alpha<\lambda\}^\bullet$, then $\mathrm{fix}(E\cup E')\subseteq\mathrm{sym}(\dot{f})$ and therefore $\dot{f}\in\mathsf{HS}$ as wanted. □

Exercise 13.18. Show that $\mathbb{1}\Vdash^{\mathsf{HS}}\mathsf{AC}_{\dot{X}}$ in the previous model.

Exercise 13.19. Show that if κ is a limit cardinal, the same construction with $\mathrm{Add}(\omega,\kappa)$ instead of $\mathrm{Add}(\omega,\kappa^+)$ cannot possibly satisfy AC_κ.

Chapter 14

Forcing without Choice: Peace is sometimes an option.

14.1 Generically preserving some choice

Definition 14.1. Let X be a set, we say that $\mathbb{P}$ is $\leq X$*-distributive* if whenever $\{D_x \mid x \in X\}$ is a family of dense open sets, $\bigcap_{x\in X} D_x$ is dense. If X can be well-ordered, we have that $\leq|X|$-distributive is the same as κ^+-distributive.

The reason we have to switch to $\leq X$ is that in ZF we have little to no control and understanding of the structure of the cardinals below X, which can have some odd effects.

Theorem 14.2. *Suppose that* AC_X *holds and* $\mathbb{P}$ *is* $\leq X$*-distributive. Then* $\mathbb{1} \Vdash \mathsf{AC}_X$*.*

Proof. Suppose that $\dot{F}$ is a $\mathbb{P}$-name such that $p \Vdash$ "$\dot{F}\colon \check{X} \to \dot{Y}$ and $\dot{F}(\check{x}) \neq \check{\varnothing}$". For each $x \in X$, let D_x be the set $\{q \leq p \mid \exists \dot{y}, q \Vdash \dot{y} \in \dot{F}(\check{x})\}$. Since $D = \bigcap_{x\in X} D_x$ is dense below p, there is some $q \leq p$ in D. Such q satisfies that for all $x \in X$, $\{\dot{y} \mid q \Vdash \dot{y} \in \dot{F}(\check{x})\}$ is a non-empty class. Using Scott's trick and AC_X in the ground model, we can choose $\dot{y}_x$ such that for all $x \in X$, $q \Vdash \dot{y}_x \in \dot{F}(\check{x})$. Therefore, $\{\langle \check{x}, \dot{y}_x\rangle^\bullet \mid x \in X\}^\bullet$ is a name for a choice function below q. Since D is dense, it follows that $p \Vdash \exists f \forall x \in \check{X}(f(x) \in \dot{F}(x))$, as wanted. □

In the other direction we can also say something.

Theorem 14.3. *Suppose that* $\mathbb{P}$ *is* $\leq X$*-distributive and* $\mathbb{1} \Vdash \mathsf{AC}_X$*, then* AC_X *holds in the ground model.*

Proof. Let $\{A_x \mid x \in X\}$ be a family of non-empty sets in the ground model. Then there is a name $\dot{f}$ and a condition p such that $p \Vdash$ "$\dot{f}(\check{x}) \in \check{A}_x$ for all $x \in X$". For each x, let D_x be the dense open set $\{q \leq p \mid \exists a \in A_x, q \Vdash \dot{f}(\check{x}) = \check{a}\}$. Since $\bigcap_{x\in X} D_x$ is dense below p, let q be a condition in this set, then $\{\langle x, a\rangle \mid q \Vdash \dot{f}(\check{x}) = \check{a}\}$ is a choice function in the ground model. □

Corollary 14.4. *Suppose that* $\mathbb{P}$ *is* $\leq X$*-distributive.* AC_X *holds if and only if* $\mathbb{1} \Vdash \mathsf{AC}_X$*.* □

Definition 14.5. We say that $\mathbb{P}$ is $\leq X$*-sequential* if whenever $\dot{f}$ is a $\mathbb{P}$-name and $p \Vdash \dot{f}\colon \check{X} \to \check{Y}$, for some Y, there is some $q \leq p$ and $g\colon X \to Y$ such that $q \Vdash \dot{f} = \check{g}$. In other words, $\mathbb{P}$ does not add new functions from X into the ground model.

Proposition 14.6. *If* $\mathbb{P}$ *is* $\leq X$*-distributive, then it is* $\leq X$*-sequential.* □

Proposition 14.7. *We can weaken Theorem 14.3 to only requiring $\leq X$-sequentiality, rather than distributivity.* □

Exercise 14.8. Suppose that $\mathbb{P}$ is a well-orderable forcing. If $\mathcal{A} = \{A_i \mid i \in I\}$ does not admit a choice function, then $\mathbb{1} \Vdash$ "$\check{\mathcal{A}}$ does not admit a choice function".

Exercise 14.9. Suppose that AC_κ fails, show that there is a generic extension in which AC_ω fails. Conclude that if AC_ω holds in all generic extensions, then $\mathsf{AC}_{\mathsf{WO}}$ holds in the ground model.

What about preserving DC? It turns out that even if a forcing is very distributive and DC_κ holds, we can still violate DC. Even worse, a sequential forcing can violate AC_ω.

14.2 Choiceless Properness

We need to be more explicit about what proper means, since even if we have countable elementary submodels, it is not entirely and immediately clear as to what is the right structure that we want to take the submodel from. We have two options: (1) we can redefine $H(\kappa)$ to be $\{x \mid \kappa \not\leq^* \operatorname{tcl}(x)\}$, this is a good approximation for $H(\kappa)$ in the context of ZFC and it can serve us for this purpose; or (2) we can simply use V_α for a sufficiently nice α, since we are really just interested in enough power sets to exist and enough recursive constructions to be doable within our countable model, and we can always find a large enough and sufficiently nice α as a consequence of the Reflection theorem.

Proposition 14.10. *The following are equivalent:*

1. DC.
2. $\operatorname{Col}(\omega, \omega_1)$ *is improper.*
3. *There is an improper forcing.*

Proof. Clearly (2) implies (3), and (3) implies (1), since if DC fails, every forcing is proper just by vacuous reasons: any $H(\kappa)$ or V_α for a tail of the ordinals will simply have no countable elementary submodels to witness the failure of properness. Finally, if DC holds, then ω_1 is regular. Therefore no suitable M will have a generic condition: if $q \in \operatorname{Col}(\omega, \omega_1)$ and $n \notin \operatorname{dom} q$, then $q \cup \{\langle n, \sup \omega_1 \cap M\rangle\}$ is incompatible with any condition in $M \cap \{p \mid n \in \operatorname{dom} p\}$. □

Just to hammer the importance of DC to the theory of proper forcing, and in general, consider the following theorem.

Theorem 14.11. DC *holds if and only if no σ-closed forcing collapses ω_1.*

Proof. If DC holds, any σ-closed forcing is σ-distributive, and therefore adds no countable sequences of ordinals. In particular, there is no surjection from ω onto ω_1. If DC fails, let T be a counterexample, that is a tree of height ω without branches and without maximal nodes, and we may assume that the elements of T are functions from a finite ordinal into some set, ordered by inclusion.

We define the forcing $\mathbb{P}$ to be the set $\{\langle t, p\rangle \mid t \in T, p \in \omega_1^{<\omega}, \operatorname{dom} p = \operatorname{dom} t\}$ with the ordering given by $\langle t, q\rangle \leq \langle s, p\rangle$ if and only if $s \subseteq t$ and $p \subseteq q$. It is not hard to check that this forcing collapses ω_1, as it projects onto $\operatorname{Col}(\omega, \omega_1)$.

On the other hand, if $\langle t_{n+1}, p_{n+1}\rangle \leq \langle t_n, p_n\rangle$ is a descending sequence of conditions, then $t_n \subseteq t_{n+1}$ is increasing in T, so the sequence must be finite, or it would define a branch. Therefore, by vacuous reasons, $\mathbb{P}$ is σ-closed. □

Theorem 14.12. *Suppose that $\mathbb{P}$ is proper and* DC *holds, then* $\mathbb{1} \Vdash \mathsf{DC}$.

Proof. Suppose that $\dot{T}$ is a name such that $p \Vdash$ "$\dot{T}$ is a tree of height $\check{\omega}$ without maximal nodes". Let M be a countable elementary submodel such that $\dot{T}, p, \mathbb{P}$ are all in M, and let $q \leq p$ be an M-generic condition. For all $n < \omega$, let $D_n = \{\bar{p} \leq p \mid \exists \dot{t}, \bar{p} \Vdash \dot{t} \in \dot{T}_n\}$, where $\dot{T}_n$ is the name for the nth level of the tree. By elementarity, $D_n \in M$ for all $n < \omega$, and so $D_n \cap M$ is predense below q. Therefore, if we let $\dot{T}_* = \dot{T} \cap M$, $q \Vdash$ "$\dot{T}_*$ is a tree of height $\check{\omega}$ without maximal nodes".

Let $\{\dot{t}_n \mid n < \omega\}$ be the set of all the names appearing in $\dot{T}_*$, then $\dot{F} = \{\langle \check{n}, \dot{t}_n \rangle^\bullet \mid n < \omega\}$ is a name such that $q \Vdash$ "$\dot{F}$ is a function from an infinite subset of $\check{\omega}$ onto $\dot{T}_*$". By combining these two facts we have that q forces that $\dot{T}$ has a countable subtree without maximal nodes, so by recursion we can show that $q \Vdash$ "$\dot{T}$ has a branch", and so by a density argument, p must already force that $\dot{T}$ has a branch as wanted. □

Exercise 14.13. Show that if $\mathbb{P}$ is σ-closed, then $\mathbb{P}$ is proper (that is, verify that Theorem 8.8 holds in ZF).

Remark. We will see later that a c.c.c. forcing, even in the presence of DC, need not be proper and may collapse ω_1. We can improve the definition of c.c.c., for example by requiring that $\mathbb{1}$ (or equivalently, every condition) is M-generic for any suitable countable elementary submodel (using Theorem 8.7 as a definition), to regain the properness of a c.c.c. forcing in ZF.

Exercise 14.14. Suppose that $\mathbb{P}$ has the property that if D is a predense set, then D contains a countable predense subset. Show that $\mathbb{P}$ is c.c.c. and proper. Moreover, assuming DC, show that this property is equivalent to "every condition is M-generic".

Exercise 14.15. Show that PFA implies DC_{ω_1}.

14.3 Kinna–Wagner Principles

Definition 14.16. We say that A is an *α-set (of ordinals)* if there is some ordinal η such that $A \subseteq \mathcal{P}^\alpha(\eta)$.

Exercise 14.17. If A and B are α-sets, then $A \times B$ is equipotent with an α-set.

Definition 14.18. Kinna–Wagner [Principle for] α is the statement "Every set is equipotent with an α-set of ordinals". We write KWP_α to denote this principle, and we write KWP to denote $\exists \alpha\ \mathsf{KWP}_\alpha$.

Exercise 14.19. KWP_0 is equivalent to AC. KWP_1 implies that every set can be linearly ordered.

Theorem 14.20. *Suppose that M and N are two models of* ZF *with the same α-sets of ordinals. If $M \models \mathsf{KWP}_\alpha$, then $M = N$.*

Proof. Since M is a model of KWP_α, given any η, we can encode $\langle V_\eta^M, \in \rangle$ as an α-set, so that set must be in N. Since the relation is still extensional and well-founded (otherwise this would be witnessed by an α-set in N), we can take the Mostowski collapse and get that $V_\eta^M \in N$ for all η, so $M \subseteq N$.

In the other direction, let η be such that $V_\eta^N \subseteq M$. By the above, we have that in fact $V_\eta^M = V_\eta^N$, therefore we can code V_η^N as an α-set in M. If $A \in V_{\eta+1}^N$, then A defines an α-set as a subset of that which codes V_η^N. Therefore $A \in M$. So, by induction $V_\eta^N = V_\eta^M$ for all η, and so $M = N$. □

Theorem 14.21. *Let α be an ordinal and let $\alpha^* = \sup\{\beta + 2 \mid \beta < \alpha\}$.*[36] *If* KWP_α *holds, then any generic extension must satisfy* KWP_{α^*}.

Proof. We separate this to three parts. For the case $\alpha = 0$, this is just the fact that AC is preserved in generic extensions. Note that if G is a V-generic filter for some $\mathbb{P} \in V$, then G is equipotent with an α-set if $\mathbb{P}$ is, since $G \subseteq \mathbb{P}$, and that for every $a \in V[G]$ there is a name $\dot{a} \in V$ and a surjection $i_G\colon \dot{a} \to a$.

In particular, if we replace $\dot{a}$ by some α-set, A, we have a function from A onto a, so $\{i_G^{-1}(x) \mid x \in a\}$ is an $(\alpha + 1)$-set. Therefore, KWP_{α^*} holds in the case that α is a successor ordinal.

Finally, if α is a limit ordinal, note that we can write α-set, $A \subseteq \mathcal{P}^\alpha(\eta)$, as the union of $\{A \cap \mathcal{P}^\beta(\eta) \mid \beta < \alpha\}$. Applying the previous argument, we can break the preimage of i_G on this stratification, getting the injection to be into $\{\{i_G^{-1}(x) \cap \mathcal{P}^\beta(\eta) \mid x \in a\} \mid \beta < \alpha\}$, but that too is an α-set when α is a limit ordinal, so KWP_{α^*} holds. □

Remark. We will see that it is quite possible that KWP_1 holds, but there is a generic extension satisfying $\neg\mathsf{KWP}_1$. By the above, that extension must satisfy KWP_2, and indeed, any further generic extension must satisfy KWP_2 as well.

Remark. We can also define KWP^*_α to mean "Every set is the surjective image of an α-set". It is not hard to see that KWP^*_α implies KWP_{α^*}, and that in fact the statement of Theorem 14.21 can be modified to "If KWP_α holds, then KWP^*_α holds in every generic extension", we will see that as a consequence, KWP^*_α can be weaker than KWP_α. It is also not hard to see from the proof of the theorem that if $\alpha = \alpha^*$, then KWP_α is equivalent to KWP^*_α.

14.4 Small Violations of Choice

Definition 14.22. We say that $\mathsf{SVC}(S)$ holds if for every set x there is an ordinal η and a surjection $\eta \times S \to x$. We write SVC to denote $\exists S\, \mathsf{SVC}(S)$.

Exercise 14.23. If $\mathsf{SVC}(S)$ holds and S is an α-set, then KWP^*_α holds, and therefore KWP_{α^*} holds.

Theorem 14.24. *Suppose that* SVC *holds, then there is a forcing $\mathbb{P}$ such that $\mathbb{1} \Vdash$* AC. *Moreover, if* $\mathsf{DC}_{<\kappa}$ *holds, we can require $\mathbb{P}$ to be κ-closed.*

Proof. Let κ be the least such that DC_κ fails, otherwise AC holds and $\mathbb{P} = \{\mathbb{1}\}$ works. We let $\mathbb{P}$ be the forcing $S^{<\kappa}$, ordered by reverse inclusion. This forcing is κ-closed, since κ must be a regular cardinal and $\mathsf{DC}_{<\kappa}$ holds. If G is a V-generic filter, it is not hard to see that $V[G] \models |S| = \kappa$, and therefore S can be well-ordered.

It follows that every $x \in V$ can be well-ordered in $V[G]$, since it is the image of $\eta \times S$ for some $\eta \in \mathrm{Ord}$, and in $V[G]$ we have a well-ordering of $\eta \times S$. Therefore G, being a subset of $\mathbb{P}$, can be well-ordered in $V[G]$. But this means that every $x \in V[G]$ is the image of a well-orderable set, and therefore $V[G] \models$ AC as wanted. □

Fact 14.25. *Suppose that there is a forcing $\mathbb{P}$ such that $\mathbb{1} \Vdash$* AC, *then* SVC *holds.*

Fact 14.26. SVC *holds if and only if the universe is a symmetric extension of a model of* ZFC.

Fact 14.27. SVC *holds if and only if the universe is $V(x)$ where x is a set and $V \models$* ZFC.

[36] So, α^* is the successor in the case where α was a successor, or α itself otherwise.

Chapter 15

Forcing over a symmetric extension

Many of the things we want to accomplish by forcing over a ZF models, the "weird counterexamples" in particular, are really situations where we constructed something very specific and we want to work that one case out. This means that in a lot of cases what we really want is to have a symmetric extension and have a forcing notion in that symmetric extension with which we want to force. Over that symmetric extension.

The key point here is that the arguments here are really about considering a two-step iteration $\mathbb{P} * \dot{\mathbb{Q}}$, but with the additional structure of the symmetries on $\mathbb{P}$.[37] One problem we are faced with, is that there is not so much "a structure" but rather just a loose collection of tricks one can use to make some arguments with. So we can just cover those by studying examples.

15.1 Forcing over the Cohen model

Recall the Cohen model, which we covered in section 10.4, is a model in which we force with $\operatorname{Add}(\omega, \omega)$ and then used permutations to "kill" any well-ordering of the set of Cohen reals, A.

Definition 15.1. If A is any set, we write $\operatorname{Col}(A, B)$ to denote the forcing whose conditions are partial functions $p\colon A \to B$ such that $\operatorname{dom} p$ is well-orderable and $|\operatorname{dom} p| < |A|$.

We will always assume implicitly that A and B are infinite in this case, otherwise the forcing is atomic and is not interesting.

Exercise 15.2. If $G \subseteq \operatorname{Col}(A, B)$ is a V-generic filter, then in $V[G]$ there is a surjection from A onto B.

Let $\langle \mathbb{P}, \mathscr{G}, \mathscr{F} \rangle$ be the symmetric system we used to define the Cohen model. We will write, as before, $\dot{a}_n$ for the nth Cohen real and $\dot{A} = \{\dot{a}_n \mid n < \omega\}^\bullet$ for the name for the Dedekind-finite set of reals. We will use M to denote the actual Cohen model given by $G \subseteq \operatorname{Add}(\omega, \omega)$ as our V-generic filter. We will be working in both V and M.

Exercise 15.3. Show that in the Cohen symmetric system, $\mathbb{1} \Vdash^{\mathsf{HS}} \aleph^*(\dot{A}) = \check{\omega}_1$.

Theorem 15.4. *Let A be the canonical set of Cohen reals in the Cohen model, then $\mathbb{Q} = \operatorname{Col}(A, \kappa)$ does not add any new sets of ordinals.*

[37] And later, also the additional structure of taking a symmetric extension of a symmetric extension.

Proof. Let $\dot{X} \in M$ be a $\mathbb{Q}$-name such that $q \Vdash \dot{X} \subseteq \check{\eta}$ for some η. We want to show that there is some $\bar{q} \leq_{\mathbb{Q}} q$ and $Y \in M$ such that $\bar{q} \Vdash \dot{X} = \check{Y}$. Let us first try and get our bearing on the situation.

If $q \in \mathbb{Q}$, then q is a finite function, since A is Dedekind-finite in M. Moreover, since q itself is a function from a finite subset of A to κ, there is some $E \subseteq \omega$ and a function $f\colon E \to \kappa$, in V, such that q is the condition given by $\{\langle \dot{a}_n, f(n)^\vee\rangle^\bullet \mid n \in E\}^\bullet$. We will write $\dot{q}_f$ to denote this canonical name.

This means that $\mathbb{Q}$ itself has the canonical name $\{\dot{q}_f \mid f \in \mathrm{Col}(\omega,\kappa)\}$.[38] Moreover, the order, being simply reverse inclusion is also canonically inherited. Finally, let us understand how these names interact with $\pi \in \mathscr{G}$:

$$\begin{aligned}
\pi\dot{q}_f &= \{\pi\langle \dot{a}_n, f(n)^\vee\rangle^\bullet \mid n \in E\}^\bullet \\
&= \{\langle \pi\dot{a}_n, f(n)^\vee\rangle^\bullet \mid n \in E\}^\bullet \\
&= \{\langle \dot{a}_{\pi n}, f(n)^\vee\rangle^\bullet \mid n \in E\}^\bullet \\
&= \{\langle \dot{a}_n, (f \circ \pi^{-1}(n))^\vee\rangle^\bullet \mid n \in \pi``E\}^\bullet \\
&= \dot{q}_{f\circ\pi^{-1}}.
\end{aligned}$$

It follows quite immediately, then, that these canonical names are all in HS. Moreover, if $\dot{X} \in M$ is a $\mathbb{Q}$-name, then we can find a $\mathbb{P} * \dot{\mathbb{Q}}$-name, and we can project that name to a "$\mathbb{P}$-name for a $\dot{\mathbb{Q}}$-name", which, since $\dot{X} \in M$, we can take to be in HS. We will make the even more general claim, that if $[\dot{X}] \in \mathsf{HS}$ is this $\mathbb{P}$-name, then its elements have the form $\langle p, \langle \dot{q}_f, [\dot{x}]\rangle^\bullet\rangle$ for some $p \in \mathbb{P}$ and $f \in \mathrm{Col}(\omega,\kappa)$.

Now, we go back to our $q \Vdash \dot{X} \subseteq \check{\eta}$ situation. Since this holds in M, there is some $p \in G$ such that

$$p \Vdash^{\mathsf{HS}} \text{``}\dot{q}_f \Vdash [\dot{X}] \subseteq [\check{\eta}]\text{''}.$$

Since in M if you are of the form $\check{\eta}$, that means that $[\check{\eta}]$ is really just a $\mathbb{P}$-name, and in the case where $\eta \in V$, we can in fact that that $\mathbb{P}$-name to be $\check{\eta}$ itself, so especially in that case we will confuse the canonical $\mathbb{P}$-name with any other names for $\check{\eta}$. Let E be a finite support for $p, \dot{q}_f$, and $[\dot{X}]$. Note that in the case of $\dot{q}_f$, this simply means that $\operatorname{dom} f \subseteq E$. Let us assume, by extending, if necessary, that $\operatorname{dom} f = E$ as well as $\operatorname{supp} p = E$.

For some $\alpha < \eta$, let $\bar{p} \leq p$ and $\dot{q}_{\bar{f}} \leq \dot{q}_f$ be such that $\bar{p} \Vdash^{\mathsf{HS}} \text{``}\dot{q}_{\bar{f}} \Vdash \check{\alpha} \in [\dot{X}]\text{''}$. Then we actually have that $\bar{p} \restriction E \Vdash^{\mathsf{HS}} \text{``}\dot{q}_{\bar{f}\restriction E} \Vdash \check{\alpha} \in [\dot{X}]\text{''}$, since for any extension of $\bar{p} \restriction E$ and $\dot{q}_{\bar{f}\restriction E}$ we can find a permutation in $\operatorname{fix}(E)$ which moves that extension to be compatible with $\bar{p}$ and with $\dot{q}_{\bar{f}}$.

However, $\dot{q}_{\bar{f}\restriction E} = \dot{q}_f$. So we can now define the $\mathbb{P}$-name,

$$\dot{Y}_f = \{\langle \bar{p}, \check{\alpha}\rangle \mid \operatorname{supp} \bar{p} = E, \bar{p} \Vdash^{\mathsf{HS}} \text{``}\dot{q}_f \Vdash \check{\alpha} \in [\dot{X}]\text{''}\}.$$

It is easy to see that $\dot{Y}_f \in \mathsf{HS}$, and therefore if $Y = \dot{Y}_f^G$, $M \models q \Vdash \dot{X} = \check{Y}$ as wanted. ☐

In a nutshell, the proof shows that the real information, at least with regards to sets of ordinals, comes from the finitely many Cohen reals that are relevant to defining the name $\dot{X}$, and once q decides where these will be mapped to, we can read off the entire set of ordinals.

Corollary 15.5. *Forcing with $\mathbb{Q}$ over M preserves cofinalities (of ordinals) and (well-ordered) cardinals.* ☐

[38] Hence the notation $\mathrm{Col}(A,\kappa)$.

Corollary 15.6. $M \models \mathbb{1}_{\mathbb{Q}} \Vdash \check{A}$ *is Dedekind-finite.*

Proof. Since A is a set of reals, any well-orderable subset of A can be coded as a set of ordinals. If A was forced to be Dedekind-infinite, we had introduced a new countably infinite subset. However, as we saw, $\mathbb{Q}$ does not add any sets of ordinals, so A must remain Dedekind-finite. □

Remark. It is in fact true that in M every Dedekind-finite set can injected into $\omega \times [A]^{<\omega}$, although this is not trivial. It follows, therefore, that $\mathbb{Q}$ preserves *all* Dedekind-finite sets over M.

Remark. If we define $\mathrm{Col}_{\mathrm{inj}}(A, B)$ to be $\{p \in \mathrm{Col}(A, B) \mid p$ is injective$\}$, we have a very different picture. Indeed, if H is M-generic for $\mathrm{Col}_{\mathrm{inj}}(A, \kappa)$, then $M[H] = V[G_\kappa]$, where G_κ is a V-generic filter for $\mathrm{Add}(\omega, \kappa)$. Indeed, A itself becomes the set of reals of size κ. So, in a sense, we can start by adding ω Cohen reals, forget their enumeration, and then re-enumerate them of any size we wish, without collapsing cardinals or changing cofinalities.

Fact 15.7. *In the Cohen model every set can be linearly ordered. Indeed,* KWP_1 *holds in the Cohen model, although also* BPI *holds in the Cohen model and the two are independent. Consequently, there are no amorphous sets.*

Theorem 15.8. *There is a forcing which adds an amorphous partition of A.*

Proof. Let $\mathbb{Q}$ be the forcing whose conditions are equivalence relations over a finite subset of A. We say that $E_1 \leq E_0$ if $E_0 \subseteq E_1$ and the extension preserves inequivalence. In other words, if $\neg(a_n \mathrel{E_0} a_m)$, then $\neg(a_n \mathrel{E_1} a_m)$.

It is easy to see that if $H \subseteq \mathbb{Q}$ is M-generic, then in $M[H]$, $\bigcup H$ is a new equivalence relation on A, which therefore defines a partition, P which is both infinite and its elements are infinite. Our goal is to show that this partition is amorphous. For convenience, we will denote the equivalence class of $a \in A$ as a/P.

As before, we can give each $e \in \mathbb{Q}$ a canonical name, which is derived by an equivalence relation on a finite subset of ω. We let $[\dot{P}]$ be the $\mathbb{P}$-name in HS for the canonical $\mathbb{Q}$-name for P. Suppose that $\dot{X} \in M$ is such that $E_0 \Vdash$ "$\dot{X} \subseteq \dot{P}$ is infinite", our goal is to show that E_0 must also force that the complement of $\dot{X}$ is finite. Suppose this is not the case, and so without loss of generality that $E_0 \Vdash$ "$\dot{X}$ is co-infinite".

Let $p \in G$ be a condition such that $p \Vdash^{\mathsf{HS}}$ "$\dot{E}_0 \Vdash [\dot{X}] \subseteq [\dot{P}]$ is infinite co-infinite". There is a finite $E \in [\omega]^{<\omega}$ which is a support for all the relevant names. As before, we may assume that p determines the canonical name for $\dot{E}_0$ and that $\operatorname{supp} p = E = \operatorname{dom} e_0$, where e_0 is the finite equivalence relation on ω defining the canonical name $\dot{E}_0$. Let $n, m \notin E$, and let $q \leq p$ and $E_1 \leq E_0$, given by an equivalence relation e_1, be conditions which satisfy the following:

1. n and m are in new equivalence classes when passing from e_0 to e_1.
2. The cardinality of n/e_1 is the same as the cardinality of m/e_1.
3. There is a bijection $\pi \in \mathrm{fix}(E)$ which respects e_1, switches n/e_1, and m/e_1 such that $\pi q = q$.
4. $q \Vdash^{\mathsf{HS}}$ "$\dot{E}_1 \Vdash [\dot{a}_n/P] \in \dot{X}, [\dot{a}_m/P] \notin \dot{X}$".

The only difficulty is in (3), however, by extending e_1 and q if necessary this is easy to achieve. This, however, is impossible, since that means that $\pi q = q$ and $\pi \dot{E}_1 = \dot{E}_1$, $\pi[\dot{X}] = [\dot{X}]$, but $\pi[\dot{a}_n/P] = [\dot{a}_m/P]$. Therefore, no such E_0 exists, and so P must be amorphous in $M[H]$. □

Corollary 15.9. KWP_1 *can be violated generically.* □

15.2 Separating closure, distributivity, and sequentiality

We know that closure and distributivity are distinct, but we can separate them even more in terms of preservation of weak choice principles. We will see that assuming $\mathsf{DC}_{<\kappa}$, each property preserves less and less choice. We start with the following exercise.

Exercise 15.10. Suppose that $\mathsf{DC}_{<\kappa}$ holds. If $\mathbb{P}$ is κ-closed, then $\mathbb{1} \Vdash \mathsf{DC}_{<\kappa}$.

Let us define the κ-Cohen model, for an uncountable regular κ. We force with $\mathbb{P} = \mathrm{Add}(\kappa,\kappa)$. We use the permutation group S_κ, acting as usual on $\mathbb{P}$. The filter is given by $\mathrm{fix}(E)$ for $E \in [\kappa]^{<\kappa}$. As we have seen before, this model will satisfy $\mathsf{DC}_{<\kappa}$. As in the Cohen model, we have $\dot{a}_\alpha$ and $\dot{A}$ to denote the generic subsets and the set collecting them. Let G be a V-generic filter, and let $M = \mathsf{HS}^G$, with a_α and A denoted the interpretation of the names in M.

Theorem 15.11. *There is a forcing $\mathbb{Q} \in M$ which is κ-distributive, but $\mathbb{1} \Vdash \neg\mathsf{DC}$.*

This is the "worst" we can expect, since we know that $\mathsf{AC}_{<\kappa}$ must be preserved by such $\mathbb{Q}$.

Proof. We let $\mathbb{Q}$ be the collection of all the well-orderable trees on a subset of A which do not have any infinite branches. We say that $t_1 \leq t_0$ if and only if t_0 is a subtree of t_1. Meaning, we can extended any maximal node, or add new splitting nodes, but nothing between pre-existing nodes in t_0. It is not hard to see that if $H \subseteq \mathbb{Q}$ is M-generic, then $\bigcup H$ is a tree on A which has height ω.

We claim that this forcing is κ-distributive, and that this in itself implies that the generic tree added by H does not have any branches. Note that any countable subset of A in M is coded into the tree by some condition. That is, if $A' \subseteq A$ is a countable set in M, and $t \in \mathbb{Q}$ is a condition, then there is some $t' \leq t$ such that $A' \subseteq \mathrm{dom}\, t'$. This means that any branch would have to be a new countable subset of A. But, since $\kappa > \omega$, if we show that $\mathbb{Q}$ is κ-distributive, then no such new subset can be added by $\mathbb{Q}$, so the tree will have no branches and therefore DC must fail in $M[H]$.

As before, we can give each $t \in \mathbb{Q}$ a canonical name which is induced by a tree on a bounded subset of κ in V, and therefore give $\mathbb{Q}$ itself a canonical name as well.

Let $\gamma < \kappa$ and $\langle D_\alpha \mid \alpha < \gamma\rangle \in M$ be a family of dense open sets. Let $\langle \dot{D}_\alpha \mid \alpha < \kappa\rangle^\bullet \in \mathsf{HS}$ be a name such that some $p \in G$ forced that to be the name for our sequence and that each $\dot{D}_\alpha$ is a dense open subset of $\dot{\mathbb{Q}}$. Note that since $\mathscr{F}$ is κ-complete, we can actually assume that the name for the sequence itself is a $\bullet$-name.

Now let $\dot{t}$ be a canonical name and let E be such that $\mathrm{fix}(E) \subseteq \mathrm{fix}(p) \cap \mathrm{sym}(\dot{t}) \cap \mathrm{sym}(\dot{D}_\alpha)$ for all $\alpha < \gamma$. By extending p and $\dot{t}$ as needed, we may assume that $E = \mathrm{supp}\, p = \mathrm{dom}\, \dot{t}$.[39] Let $p' \leq p$ be an extension such that for all $\alpha < \gamma$, there is some canonical condition $\dot{t}_\alpha$ such that $p' \Vdash \dot{t} \geq \dot{t}_\alpha \in \dot{D}_\alpha$. We can find such p' since $\mathbb{P}$ itself is κ-closed and $\gamma < \kappa$. Note that by the assumption that the names are canonical, this is really a statement about the trees which define them in the ground model as well.

Let $E' \in [\kappa]^{<\kappa}$ be a large enough set such that $\mathrm{fix}(E') \subseteq \mathrm{sym}(\dot{t}_\alpha)$ for all $\alpha < \gamma$ and that $E \cup \mathrm{supp}\, p' \subseteq E'$. Next, choose for each $\alpha < \gamma$ a permutation $\pi_\alpha \colon \kappa \to \kappa$ such that $\pi_\alpha \in \mathrm{fix}(E)$ and $\{\pi_\alpha``(E' \setminus E) \mid \alpha < \gamma\}$ is a pairwise disjoint family. This is possible, again, since κ is regular and $\gamma < \kappa$ and $E' \in [\kappa]^{<\kappa}$. The following hold:

1. $q = \bigcup_{\alpha<\gamma} \pi_\alpha p'$ is a condition, since $\mathrm{dom}\, \pi_\alpha p' \cap \mathrm{dom}\, \pi_\beta p' = E$ for all $\alpha \neq \beta$.

[39] We abuse the notation here, confusing between $\dot{t}$ and the tree defining it, since the condition is assumed to be of the canonical form.

2. $\pi_\alpha p' \Vdash^{\mathsf{HS}} \dot{t} \geq \pi_\alpha \dot{t}_\alpha \in \dot{D}_\alpha$.

3. If $\dot{a}_\xi \in \operatorname{dom} \pi_\alpha \dot{t}_\alpha \cap \operatorname{dom} \pi_\beta \dot{t}_\beta$, for $\alpha \neq \beta$, then $\xi \in E$.

It follows from the three conditions that $\dot{s} = \bigcup_{\alpha<\gamma} \pi_\alpha \dot{t}_\alpha$ must be a condition as well, and indeed, it is a canonical name for a condition. If $\dot{s}$ was not a tree, this must be witnessed by a pair which is in a single $\pi_\alpha \dot{t}_\alpha$ or in $\dot{t}$ itself, which is impossible. Similarly, if the tree defining $\dot{s}$ had a branch, this would have to be witnessed by one of the conditions.

But, since $\dot{s}$ clearly extends any and all $\pi_\alpha \dot{t}_\alpha$, $q \Vdash^{\mathsf{HS}} \dot{s} \in \dot{D}_\alpha$ for all $\alpha < \gamma$, and therefore we found a condition $q \leq p$ and $\dot{s}$ such that $q \Vdash^{\mathsf{HS}}$ "$\dot{s} \leq \dot{t}$ and $\dot{s} \in \bigcap_{\alpha<\gamma} \dot{D}_\alpha$", and so $\mathbb{1} \Vdash^{\mathsf{HS}}$ "$\dot{\mathbb{Q}}$ is $\check{\kappa}$-distributive" as wanted. □

Theorem 15.12. *There is a forcing $\mathbb{Q} \in M$ such that $\mathbb{Q}$ is κ-sequential, but $\mathbb{1} \Vdash \neg\mathsf{AC}_\omega$.* □

$\mathbb{Q}$ is given by finite partitions of well-orderable subsets of A. This is a generalisation of Theorem 15.8 and the proof that it adds an amorphous set is very similar. The proof that this forcing is κ-sequential is the same idea as Theorem 15.11.

15.3 Preserving violations of choice: caring for initial segments

So, after we have worked so terribly hard to violate choice, how can we ensure that our forcing is not going to destroy what we have already done? Clearly, if we make some large enough initial segment of the universe countable, or generally well-orderable, this will happen. Indeed, the fact that SVC holds in all symmetric extensions (of ZFC models) already tell us that it is possible to force ZFC to hold again and destroy the beautiful chaos that we created.

But we can try and be a bit more subtle. Perhaps we added one failure at the level of the real numbers or so, and now we wish to add another. How can we ensure that these two objects do not interact?

Exercise 15.13. If κ is uncountable, then $\operatorname{Add}(\kappa, 1)$ well-orders $[\kappa]^{<\kappa}$.

The problem, therefore, is finding a way to add new generic subsets to the universe without adding bounded sets. In general, this is a very difficult task and it may very well be that it is not even provable from ZF that such partial orders always exist. However, if we took a symmetric extension of a model of ZFC, or equivalently, if SVC holds, we can indeed find such forcing.

Theorem 15.14. *Suppose that* SVC *holds in V, then for every α there is a regular cardinal κ and partial order $\mathbb{P}_\kappa$ such that $\mathbb{P}_\kappa$ adds generic subsets to κ without adding new sets of rank α.*

Proof. Since SVC holds, there is some $W \subseteq V$ such that V is a symmetric extension of W by some symmetric system $\langle \mathbb{P}, \mathscr{G}, \mathscr{F} \rangle$ and $W \models \mathsf{ZFC}$. Let κ be a regular cardinal in W large enough such that $\mathbb{1}_{\mathbb{P}} \Vdash (|\dot{V}_\alpha| + |\mathbb{P}|^W) < \check{\kappa}$.

Let $\mathbb{P}_\kappa$ be $\operatorname{Add}(\kappa, 1)^W$. By the choice of κ, it is easy to see that $\mathbb{P}$ is κ-c.c. in W, and by the choice of $\mathbb{P}_\kappa$ we know it is κ-closed in W. Therefore, $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}}$ "$\check{\mathbb{P}}_\kappa$ is κ-distributive", as we saw in Theorem 5.9.

Therefore, if $G \subseteq \mathbb{P}$ is W-generic which is used to define V, we have that in $W[G]$, $\mathbb{P}_\kappa$ is still κ-distributive. Suppose that in V we have $\{D_x \mid x \in V_\alpha\}$ as a family of dense open subsets of $\mathbb{P}_\kappa$, then in $W[G]$ this family has size $< \kappa$, therefore its intersection is dense in $\mathbb{P}_\kappa$. However, as the intersection is defined from the family, rather from its well-ordering, this intersection is in V. Therefore $\mathbb{P}_\kappa$ is $\leq V_\alpha$-distributive in V, so it does not add subsets of V_α. □

Chapter 16

Intermezzo: One and a half step towards Bristol

The Bristol model is a model in which KWP fails, and consequently, SVC must fail too. The interesting thing about the Bristol model is that it is intermediate between L and $L[c]$ where c is a Cohen real. The general construction of the Bristol model is a class-long iteration of symmetric extensions, and we will not cover it here. But it is an interesting example for both a symmetric extension and forcing over a symmetric extension. Through this section we work in L, or at least in a model of GCH, or at least in a model where $2^{2^{\aleph_0}} = \aleph_2$. Throughout this section, $=^*$ and $\subseteq^*$ mean "up to a finite difference".

Definition 16.1. Fix $\{A_\alpha \mid \alpha < \omega_1\}$ to be a family of almost disjoint subsets of ω. We say that $\Pi\colon \omega_1 \to \omega_1$ can be *implemented* if there is $\pi\colon \omega \to \omega$ such that for all $\alpha < \omega_1$, $\pi``A_\alpha =^* A_{\Pi(\alpha)}$. We will use $\iota(\pi)$ to denote the permutation that π implements.

Definition 16.2. Let $\{A_\alpha \mid \alpha < \omega_1\}$ be an almost disjoint family. For a countable $I \subseteq \omega_1$, we say that $\mathcal{B} = \{B_\alpha \mid \alpha \in I\}$ is a *disjoint approximation* of $\{A_\alpha \mid \alpha \in I\}$ if $\mathcal{B}$ is a pairwise disjoint family satisfying that $B_\alpha =^* A_\alpha$ for all $\alpha \in I$ and

$$A_\xi \cap \bigcup \mathcal{B} \text{ is infinite } \iff \xi \in I.$$

If $B_\alpha \subseteq A_\alpha$ for all $\alpha \in I$, we say that $\mathcal{B}$ is a *disjoint refinement.* If for any countable I there is a disjoint approximation, we say that $\{A_\alpha \mid \alpha < \omega_1\}$ is a *permutable family.*

Proposition 16.3. *If $\{A_\alpha \mid \alpha < \omega_1\}$ is permutable, then every bounded permutation of ω_1 can be implemented.*

Proof. Let Π be a bounded permutation and let I be its domain, i.e. $\Pi \upharpoonright \omega_1 \setminus I = \mathrm{id}$. Let $\{B_\alpha \mid \alpha \in I\}$ be a disjoint approximation, and let π be the order isomorphism mapping B_α to $B_{\Pi(\alpha)}$ and the identity on $\omega \setminus \bigcup_{\alpha \in I} B_\alpha$. Easily, $\iota(\pi) = \Pi$. ☐

Exercise 16.4. $\{A_\alpha \mid \alpha < \omega_1\}$ is a permutable family if and only if there is $\{B_\alpha \mid \alpha < \omega_1\}$ such that $B_\alpha \subseteq^* B_\beta$ for all $\alpha < \beta$ and $A_\alpha = B_{\alpha+1} \setminus B_\alpha$. In particular, there exists a permutable family.

Our forcing is, as we mentioned, $\mathrm{Add}(\omega, 1)$. Let us fix a permutable family $\{A_\alpha \mid \alpha < \omega_1\}$ and let $\mathscr{G}$ be the group of all permutations of ω which implement a bounded permutation of ω_1. Then $\mathscr{G}$ acts on $\mathrm{Add}(\omega, 1)$ by $\pi p(\pi n) = p(n)$.

For a disjoint approximation, $\mathcal{B}$, we let $\operatorname{fix}(\mathcal{B}) = \{\pi \in \mathscr{G} \mid \pi \upharpoonright \bigcup\mathcal{B} = \mathrm{id}\}$. Note that if $\pi \in \operatorname{fix}(\mathcal{B})$, then $\iota(\pi) \upharpoonright I = \mathrm{id}$. Our filter of groups is generated by $\operatorname{fix}(\mathcal{B})$ for disjoint approximations $\mathcal{B}$.

For $A \subseteq \omega$, we write $\mathbb{P} \upharpoonright A$ as $\{p \in \mathbb{P} \mid \operatorname{dom} p \subseteq A\}$. Then, letting $\dot{c}$ be the canonical name for the Cohen real, $\dot{c}_A$ is the canonical name for the Cohen real restricted to $\mathbb{P} \upharpoonright A$. Note that if $\pi\colon \omega \to \omega$ is a permutation, then π induces an isomorphism between $\mathbb{P} \upharpoonright A$ and $\mathbb{P} \upharpoonright \pi``A$, and in particular, $\pi\dot{c}_A = \dot{c}_{\pi``A}$.

Let us say that a $\mathbb{P}$-name is an *A-name* if it is a $\mathbb{P} \upharpoonright A$-name, and it is *almost A-name* if it is a B-name for some B such that $B =^* A$. Finally, let us say that a name for a set of ordinals is "decent" if every name inside of it has the form $\check{\xi}$.

Proposition 16.5. *For all $\alpha < \omega_1$, $\dot{c}_{A_\alpha} \in \mathsf{HS}$.* □

Proposition 16.6. *If $\dot{x} \in \mathsf{HS}$ and $\mathbb{1} \Vdash^{\mathsf{HS}} \dot{x} \subseteq \check{\omega}$, then there is a disjoint approximation $\mathcal{B}$ and a decent $\bigcup\mathcal{B}$-name $\dot{x}_* \in \mathsf{HS}$ such that $\mathbb{1} \Vdash \dot{x} = \dot{x}_*$.*

Proof. We let $\mathcal{B}$ be a disjoint approximation such that $\operatorname{fix}(\mathcal{B}) \subseteq \operatorname{sym}(\dot{x})$. Let $B = \bigcup\mathcal{B}$, then for any p and $n < \omega$, $p \Vdash \check{n} \in \dot{x}$ if and only if $\pi p \Vdash \check{n} \in \dot{x}$ for $\pi \in \operatorname{fix}(\mathcal{B})$. In particular, by the usual homogeneity argument, $p \upharpoonright B \Vdash \check{n} \in \dot{x}$. Therefore, $\dot{x}_* = \{\langle p, \check{n}\rangle \mid \operatorname{dom} p \subseteq B, p \Vdash \check{n} \in \dot{x}\}$ is a decent $\mathcal{B}$-name, and easily $\mathbb{1} \Vdash \dot{x} = \dot{x}_*$. □

Corollary 16.7. *If $\dot{x} \in \mathsf{HS}$ then $\mathbb{1} \Vdash \dot{x} \neq \dot{c}$.* □

Proposition 16.8. $\mathbb{1} \Vdash^{\mathsf{HS}} \neg\mathsf{AC}$.

Proof. Let G be a V-generic filter and let $M = \mathsf{HS}^G$. Since $V[G]$ is a Cohen extension, if $M \models \mathsf{AC}$, then $M = V[r]$ for some real number $r \in M$. Therefore, there is a name $\dot{r} \in \mathsf{HS}$ which is a decent $\bigcup\mathcal{B}$-name for some disjoint refinement $\mathcal{B}$. It follows that M must satisfy that every set has a $\bigcup\mathcal{B}$-name, but if I is the index set for $\mathcal{B}$, and $\alpha \notin I$, then $\dot{c}_{A_\alpha}$ is not equivalent to any $\bigcup\mathcal{B}$-name □

For each $\alpha < \omega_1$, let $\dot{R}_\alpha = \{\dot{x} \in \mathsf{HS} \mid \dot{x}$ is a decent almost A_α-name for a real$\}$. We now have that $\pi\dot{R}_\alpha = \dot{R}_{\iota(\pi)(\alpha)}$. So, in particular, $\dot{R} = \{\dot{R}_\alpha \mid \alpha < \omega_1\}^\bullet \in \mathsf{HS}$. In terms of objects, each $\dot{R}_\alpha$ is the "well-behaved name" for the reals of $V[c_{A_\alpha}]$.

Exercise 16.9 (*). $\mathbb{1} \Vdash^{\mathsf{HS}} \dot{R}$ is $\aleph_1$-amorphous. (Every subset of $\dot{R}^G$ is countable or co-countable.)

The next step in the construction is actually to force a well-ordering of $\dot{R}$. We have a good candidate for that well-ordering, namely $\dot{\rho} = \langle \dot{R}_\alpha \mid \alpha < \omega_1\rangle^\bullet$. Moreover, as the following proposition shows, it is a very good candidate.

Proposition 16.10. *$\dot{\rho} \notin \mathsf{HS}$, but for any countable $I \subseteq \omega_1$, $\dot{\rho} \upharpoonright I = \langle \dot{R}_\alpha \mid \alpha \in I\rangle^\bullet \in \mathsf{HS}$.* □

Moreover, if $\pi \in \mathscr{G}$, since $\pi\dot{R}_\alpha = \dot{R}_{\iota(\pi)(\alpha)}$, we get that $\pi\dot{\rho} \upharpoonright I$ is a "shuffled sequence" which is given by a $\bullet$-name. We can therefore define the name $\dot{\mathbb{Q}}$ for the forcing whose conditions are these sort of initial segments. Namely,

$$\dot{\mathbb{Q}} = \{\pi\dot{\rho} \upharpoonright I \mid I \in [\omega_1]^{<\omega_1}, \pi \in \mathscr{G}\}^\bullet.$$

The order is given by inclusion, of course. This process is a "symmetrisation of a name", where we take a name ($\dot{\rho}$ in this case), look at which parts of it are symmetric. Of course, these parts are not likely to form a symmetric name themselves, since that would often imply we started with a name in HS, so we close this collection under the automorphism group action. This will often yield a "nice enough" name for a forcing which will add the name we started with.

Theorem 16.11. $\mathbb{1} \Vdash \dot\rho$ *is* HS*-generic for* $\dot{\mathbb{Q}}$.

Proof. Suppose that $\dot D \in \mathsf{HS}$ is a name such that $p \Vdash^{\mathsf{HS}}$ "$\dot D \subseteq \dot{\mathbb{Q}}$ is a dense open set". We will find some η such that $p \Vdash^{\mathsf{HS}} \dot\rho \restriction \eta \in \dot D$. Let $\mathcal{B}$ be a disjoint approximation such that $\operatorname{fix}(\mathcal{B}) \subseteq \operatorname{sym}(\dot D)$, and we may assume without loss of generality that $\operatorname{fix}(\mathcal{B}) \subseteq \operatorname{fix}(p)$ as well. We can assume, by extending $\mathcal{B}$ if need be, that α is the index set of $\mathcal{B}$. We start by noting that if $\alpha \subseteq A$ and $q \leq p$ is a condition such that $q \Vdash^{\mathsf{HS}} \pi\dot\rho \restriction A \in \dot D$ for some π satisfying $\iota(\pi) \restriction \alpha = \mathrm{id}$, then $q^{\mathsf{HS}} \Vdash \dot\rho \restriction A \in \dot D$. This is because we can find some $\tau \in \operatorname{fix}(\mathcal{B})$ such that $\iota(\tau) = \iota(\pi)^{-1}$ and $\tau q = q$. We can do that since $\iota(\pi)$ is the identity on α, so $\iota(\pi)^{-1}$ is also the identity on α, so it can be implemented by some $\tau \in \operatorname{fix}(\mathcal{B})$. The requirement $\tau q = q$ is easy to adjust for.

Now, let $q \leq p$ be such an extension, since p forced that $\dot D$ is dense, there will be one. By the observation, $q \Vdash^{\mathsf{HS}} \dot\rho \restriction A \in \dot D$. Since $\dot D$ is forced to be open, we let $\eta_q = \sup A$, and then $q \Vdash^{\mathsf{HS}} \dot\rho \restriction \eta_q \in \dot D$. Finally, there is a maximal antichain below p of such q, let η be the supremum of the η_q, since $\operatorname{Add}(\omega, 1)$ is a c.c.c. forcing, $\eta < \omega_1$. So $p \Vdash^{\mathsf{HS}} \dot\rho \restriction \eta \in \dot D$ as wanted. □

Finally, we want to claim that even though we well-ordered the set, we did not add any new reals or sets of ordinals. And in particular, we could not have forced back the axiom of choice. To do that, note that in $V[c]$, $\mathbb{Q}$ is in fact isomorphic to $\operatorname{Add}(\omega_1, 1)^V$. Similar to the argument we have seen in Theorem 15.14, we have that $\mathbb{1} \Vdash^{\mathsf{HS}}$ "$\dot{\mathbb{Q}}$ is σ-distributive". We will see later a deeper reason for which $\mathbb{Q}$ does not add any sets of ordinals as well.

But before that, let us take a moment to think about all of those generic filters. In the first example where we forced over the Cohen model with $\operatorname{Col}(A, \kappa)$, it is clear that the forcing required adding generics over $V[G]$. In both the forcing violating KWP_1, as well as in the example of Theorem 15.11, we can slightly modify the symmetric system to obtain the resulting model directly.[40] But, in the case of the Bristol model, this is now very surprising, since the forcing we are using is actually one that should have no generics in $V[c]$ whatsoever.

[40]This should not be very surprising, after all, if $a \subseteq V$ and $a \in V[G]$, then $V[a]$ is a generic extension of V.

Chapter 17

Mixing symmetric systems: products and iterations

17.1 Products

Suppose that $\langle\mathbb{Q}_0,\mathscr{G}_0,\mathscr{F}_0\rangle$ and $\langle\mathbb{Q}_1,\mathscr{G}_1,\mathscr{F}_1\rangle$ are two symmetric systems. We can define a symmetric system on $\mathbb{P}=\mathbb{Q}_0\times\mathbb{Q}_1$ by taking $\mathscr{G}=\mathscr{G}_0\times\mathscr{G}_1$, with $(\pi_0,\pi_1)(q_0,q_1)=\langle\pi_0 q_0,\pi_1 q_1\rangle$, and taking $\mathscr{F}$ to be the filter generated by $\{H_0\times H_1\mid H_i\in\mathscr{F}_i\}$. We will often abuse the notation and simply write $\langle\mathbb{Q}_0,\mathscr{G}_0,\mathscr{F}_0\rangle\times\langle\mathbb{Q}_1,\mathscr{G}_1,\mathscr{F}_1\rangle$.

Exercise 17.1. Show that $\langle\mathbb{P},\mathscr{G},\mathscr{F}\rangle$ is a symmetric system.

Theorem 17.2. *Let $\langle\mathbb{P},\mathscr{G},\mathscr{F}\rangle=\langle\mathbb{Q}_0,\mathscr{G}_0,\mathscr{F}_0\rangle\times\langle\mathbb{Q}_1,\mathscr{G}_1,\mathscr{F}_1\rangle$ and let $G=G_0\times G_1$ be a V-generic filter for $\mathbb{P}$. Then $\mathsf{HS}^G_{\mathscr{F}}$ is a symmetric extension of $\mathsf{HS}^{G_0}_{\mathscr{F}_0}$ by $\langle\mathbb{Q}_1,\mathscr{G}_1,\mathscr{F}_1\rangle$ using G_1 as the generic filter.*

Proof. Let us denote by M_0 the model $\mathsf{HS}^{G_0}_{\mathscr{F}_0}$. Suppose that $\dot{x}\in M_0$ is hereditarily $\mathscr{F}_1$-symmetric, we want to claim that there is some $\dot{x}_*\in\mathsf{HS}_{\mathscr{F}}$ such that $\dot{x}^{G_1}=\dot{x}_*^G$.

We prove this by induction on the rank of $\dot{x}$. Let $[\dot{x}]$ be a name in $\mathsf{HS}_{\mathscr{F}_0}$ such that $[\dot{x}]^{G_0}=\dot{x}$. We define $\dot{x}_*$ to be

$$\dot{x}_*=\{\langle\langle p,q\rangle,\dot{y}_*\rangle\mid\langle p,\langle\check{q},[\dot{y}]\rangle\rangle\in[\dot{x}]\}.$$

It is not hard to see that $\dot{x}_*^G=\dot{x}^{G_1}$. Let H_0 be $\operatorname{sym}_{\mathscr{G}_0}([\dot{x}])$ and let $H_1\in\mathscr{F}_1$ be such that $\mathbb{1}_{\mathbb{Q}_0}\Vdash\check{H}_1=\operatorname{sym}_{\mathscr{G}_1}(\dot{x})$,[41] which while is computed in M, is itself an element of V. We claim that if $(\pi_0,\pi_1)=\pi\in H=H_0\times H_1$, then $\pi\dot{x}_*=\dot{x}_*$.

To see that, first note that

$$\pi\dot{x}_*=(\pi_0,\pi_1)\dot{x}_*=\{\langle\langle\pi_0 p,\pi_1 q\rangle,\pi\dot{y}_*\rangle\mid\langle\langle p,q\rangle,\dot{y}_*\rangle\in\dot{x}_*\},$$

so it is enough to understand how $\pi\dot{y}_*$ behaves. Note that we can add to our induction hypothesis that $\pi\dot{y}_*$ defined with the above recursive definition, also satisfies that $\pi_0[\pi_1\dot{y}]$ translates exactly to $\pi\dot{y}_*$, in which case the above computation of $\pi\dot{x}_*$ simplifies back to $\dot{x}_*$. □

Note that by a symmetry argument we can change the order of the product, so this holds in the other direction as well.

[41] This is for simplicity, of course, the complete course of action would be to pick a tenacious condition in G_0 which decided the value of H_1 and then restrict H_0 to also fix that condition. But for readability and general concept purposes, we will pretend that this condition was simply $\mathbb{1}_{\mathbb{Q}_0}$.

Exercise 17.3. Suppose that $\langle\mathbb{Q}_0, \mathscr{G}_0, \mathscr{F}_0\rangle$ and $\langle\mathbb{Q}_1, \mathscr{G}_1, \mathscr{F}_1\rangle$ are both homogeneous systems, then $\langle\mathbb{Q}_0, \mathscr{G}_0, \mathscr{F}_0\rangle \times \langle\mathbb{Q}_1, \mathscr{G}_1, \mathscr{F}_1\rangle$ is homogeneous as well.

We can also talk about infinitary product, although we now need to be more careful about our supports. This, however, opens a door for a wonderful interplay of three different supports for three different products: forcing notions, groups, and filters. In the context of the forcing notions we already understand what a support is: it is the set of coordinates where the information is not $\mathbb{1}$. In the context of the groups, this is the set of coordinates where the automorphism is not id. In the context of the filters, this is the set of coordinates where the group is not $\mathscr{G}_\alpha$.

Theorem 17.4. *It is consistent with* ZF *that every ultrafilter on ω is principal.*

Proof. Let us consider the symmetric system given by $\operatorname{Add}(\omega, 1)$ with its full automorphism group as $\mathscr{G}$, and with the improper filter, namely, the filter of subgroups containing the trivial group. It is easy to see that this symmetric system is degenerate in the sense that every name is in HS.

Let $\langle\mathbb{P}, \mathscr{G}, \mathscr{F}\rangle$ be the finite support product of countably many copies of this system. We claim that $\mathbb{1} \Vdash^{\mathsf{HS}}$"There are no free ultrafilters on $\check{\omega}$".

Suppose that $\dot{U} \in \mathsf{HS}$ is a name such that $p \Vdash^{\mathsf{HS}}$ "$\dot{U}$ is an ultrafilter". Since the product is finitely supported, we let n be such that the support of the groups and p is below n. Let $\dot{c}_n$ be the name of the nth Cohen real. If $q \leq p$ is such that $q \Vdash \dot{c}_n \in \dot{U}$, let k be such that $\operatorname{dom} q(n) \subseteq k$, and let σ_k be the automorphism of $\operatorname{Add}(\omega, 1)$ defined by

$$\sigma_k \bar{p}(i) = \begin{cases} \bar{p}(i) & i < k \\ 1 - \bar{p}(i) & k \leq i \end{cases}$$

and let $\pi \in \mathscr{G}$ be such that $\pi_n = \sigma_k$ is the above and otherwise any other coordinate is the identity.

Easily, $\pi\dot{U} = \dot{U}$ and $\pi q = q$, therefore $q \Vdash \pi\dot{c}_n \in \dot{U}$. But $q \Vdash \dot{c}_n \cap \sigma_k\dot{c}_n \subseteq \check{k}$. Therefore q must force that $\dot{U}$ is principal, since it contains a finite set. □

Remark. What the above shows is that now that we have introduced products of symmetric systems, the improper filter allows us to combine models of ZFC into a model of ZF in a somewhat coherent way. Indeed, it is a way to present symmetric systems given by our usual wreath product construction by considering them as products of pointwise "degenerate" systems, in a way.

Lemma 17.5. *Suppose that $\langle\mathbb{Q}_\alpha, \mathscr{G}_\alpha, \mathscr{F}_\alpha\rangle$, for $\alpha < \kappa$, are homogeneous systems. Then the tenacious conditions of any support product are determined by the supports of the conditions and the filters.* □

So, for example, if we took a full support product of the forcing notions, but finite support product of the filters, then the conditions that are finitely supported are exactly the tenacious ones.

Theorem 17.6. *Suppose that $\langle\mathbb{Q}_\alpha, \mathscr{G}_\alpha, \mathscr{F}_\alpha\rangle$, for $\alpha < \kappa$ are homogeneous systems. Let $I_\mathbb{P}, I_\mathscr{G}, I_\mathscr{F}$ be three ideals over κ containing all the singletons, then taking $\mathbb{P} = \prod^{I_\mathbb{P}}_{\alpha<\kappa} \mathbb{Q}_\alpha$, $\mathscr{G} = \prod^{I_\mathscr{G}}_{\alpha<\kappa} \mathscr{G}_\alpha$, and $\mathscr{F} = \prod^{I_\mathscr{F}}_{\alpha<\kappa} \mathscr{F}_\alpha$, we have that $\langle\mathbb{P}, \mathscr{G}, \mathscr{F}\rangle$ is equivalent to the case where $I = I_\mathbb{P} \cap I_\mathscr{F}$ is used for all three.*

Proof. By Lemma 17.5 we have that the tenacious conditions are exactly those with support in I. We claim that if $p \Vdash^{\mathsf{HS}} \varphi(\dot{x})$ and S is the support of $\operatorname{sym}(\dot{x})$, then $p \restriction S \Vdash^{\mathsf{HS}} \varphi(\dot{x})$. In the case that $S \in I_{\mathscr{G}}$, this is trivial, since any extension of $p \upharpoonright S$ can be made compatible with p by the pointwise homogeneity.

However, in the general case, it is enough to note that if $\alpha \notin S$, we can assume without loss of generality that $p(\alpha) = \mathbb{1}_{\mathbb{Q}_\alpha}$. This is essentially the same argument, take any extension of $p^\alpha = p \upharpoonright \kappa \setminus \{\alpha\}$, and it can be made compatible with p, therefore we may assume that α is not in the support of p.

Note that the support of $p \upharpoonright S$ lies in I, as a subset of $S \in I_{\mathscr{F}}$ and $\operatorname{supp} p \in I_{\mathbb{P}}$. It follows, therefore, that if $\dot{x} \in \mathsf{HS}$, then we can assume that every condition which appears in $\dot{x}$ is one where every condition has a support in I, and therefore the I-support product will have the same names which are hereditarily symmetric, and so the two systems are equivalent in the sense of Theorem 10.28. □

Remark. One is left to wonder why bother with the above theorem. It may not seem like much, but it is often easy to prove some structural preservation theorems (e.g. cardinals are not collapsed; Cohen reals are not added) by using a full-support product, or an Easton support product, etc. This allows us to still take $I_{\mathscr{F}}$ to be finite, which means that the preservation theorems will now hold for the finite support product, assuming homogeneity.

We will not discuss products any further, since unfortunately, their major usefulness is in class-sized products, rather than set-sized product. For example, we can arrange a Dedekind-finite set A which can be mapped onto κ, for any fixed κ. If we want one that maps onto κ and one that maps onto κ^+, then simply taking the one that maps onto κ^+ is sufficient. But, if we wish to arrange that for *all* κ, then a class-sized product must be used.[42]

17.2 First example: Morris' local step

Morris constructed a model in which ZF holds, but for any α, there is a set A_α which is the countable union of countable sets, and $\aleph_\alpha < \aleph^*(\mathcal{P}(A_\alpha))$. Such a model must fail SVC, since if we force AC to hold, any such A_α must become countable and so $\aleph^*(\mathcal{P}(A_\alpha)) = (2^{\aleph_0})^+$ must be larger than all the ordinals. Indeed, this model has a much more severe failure of SVC, there is no extension of this model to a model of ZFC with the same ordinals.

The original proof by Morris was written in his Ph.D. and was never fully published as a paper. It involved a class length iteration of symmetric extensions that was defined before forcing, symmetric extensions, or iterations were properly clarified as we know them today. Nevertheless, we can still study a local failure of the above. Namely, for a fixed κ, which we can assume is regular, we can still ask to find a symmetric system which preserves "enough cardinals" and adds some A is the countable union of countable sets with the above property, whose power set can be mapped onto κ. For the usual simplicity, we will assume that $2^\kappa = \kappa^+$.

Adding A is fairly simple. Consider $\omega \times \omega$ with the preorder $\langle i, n\rangle \leq \langle j, m\rangle \iff i \leq j$. With its automorphism group, and the ideal of sets generated by $\{n \times \omega \mid n < \omega\}$ we have, by Theorem 12.13, a symmetric copy of this structure. Since each level is fixed pointwise, those will be countable, and since the levels are not moved, their enumeration is also fixed, and so we have that the symmetric copy is a countable union of countable sets.

More explicitly, $\mathbb{P} = \operatorname{Add}(\kappa, \omega \times \omega \times \kappa)$, $\mathscr{G}$ is the group $(\{\mathrm{id}\} \wr S_\omega) \wr S_\kappa$ with the standard action $\pi p(\pi(n, m, \alpha), \beta) = p(n, m, \alpha, \beta)$, and the filter is generated by groups of the form

$$\operatorname{fix}(n, E) = \{\pi \in \mathscr{G} \mid \pi \upharpoonright n \times \omega \times E = \mathrm{id}\}$$

[42]This is, of course, a gross oversimplification, and we can find quite a few natural situations where set-sized products are just as useful and interesting.

for $n < \omega$ and $E \in [\kappa]^{<\kappa}$. We will often ignore E and simply write $\operatorname{fix}(n)$ to indicate that for some suitable E the group is $\operatorname{fix}(n, E)$.[43]

We can now define names, $\dot{x}_{n,m,\alpha} = \{\langle p, \check{\beta}\rangle \mid p(n,m,\alpha,\beta) = 1\}$, $\dot{a}_{n,m} = \{\dot{x}_{n,m,\alpha} \mid \alpha < \kappa\}^\bullet$, $\dot{A}_n = \{\dot{a}_{n,m} \mid m < \omega\}^\bullet$ and $\dot{A} = \{\dot{a}_{n,m} \mid n, m < \omega\}^\bullet$.

Proposition 17.7. $\pi\dot{x}_{n,m,\alpha} = \dot{x}_{\pi(n,m,\alpha)}$, $\pi\dot{a}_{n,m} = \dot{a}_{\pi^*(n,m)} = \dot{a}_{n,\pi^*_n m}$, $\pi\dot{A}_n = \dot{A}_n$, $\pi\dot{A} = \dot{A}$. □

As a corollary from that, all of these names are in HS. But also $\langle \dot{a}_{n,m} \mid m < \omega\rangle^\bullet$ and $\langle \dot{A}_n \mid n < \omega\rangle^\bullet$ are in HS. In other words, $\mathbb{1} \Vdash^{\mathsf{HS}}$ "$\dot{A}$ is a countable union of countable sets".

Proposition 17.8. $\mathbb{1} \Vdash^{\mathsf{HS}} \dot{A}$ *is not countable.*

Proof. Suppose that $\dot{f} \in \mathsf{HS}$ and $p \Vdash^{\mathsf{HS}} \dot{f}\colon \check{\omega} \to \dot{A}$, let n be such that $\operatorname{fix}(n, E) \subseteq \operatorname{sym}(\dot{f})$. Suppose that $k < \omega$ and $p \Vdash \dot{f}(\check{k}) = \dot{a}_{n,0}$. Then there is some $\pi \in \operatorname{fix}(n, E)$ such that πp is compatible with p and $\pi\dot{a}_{n,0} = \dot{a}_{n,1}$. This is, of course, impossible, as this implies p must force that $\dot{f}$ is not a function. Therefore, p must force that $\operatorname{rng} \dot{f} \subseteq \bigcup\{\dot{A}_k \mid k < n\}$. In particular, it is impossible for $\dot{f}$ to be surjective. □

Let M denote the symmetric extension, and let us work inside this model. We will define a symmetric system in M which forces that $\kappa < \aleph^*(\mathcal{P}(A))$. Moreover, we will do so without adding any new sets of ordinals, so κ will not be collapsed, nor the real numbers change their size.

We let T denote the "choice tree" of the sequence $\langle A_n \mid n < \omega\rangle$. Namely, $T = \bigcup_{n<\omega} \prod_{k<n} A_k$. Note that the proposition above really shows that T has no branches, as a branch in the tree would be a function from ω into an unboundedly many of the A_n. The forcing $\mathbb{Q}$ is going to be a variation of the finite support product of $\kappa \times \omega$ copies of T, we want to ensure that the branches that are added by the different copies of T are almost disjoint. This will naturally form a family of $\kappa \times \omega$ different subsets of A, which we will then make into a Dedekind-finite set that can be mapped onto κ.

So, a condition in $\mathbb{Q}$ is a finite sequence, $\langle t_i \mid i \in E\rangle$ such that $t_i \in T$ and $E \in [\kappa \times \omega]^{<\omega}$, we say that E is the *support* of the condition. We say that $t \leq s$ if the two conditions hold:

1. $\operatorname{supp} s \subseteq \operatorname{supp} t$ and for all $i \in \operatorname{supp} s$, $s_i \subseteq t_i$.
2. For all $i, j \in \operatorname{supp} s$, if $t_i(k) = t_j(k)$, then $k \in \operatorname{dom} s_i \cap \operatorname{dom} s_j$.

In plain words, we are extending the support, and on each coordinate of the support we may extend along the tree, but any new information must be pairwise distinct.

Our automorphism group, $\mathscr{H}$, is the permutation group of $\kappa \times \omega$ given by $\{\mathrm{id}\} \wr S_\omega^{\mathrm{fin}}$, where S_ω^{fin} is the group of finitary permutations of ω, with the action, as expected, given by

$$\pi\langle t_i \mid i \in E\rangle = \langle t_{\pi i} \mid i \in E\rangle.$$

The filter $\mathscr{K}$ is generated by the subgroups $\operatorname{fix}(E) = \{\pi \in \mathscr{H} \mid \pi \restriction E = \mathrm{id}\}$ with $E \in [\kappa \times \omega]^{<\omega}$.

We define $\dot{b}_{\alpha,n} = \{\langle t, \check{a}\rangle \mid \exists i, t_{\alpha,n}(i) = a\}$ and $\dot{B}_\alpha = \{\dot{b}_{\alpha,n} \mid n < \omega\}^\bullet$. It is not hard to verify, as usually do, that $\pi\dot{b}_{\alpha,n} = \dot{b}_{\pi(\alpha,n)}$ and consequently, $\pi\dot{B}_\alpha = \dot{B}_\alpha$. Finally, we let $\dot{B} = \{\dot{b}_{\alpha,n} \mid \langle \alpha, n\rangle \in \kappa \times \omega\}^\bullet$, then all these names are in HS, as is the canonical name for the enumeration $\langle \dot{B}_\alpha \mid \alpha < \kappa\rangle^\bullet$. So, trivially, $\mathbb{1} \Vdash^{\mathsf{HS}}_{\mathbb{Q}} \check{\kappa} \leq^* \dot{B}$.

[43] In reality, we can even allow E to vary over some countably many different sets, but since κ is regular this will only make a difference if $\kappa = \omega$.

Proposition 17.9. $\mathbb{1} \Vdash^{\mathsf{HS}}_{\mathbb{Q}} \dot{B}$ *is Dedekind-finite.*

Proof. Suppose that $t \Vdash \dot{f}\colon \check{\omega} \to \dot{B}$ and let E be a finite support for t and $\dot{f}$. We let $\langle\alpha, n\rangle \notin E$ and let $t' \leq t$ be a condition such that $t' \Vdash \dot{f}(\check{k}) = \dot{b}_{\alpha,n}$. We can find some m large enough such that $\langle\alpha, m\rangle \notin E \cup \operatorname{supp} t'$ and take π to be the permutation which only exchanges between $\langle\alpha, n\rangle$ and $\langle\alpha, m\rangle$. Clearly, $\pi \in \operatorname{fix}(E)$ so $\pi t = t$ and $\pi\dot{f} = \dot{f}$ as well. Moreover, $\pi t'$ and t' are compatible, as the second condition holds vacuously (as we increasing the support of either one when considering $t' \cup \pi t'$). But this means that t' cannot force that $\dot{f}$ is injective, but it was arbitrary, so t must force that $\dot{f}$ is not injective. □

We wish to argue that no sets of ordinals were added when taking the symmetric extension of M. And for this we will need to go back to V and consider the whole structure. We have very natural canonical names for the elements of T given by functions in $\omega^{<\omega}$, as well as the conditions in $\mathbb{Q}$, etc. Note that also $\dot{\mathscr{H}}$ and $\dot{\mathscr{K}}$ have canonical names, since we cleverly chose only finitary permutations of ω and finite sets, which are not changed from the ground model to M.

It will be helpful to understand the canonical names for the conditions in $\mathbb{Q}$, as well. These are essentially functions from $\kappa \times \omega \to \omega^{<\omega}$ with finite domain, so if f is such function,

$$\dot{t}_f = \left\{\langle\check{\alpha}, \check{n}, \langle\dot{a}_{f(\alpha,n)(m)} \mid m < |f(\alpha,n)|\rangle^\bullet\rangle^\bullet \;\middle|\; \langle\alpha, n\rangle \in \operatorname{dom} f\right\}^\bullet.$$

Therefore, if $\pi \in \mathscr{G}$, the action $\pi\dot{t}_f$ is given by considering the action of π^* on the elements of $\omega^{<\omega}$, as subsets of $\omega \times \omega$.

Theorem 17.10. *Suppose that in M, $t \Vdash^{\mathsf{HS}} \dot{X} \subseteq \check{\eta}$ for some ordinal η. Then there is some $Y \in M$ and $t' \leq t$ such that $t' \Vdash \dot{X} = \check{Y}$.*

Proof. First, let us argue that in M we can assume that $\operatorname{fix}(\operatorname{supp} t) \subseteq \operatorname{sym}(\dot{X})$, since by homogeneity, if $t' \leq t$ and $t' \Vdash \check{\xi} \in \dot{X}$, then we can eliminate the information outside $\operatorname{supp} t$ from t'. To complete the argument, we will have to finish the argument working in V.

Let $[\dot{X}]$ denote a name in HS for $\dot{X}$, and let $\operatorname{fix}(n)$ be a such that $\operatorname{fix}(n) \subseteq \operatorname{sym}([\dot{X}])$. Suppose that $p \Vdash^{\mathsf{HS}}$ "$\dot{t}_f \Vdash_{\mathbb{Q}} \check{\xi} \in [\dot{X}]$". Without loss of generality we may assume that $\operatorname{dom} f = E_1$ is such that $p \Vdash \operatorname{fix}(E_1) = \operatorname{sym}([\dot{X}])$.

Say that f is a *short function* if the condition it defines has the property that it cannot be weakened without shrinking its support, and f is *k-short* if the condition holds above the kth level of the tree. Let $g \subseteq f$ be n-short with $\operatorname{dom} f = \operatorname{dom} g$, where n is as above,[44] then we claim that $p \Vdash^{\mathsf{HS}}$ "$\dot{t}_g \Vdash_{\mathbb{Q}} \check{\xi} \in [\dot{X}]$".

To see that, if h is such that $\dot{t}_h \leq \dot{t}_g$ with $\operatorname{dom} h = \operatorname{dom} g$, then for any k where h add choices to level k, these must be pairwise different. Therefore, we can find a permutation of ω, π^*_k which maps these values to the values of f at that level. And therefore we can find some $\pi^* \in \operatorname{fix}(n)$ for which $\pi^* p$ is compatible with p and $\pi^*\dot{t}_h$ is compatible with $\dot{t}_f$. So p cannot force that an extension of $\dot{t}_g$ disagrees with $\dot{t}_f$ on the truth of $\check{\xi} \in [\dot{X}]$.

Therefore, any n-short condition whose support is E_1 must decide the entire content of $[\dot{X}]$, which is what we wanted. □

[44]Note that it is not necessary for g to be only defined up to n on its various coordinates!

17.3 Second example: Cohen model for X and Monro's models

Let X be any set such that $\omega \not< |X|$,[45] in some model of ZF, we define the Cohen model for X by repeating the construction of the Cohen model over X. Namely, $\mathbb{P}$ is $\mathrm{Add}(X, \omega)$, whose conditions are finite $p\colon \omega \times X \to 2$ ordered by reverse inclusion; $\mathscr{G}$ is S_ω^{fin}, acting on $\mathbb{P}$ by $\pi p(\pi n, x) = p(n, x)$; and $\mathscr{F}$ is the filter generated by $\mathrm{fix}(E)$ for $E \in [\omega]^{<\omega}$. We let $\dot{a}_n = \{\langle p, \check{x}\rangle \mid p(n, x) = 1\}$ and $\dot{A} = \{\dot{a}_n \mid n < \omega\}^\bullet$. It is the standard argument as we have seen it many times by now that $\dot{a}_n, \dot{A}$ are all in HS.

Proposition 17.11. *For any infinite $Y \in V$, $\mathbb{1} \Vdash^{\mathsf{HS}} |\check{Y}| \nleq |\dot{A}| \nleq |\check{Y}|$.*

Proof. Let $\dot{f} \in \mathsf{HS}$ and p be such that $p \Vdash \dot{f}\colon \check{Y} \to \dot{A}$, and let $E \in [\omega]^{<\omega}$ be such that $\mathrm{fix}(E) \subseteq \mathrm{sym}(\dot{f}) \cap \mathrm{fix}(p)$. Let $n \notin E$ and let $q \leq p$ be such that $q \Vdash \dot{f}(\check{y}) = \dot{a}_n$ for some $y \in Y$. We can now find $m \notin E \cup \{n\} \cup \mathrm{supp}\, q$ and consider the permutation π which is the cycle $(n\ m)$, then πq is compatible with q, while also forcing $\dot{f}(\check{y}) = \dot{a}_m$, which is impossible. Therefore p must force that $\mathrm{rng}(\dot{f}) \subseteq \{\dot{a}_n \mid n \in E\}^\bullet$, which is a finite set so $\dot{f}$ cannot be injective.

In the other direction the proof is similar: if $p \Vdash^{\mathsf{HS}} \dot{f}\colon \dot{A} \to \check{Y}$ instead, taking $n \notin E$ and let $q \leq p$ be such that $q \Vdash \dot{f}(\dot{a}_n) = \check{y}$, apply the same argument to obtain that $q \cup \pi q$ force that $\dot{f}$ is not injective.[46] In particular, since any condition extending p has an extension which forces that $\dot{f}$ is not injective, it follows that no extension of p can force that $\dot{f}$ is injective, and therefore p must force that $\dot{f}$ is not injective. □

What this shows us is that by taking a Cohen model for X, we have added a new cardinality to the universe. The Monro iteration is a sequence of models, starting with $V \models \mathsf{ZFC}$.[47] We can set $M_0 = V$ and $A_0 = \omega$; then M_n is the Cohen model for A_n over M_n.

Theorem 17.12. *For any n, M_{n+1} and M_{n+2} have the same n-sets of ordinals, and more generally, going from M_{n+1} to M_{n+2} does not add any subsets of M_n.*

Proof. First, note that the general claim implies the claim about n-sets of ordinals: since ordinals are all in M_0, any 0-sets of ordinals are subsets of M_0, so these can only be added by going into M_1; by induction, M_{n+1} contains all the n-sets of ordinals, etc.

Let us see that M_1 and M_2 have the same sets of ordinals. For that, we let $\mathbb{P} = \mathrm{Add}(\omega, \omega)$ and $\mathbb{Q} = \mathrm{Add}(A_1, \omega)$, along with the group and filter, which are the same for both of these forcings, so we will use $\mathsf{HS}_\mathbb{P}$ and $\mathsf{HS}_\mathbb{Q}$ to differentiate the symmetric names. We will implicitly assume that any name for a condition in $\mathbb{Q}$ is a canonical name. That is, for some condition $f \in \mathrm{Add}(\omega, \omega)$, $\dot{q}_f$ is a canonical name for a condition if it has the form $\{\langle \check{n}, \dot{a}^1_m, \check{\varepsilon}\rangle^\bullet \mid f(n, m) = \varepsilon\}^\bullet$. These, as we have seen in the past, are all in $\mathsf{HS}_\mathbb{P}$. Indeed, if we have a condition $\langle p, \dot{q}\rangle$, we will always assume that p is strong enough to decide that $\dot{q} = \dot{q}_f$ for some f.

Suppose that $X \in M_2$ is a set of ordinals and let $\dot{X}$ be a $\mathbb{P} * \dot{\mathbb{Q}}$-name for X. Since $X \in M_2$, we may assume, as we did in the Morris' model, that $[\dot{X}]$, the $\mathbb{P}$-name projection of $\dot{X}$, is in $\mathsf{HS}_\mathbb{P}$ and $\mathbb{1} \Vdash^{\mathsf{HS}}_\mathbb{P} [\dot{X}] \in \mathsf{HS}^\bullet_\mathbb{Q}$.

Suppose that $\langle p, \dot{q}\rangle \Vdash \check{\xi} \in \dot{X}$, let us moreover assume, without loss of generality that p has decided the set $E_1 \in [\omega]^{<\omega}$ such that $p \Vdash^{\mathsf{HS}}_\mathbb{P} \mathrm{fix}(\check{E}_1) \subseteq \mathrm{sym}([\dot{X}])$.

By the usual homogeneity arguments, we may assume that $\mathrm{supp}\, q = E_1$ as well, and that $\mathrm{supp}\, p = E_0$ is such that $E_1 \subseteq E_0$ and $\mathrm{fix}(E_0) \subseteq \mathrm{sym}([\dot{X}])$ as a $\mathbb{P}$-name in M_0. For readability, let us denote by $E_0^\bullet$ the name $\{\dot{a}^1_n \mid n \in E_0\}^\bullet$.

[45] We can actually define this for arbitrary X, it is just more complicated and we will not need it here.
[46] Note that there is no problem for $\dot{f}$ to be surjective!
[47] Or, as was tradition, $V = L$.

Claim. $p \Vdash^{\mathsf{HS}}_{\mathbb{P}}$ "$\dot{q} \upharpoonright E_1 \times E_0^\bullet \Vdash_{\mathbb{Q}} \check{\xi} \in [\dot{X}]$".

The claim is not meaningless, since it may very well be that the domain of $\dot{q}$ is some $E_1 \times F^\bullet$, where F is much larger than E_0.

Proof of Claim. Suppose that $\dot{q}'$ is a name for a condition such that $p \Vdash_{\mathbb{P}} \dot{q}' \leq_{\mathbb{Q}} \dot{q} \upharpoonright E_1 \times E_0^\bullet$. As we already know that E_1 is the support of q, we may assume that $p \Vdash^{\mathsf{HS}}_{\mathbb{P}} \operatorname{supp} \dot{q}' = \check{E}_1$, as outside that domain the conditions are compatible and there is no influence on the statement $\check{\xi} \in [\dot{X}]$. So we can write $\operatorname{dom} q'$ as $E_1 \times E_0'$.

Next, we can find some $\pi \in \operatorname{fix}(E_0)$ such that $E_1 \times \pi``(E_0' \setminus E_0)^\bullet \cap \operatorname{dom} \dot{q} = \varnothing$. This means that $\pi p = p$ and $\pi \dot{q}'$ is compatible with $\dot{q}$. In particular, p cannot force that $\dot{q}' \Vdash \check{\xi} \notin [\dot{X}]$. This, by the usual forcing arguments, implies that p in fact forces that any extension of $\dot{q} \upharpoonright E_1 \times E_0^\bullet$ must agree with $\dot{q}$ on the truth of $\xi \in X$. □

Using the claim we get that in M_1, once a condition contains enough information, any $\dot{X} \in \mathsf{HS}_{\mathbb{Q}}$ for a set of ordinals will be fully determined. In particular, it will be in a name for a set in M_1. Analysing this proof shows that all we used was the fact that ξ was an element of M_0. Therefore, for a general M_n, if $m \in M_n$, we can repeat the same argument using $\mathbb{P} = \operatorname{Add}(A_n, \omega)$ and $\mathbb{Q} = \operatorname{Add}(A_{n+1}, \omega)$, computed in M_{n+1} to obtain the general result. □

Corollary 17.13. *For all $k > n$, $M_k \models \neg\mathsf{KWP}_n$.*

Proof. Since M_k and M_{n+1} have the same n-sets of ordinals, if KWP_n was true in one of them, by Theorem 14.20 we would have that $M_k = M_{n+1}$, but this is clearly false if $k > n+1$. □

Fact 17.14. *M_n is defined as $V(A_n)$, and it is a model of KWP_n.*

This approach can be extended transfinitely, although the limit steps are not as simple to deal with, to construct models where $\mathsf{KWP}_{\alpha+1}$ holds and KWP_α fails for all $\alpha < \omega_1$. The thing to note here is that for any countable length, the iteration will be isomorphic, externally of course, to $\operatorname{Add}(\omega, \alpha) \cong \operatorname{Add}(\omega, 1)$.

17.4 Some words on "upwards homogeneity"

In pretty much all the examples we have seen of forcing over symmetric extensions, to an extent, we had some very nice and canonical names for conditions, and we can modify these using the permutations of the first. In other words, if we have $\langle p, \dot{q} \rangle$, we could normally find some weak $\dot{q}_*$ such that any two extensions of $\langle p, \dot{q}_* \rangle$ can be made compatible using automorphisms of $\mathbb{P}$ which fix p.

Let us attempt at the following definition.

Definition 17.15. We say that $\mathbb{P} * \dot{\mathbb{Q}}$ is *upwards homogeneous* if for any $\langle p, \dot{q} \rangle$ and $\langle p, \dot{q}' \rangle$ there is some $\pi \in \operatorname{Aut}(\mathbb{P})$ such that $\pi p = p$ and $\langle p, \dot{q} \rangle$ is compatible with $\langle p, \pi \dot{q}' \rangle$. If $\mathbb{P}$ is part of a symmetric system $\langle \mathbb{P}, \mathscr{G}, \mathscr{F} \rangle$, we will require that $\pi \in \mathscr{G}$.

We are ignoring the fact that we need $\pi \dot{\mathbb{Q}} = \dot{\mathbb{Q}}$, but we will discuss this in the following section. Let us ignore that for now. Ideally, looking at the Morris and Monro iterations, we would like to argue that if $\mathbb{P} * \dot{\mathbb{Q}}$ is upwards homogeneous, to an extent, any subset of V is added during the $\mathbb{P}$-step of the iteration.

This seems like this should work, but it does not. First and foremost, it is easy to show that without their symmetries, the Morris and Monro iterations will certainly add Cohen reals.[48]

Secondly, we can show that in the majority of the cases we have seen so far,[49] we cannot have $\dot{q}'$ be any condition, but rather there will be some weakening of $\dot{q}$ which depends on p, and $\dot{q}$ and $\dot{q}'$ must both extend that weaker condition.

Finally, going through the proofs that no new sets of ordinals are added, which are generally the same as "no ground model sets are added by $\mathbb{Q}$" in the more general context,[50] we see that we actually need even more strength. Namely, we need to preserve some name $[\dot{X}]$ as well. So really we need to have some sufficiently well-behaved filter of subgroups, or, in other words, given any H in the filter and any condition p_0, there is some $p \leq_{\mathbb{P}} p_0$ such that in $\operatorname{fix}(p) \cap H$ we can find an automorphism making $\dot{q}$ and $\dot{q}'$ compatible (assuming they agree on some condition that depends on p and H).

This is all quite terrible. But it turns out that a lot of the natural constructions will satisfy these conditions, at the very least up to a point. This is something that is very clear in the Monro iteration, if our conditions has a sense of "support" and "content" (this would be the information a condition carries on each new subset in its support, for example), the $\mathbb{Q}$-side permutations restrict the support to be manageable, whereas the $\mathbb{P}$-side permutations allow us to restrict the "content" in a relatively uniform manner. So, if we combine these two properties we get that any symmetric name for a set of ordinals, or a subset of V in general, will be determined by something that is bounded in both "support" and "content" and therefore can be fully determined by a single condition.

Nevertheless, a full of a general definition of upwards homogeneity is much harder to get right (if we want to have a ground model preservation theorem[51]), even in some considerable generality, without it looking very artificial.

17.5 The two-step iterations

Let us now explore some of the basics of approaching a general two-step iteration of symmetric extensions. In this scenario, we have a symmetric system, $\langle \mathbb{Q}_0, \mathscr{G}_0, \mathscr{F}_0 \rangle$ and a name $\langle \dot{\mathbb{Q}}_1, \dot{\mathscr{G}}_1, \dot{\mathscr{F}}_1 \rangle^\bullet \in \mathsf{HS}_{\mathscr{F}_0}$. We want to explore how to find the correct class of $\mathbb{Q}_0 * \dot{\mathbb{Q}}_1$-names which "predicts" the iterated symmetric extension. Clearly, these are exactly the names which have a "projection" to $\mathbb{Q}_0$ which is symmetric and is forced to be symmetric.

As we are clearly looking at $\mathbb{Q}_0 * \dot{\mathbb{Q}}_1$, the first step is to try and to understand how to turn $\dot{\mathscr{G}}_1$ and $\mathscr{G}_0$ into automorphisms of the iteration. Let us define a concept that had been hidden under the radar through this entire part of the notes.

Definition 17.16. Suppose that $\mathbb{P}$ is a forcing notion, $\pi \in \operatorname{Aut}(\mathbb{P})$, and $\dot{A}$ is a name. We say that π *respects* $\dot{A}$ if $\mathbb{1} \Vdash \pi\dot{A} = \dot{A}$. If $\dot{A}$ carries an implicit structure, we require that structure to be preserved as well. If $\dot{A}$ is respected by all the automorphisms in a group $\mathscr{G}$ we will say that $\mathscr{G}$ respects $\dot{A}$.

Exercise 17.17. $\{\pi \in \operatorname{Aut}(\mathbb{P}) \mid \pi \text{ respects } \dot{A}\}$ is a group.

Proposition 17.18. *If $\pi \in \operatorname{Aut}(\mathbb{Q}_0)$, then $\langle q_0, \dot{q}_1 \rangle \mapsto \langle \pi q_0, \pi \dot{q}_1 \rangle$ is an automorphism of $\mathbb{Q}_0 * \dot{\mathbb{Q}}_1$ if and only if π respects $\dot{\mathbb{Q}}_1$.* □

[48] E.g., $\{n \mid x \in a_n\}$ in the Morris case, once we no longer are able to enumerate the new subsets this problem "goes away".

[49] The Bristol model is the not-so-obvious exception.

[50] Although this may not always be the case!

[51] We do, just to make it absolutely clear.

So the first and foremost requirement is that $\mathscr{G}_0$ respects $\langle\dot{\mathbb{Q}}_1, \dot{\mathscr{G}}_1, \dot{\mathscr{F}}_1\rangle^\bullet$. Let us assume that this condition holds, then.

Proposition 17.19. *Suppose* $\mathbb{1} \Vdash_{\mathbb{Q}_0} \dot\sigma \in \operatorname{Aut}(\dot{\mathbb{Q}}_1)$*, then* $\langle q_0, \dot q_1\rangle \mapsto \langle q_0, \dot\sigma \dot q_1\rangle$ *is an automorphism of* $\mathbb{Q}_0 * \dot{\mathbb{Q}}_1$. □

Definition 17.20. The *generic semidirect product* is the group $\mathscr{G}_0 * \dot{\mathscr{G}}_1$ which is generated by $\mathscr{G}_0$ and $\dot{\mathscr{G}}_1$ in the aforementioned way. Given $\langle\pi, \dot\sigma\rangle$ such that $\pi \in \mathscr{G}_0$ and $\mathbb{1} \Vdash^{\mathsf{HS}}_{\mathbb{Q}_0} \dot\sigma \in \dot{\mathscr{G}}_1$, the action is given by $\langle\pi, \dot\sigma\rangle(p, \dot q) = \langle\pi p, \pi(\dot\sigma \dot q)\rangle = \langle\pi p, \pi\dot\sigma\pi\dot q\rangle$.

Note that we required that $\dot\sigma \in \dot{\mathscr{G}}_1$ is a symmetric statement, which means that $\dot\sigma \in \mathscr{G}_1$. This is not a trivial requirement. However, on a dense set, it becomes very manageable. If $\dot\sigma \in \mathsf{HS}$ and $p \Vdash \dot\sigma \in \dot{\mathscr{G}}_1$, then we can define a name $\dot\sigma_* \in \mathsf{HS}$ given by taking $\dot\sigma$ below p and $\mathrm{id}^\bullet$ on any incompatible condition. If p is tenacious, $\operatorname{fix}(p) \cap \operatorname{sym}(\dot\sigma)$ witnesses that $\dot\sigma_* \in \mathsf{HS}$,[52] which allows us enough flexibility for the argument to work, at least when iterating finitely many steps.[53]

Exercise 17.21. Verify that $\mathscr{G}_0 * \dot{\mathscr{G}}_1$ is indeed an automorphism group of $\mathbb{Q}_0 * \dot{\mathbb{Q}}_1$ and show that $\langle\pi_1, \dot\sigma_1\rangle \circ \langle\pi_0, \dot\sigma_0\rangle = \langle\pi_1\pi_0, \pi_0^{-1}(\dot\sigma_1)\dot\sigma_0\rangle$ and $\langle\pi, \dot\sigma\rangle^{-1} = \langle\pi^{-1}, \pi(\dot\sigma^{-1})\rangle$.

Exercise 17.22. Assume that $\langle\mathbb{Q}_1, \mathscr{G}_1, \mathscr{F}_1\rangle$ is in the ground model, show that $\mathscr{G}_0 * \check{\mathscr{G}}_1 \cong \mathscr{G}_0 \times \mathscr{G}_1$.

Exercise 17.23. Suppose that $\mathscr{G}_0$ witnesses the homogeneity of $\mathbb{Q}_0$ and $\mathbb{1} \Vdash_{\mathbb{Q}_0}$ "$\dot{\mathscr{G}}_1$ witnesses the homogeneity of $\dot{\mathbb{Q}}_1$", then $\mathscr{G}_0 * \dot{\mathscr{G}}_1$ witnesses the homogeneity of $\mathbb{Q}_0 * \dot{\mathbb{Q}}_1$.

This definition extends naturally to the general case, by simply iterating the construction and taking a suitable notion of a limit relative to the limit steps of the iteration. This may require us to be slightly careful in how we define this, but we will not deal with the general iteration case (even the finite support one) so we can just move on. The next step, of course, is to identify a filter on $\mathscr{G}_0 * \dot{\mathscr{G}}_1$.

In the case of simply two steps, we can significantly simplify much of the problem.

Definition 17.24. We write $\mathscr{F}_0 * \dot{\mathscr{F}}_1$ to denote the filter generated by subgroups of the form $H_0 * \dot H_1$ where $H_0 \in \mathscr{F}_0$, $\mathbb{1} \Vdash^{\mathsf{HS}}_{\mathbb{Q}_0} \dot H_1 \in \dot{\mathscr{F}}_1$, and H_0 respects $\dot H_1$.

The same "trick" that we have used for the automorphisms will work here as well to guarantee "enough" names are in HS which are guaranteed to be in $\dot{\mathscr{F}}_1$.

Proposition 17.25. *$\mathscr{F}_0 * \dot{\mathscr{F}}_1$ is a normal filter of subgroups.*

Proof. Let $\langle\pi, \dot\sigma\rangle \in \mathscr{G}_0 * \dot{\mathscr{G}}_1$ and let $\langle\tau, \dot\varphi\rangle \in H_0 * \dot H_1$. We can compute that

$$\langle\pi, \dot\sigma\rangle \circ \langle\tau, \dot\varphi\rangle \circ \langle\pi, \dot\sigma\rangle^{-1} = \langle\pi\tau\pi^{-1}, \pi((\tau^{-1}\dot\sigma)\dot\varphi\dot\sigma^{-1})\rangle.$$

In particular, if τ respects $\dot\sigma$, then we can replace $\tau^{-1}\dot\sigma$ by $\dot\sigma$ and obtain that the conjugation is $\langle\pi\tau\pi^{-1}, \pi(\dot\sigma\dot\varphi\dot\sigma^{-1})\rangle$. In particular, in that case, the conjugation is $\pi H_0\pi^{-1} * \pi(\dot\sigma\dot H_1\dot\sigma^{-1})$.

But since $\dot\sigma \in \mathsf{HS}$ we can simply shrink H_0 to $H_0' = H_0 \cap \operatorname{sym}(\dot\sigma)$, which means that the conjugation contains the group $\pi H_0'\pi^{-1} * \pi(\dot\sigma\dot H_1\dot\sigma^{-1})$, as wanted. □

Theorem 17.26. *Let $G_0 * G_1$ be V-generic for $\mathbb{Q}_0 * \dot{\mathbb{Q}}_1$, then $\mathsf{HS}^{G_0 * G_1}_{\mathscr{F}_0 * \dot{\mathscr{F}}_1}$ is the same model as the symmetric extension of $\mathsf{HS}^{G_0}_{\mathscr{F}_0}$ by $\langle\dot{\mathbb{Q}}_1^{G_0}, \dot{\mathscr{G}}_1^{G_0}, \dot{\mathscr{F}}_1^{G_0}\rangle$, using G_1 as the generic filter.* □

[52] And if it is not tenacious we can simply take $\dot\sigma$ on the orbit of p under $\operatorname{sym}(\dot\sigma)$.
[53] And with some work, also finite support iterations in general.