Lecture Notes: Large Cardinals

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Prologue

What are large cardinals? Well, the concept is not a mathematical one, and we can even prove that. But for the most part, these are axioms that we can add to the standard axioms of set theory (ZFC or ZF for the most part) and study their impact.

One of the general understanding is that large cardinal axioms will prove the consistency of set theory in a strong sense: not only that models can be produced, transitive set-sized models can be produced. For the most part, we get that if κ is the least large cardinal of type X, then in V_κ there will be a proper class, or at the very least one, large cardinals of type Y.[1]

Many of the properties we study begin by looking at ω and asking "what happens if we add *uncountable* to this property?", as such these axioms are often called "strong axioms of infinity", as they can be seen as a natural strengthening of the Axiom of Infinity, positing the existence of an uncountable cardinal.

So, what are these large cardinal axioms good for? We use them to study how strong a mathematical statement might be. For example, "every projective set is Lebesgue measurable" is equiconsistent with "there exists an inaccessible cardinal", and "there is no Aronsazjn tree on ω_2 and ω_3 at the same time" follows from the existence of a supercompact cardinal and a weakly compact cardinal above it, and it implies the consistency of a Woodin cardinal.

We will, in the following few weeks, climb up the hierarchy of large cardinal axioms, from the small to the large, to the huge. This prologue will evolve and change by the end of the course to reflect more accurately the content of these notes.

[1]Some counterexamples, of course, are inevitable, and not just contrived ones. $0^\#$ does not imply the existence of an inaccessible cardinal, but it implies a proper class of inaccessible cardinals exists in L; the least huge cardinal is below the least supercompact, but it implies the consistency of many supercompact cardinals.

Chapter 1

Accessing the inaccessible

1.1 The inaccessible world

Definition 1.1. We say that κ is a *strong limit cardinal* if whenever $\alpha < \kappa$, $2^\alpha < \kappa$.

Definition 1.2. We say that κ is an *inaccessible cardinal* if it is a regular strong limit cardinal.

Exercise 1.3. The following are equivalent:

1. κ is inaccessible.
2. κ is a limit cardinal and $\kappa^{<\kappa} = \kappa$.

Exercise 1.4. Prove that $\kappa = 2^{<\kappa}$ does not imply inaccessibility even in conjunction with any one of these property:

1. κ is a limit cardinal.
2. κ is regular.

Exercise 1.5. ω is an inaccessible cardinal.

From this point onwards, we will always assume that an inaccessible cardinal is uncountable.

Theorem 1.6. *The following are equivalent:*

1. *κ is inaccessible.*
2. *For every $x \in V_\kappa$, if $f\colon x \to \kappa$, then* $\sup \operatorname{rng} f < \kappa$.
3. *For every $\alpha < \kappa$, if $f\colon 2^\alpha \to \kappa$, then* $\sup \operatorname{rng} f < \kappa$.

Proof. (2) $\to$ (3): Trivial.

(3) $\to$ (1): To see that κ is regular, note that otherwise there is some $\alpha < \kappa$ and $f\colon \alpha \to \kappa$ with $\sup \operatorname{rng} f = \kappa$, this f extends trivially to 2^α. If κ is not a strong limit, let α be the least such that $\kappa \leq 2^\alpha$, then there is an injection from κ into 2^α, which can be reversed and therefore there is $f\colon 2^\alpha \to \kappa$ which is onto, so $\sup \operatorname{rng} f = \kappa$.

(1) $\to$ (2): It is enough to check that for $x = V_\alpha$ for some $\alpha < \kappa$, since if $x \in V_\kappa$ there is some $\alpha < \kappa$ such that $x \in V_\alpha$. For this it is enough to show that if $\alpha < \kappa$, $|V_\alpha| < \kappa$, since κ is

a regular cardinal. We prove this by induction on α, suppose that this holds for all $\beta < \alpha$. If $\alpha = \beta + 1$, then $|V_\beta| = \lambda < \kappa$, so $|V_\alpha| = |V_{\beta+1}| = |\mathcal{P}(V_\beta)| = 2^\lambda < \kappa$. If α is a limit, then for all $\beta < \alpha$, $|V_\beta| = \lambda_\beta < \kappa$, and sine $\alpha < \kappa$, $|V_\alpha| = |\bigcup\{V_\beta \mid \beta < \alpha\}| = \sup\{\lambda_\beta \mid \beta < \alpha\} < \kappa$. □

Exercise 1.7. If κ is inaccessible, then $\kappa = \aleph_\kappa = \beth_\kappa$.

Theorem 1.8. *If κ is inaccessible, then $V_\kappa \models \mathsf{ZFC}$.*

Proof. Since V_κ is transitive it satisfies the axioms of Extensionality and Foundation. Since κ is uncountable, Infinity holds as well. Since κ is a limit ordinal, Power Set, Union, and Choice hold. Finally, let $x \in V_\kappa$ and φ such that $V_\kappa \models \varphi(u, v)$ defines a function on x. Defining $f(u) = \operatorname{rank}(v)$, such that $\varphi(u, v)$ holds, is a function from x to κ, so by (2) in Theorem 1.6 it must be bounded, so $\{v \mid \exists u \in x\varphi(u, v)\} \in V_\kappa$. □

The proof actually shows that $V_\kappa \models \mathsf{ZFC}_2$, where Replacement is a single second-order axiom.

Exercise 1.9. If $V_\kappa \models \mathsf{ZFC}_2$, then κ is inaccessible.

If we want to keep going on and grow larger and stronger, then, perhaps the first natural step would be this.

Definition 1.10. A cardinal κ is 2-*inaccessible* if it is an inaccessible cardinal which is the limit of inaccessible cardinals. More generally, for $\alpha > 0$, κ is α-*inaccessible* if it is inaccessible and the set of β-inaccessible cardinals below it is unbounded in κ for all $\beta < \alpha$.

Remark. The above condition just ignores 0-inaccessible cardinals, and we will often just define those to be the regular cardinals, or the strong limit cardinals, in whatever way suits our needs.

Exercise 1.11. Suppose that κ is 2-inaccessible, then in V_κ there is a proper class of inaccessible cardinals.

However, this is not a particularly satisfying way of increasing the strength of our large cardinal axioms, it is overly tedious and technical, and we want a stronger axiom to be more than just "the limit of weaker cardinals", but indeed to embody that limit in a strong sense. For this, let us look downwards for a moment.

Theorem 1.12. *Let $R \subseteq V_\kappa$, then there is $\alpha < \kappa$ such that $\langle V_\alpha, \in, R \cap V_\alpha\rangle \prec \langle V_\kappa, \in, R\rangle$.*

Proof. We define by recursion a sequence, $\alpha_0 = 0$, and let M_0 be the elementary substructure generated by V_{α_0}. Let $\alpha_{n+1} = \operatorname{rank}(M_0)$, and let M_{n+1} be the elementary substructure generated by V_{α_n}. We let $\alpha = \sup \alpha_n$, then $V_\alpha = \bigcup V_{\alpha_n} = \bigcup M_n$, and therefore $V_\alpha \prec V_\kappa$. □

Remark. We can ask why fail for V_{ω_1}, for example. The reason is that if $V_\kappa \not\models \mathsf{ZFC}$, then there is a formula $\varphi(u, v)$ and $x \in V_\kappa$ such that $V_\kappa \models (\forall u \in x)\exists! v\varphi(u, v) \land \forall y \exists v(v \notin y \land (\exists u \in x)\varphi(u, v)$. This means that any elementary submodel of V_κ will invariably have rank κ (since the existence of such an x is expressible in first-order logic).

We will say that α is a *reflection point* of R if $\langle V_\alpha, \in, R \cap V_\alpha\rangle \prec \langle V_\kappa, \in, R\rangle$.

Exercise 1.13. Let κ be an inaccessible cardinal. Show that for any $R \subseteq V_\kappa$, the set of reflection points of R is a club in κ.

Corollary 1.14. *If κ is inaccessible, then there is some $\alpha < \kappa$ such that $V_\alpha \models \mathsf{ZFC}$. In particular, if α is the least such that $V_\alpha \models \mathsf{ZFC}$, then α is singular.*

Proof. The first part is trivial, for the second part, if α is inaccessible, then it is not the least cardinal for which ZFC holds in V_α. Therefore, it is either singular or not a strong limit. However, if α is not a strong limit, then there is some $\beta < \alpha$ such that $|\mathcal{P}(\beta)| \geq \alpha$, and in that case V_α does not satisfy "Every well-ordered set is bijective with an ordinal", which is a theorem of ZF. $\square$

Definition 1.15. We say that κ is a *worldly* cardinal if $V_\alpha \models \mathsf{ZFC}$.

Exercise 1.16. If κ is a worldly cardinal and $\mathrm{cf}(\kappa) > \omega$, then $V_\kappa \models$"There is a proper class of worldly cardinals". Consequently, the ω_1th worldly cardinal has cofinality ω.

Exercise 1.17. If $\alpha < \beta$ and $V_\alpha \prec V_\beta$, then α is a worldly cardinal that is the limit of worldly cardinals.

Theorem 1.18. *Suppose that κ is a cardinal such that whenever $R \subseteq V_\kappa$, R reflects at some $\alpha < \kappa$, then κ is an inaccessible cardinal.*

Proof. We use (2) from Theorem 1.6. Suppose that $x \in V_\kappa$ and $f\colon x \to \kappa$, then $f \subseteq V_\kappa$, so there is some α such that $\langle V_\alpha, \in, f \cap V_\alpha\rangle \prec \langle V_\kappa, \in, f\rangle$. Since $\langle V_\kappa, \in, f\rangle \models$ "$\operatorname{dom} f$ is a set", this holds at the reflection point. Let $x' \in V_\alpha$ be $\operatorname{dom} f \cap V_\alpha$, then by elementarity it must be that $V_\kappa \models x' = \operatorname{dom} f$, and therefore $x = x'$. Therefore $f \cap V_\alpha = f$, so $\sup \operatorname{rng} f < \alpha < \kappa$. $\square$

From this we can first of all define a hierarchy of worldly cardinals by judging at what point they reveal to be singular. Namely, how simple of a set R can we find such that $\langle V_\kappa, \in, R\rangle$ does not reflect? But we can also create strong and strong extensions of inaccessibility, by requiring not only that we are the limit of inaccessible cardinals, but also that we reflect at inaccessible cardinals.

1.2 And yet it does not move

Definition 1.19. We say that κ is a *Mahlo cardinal* if it is a strong limit cardinal such that $\{\alpha < \kappa \mid \alpha = \mathrm{cf}(\alpha)\}$ is a stationary set.[2]

Proposition 1.20. *If κ is Mahlo, then κ is regular.*

Proof. Suppose that κ is singular, let $A \subseteq \kappa$ be a club of minimal order type with $\min A > \mathrm{cf}(\kappa)$, then $\mathrm{acc}(A)$ is also a club of minimal order type, and every $\lambda \in \mathrm{acc}(A)$ is singular, as it is the limit of a sequence shorter than $\mathrm{cf}(\kappa)$. Therefore $\mathrm{acc}(A) \cap \{\alpha < \kappa \mid \alpha = \mathrm{cf}(\alpha)\} = \varnothing$. $\square$

Theorem 1.21. *For a strong limit cardinal, κ, the follow are equivalent:*

1. *κ is Mahlo.*
2. *$\{\alpha < \kappa \mid \alpha$ is inaccessible$\}$ is stationary.*
3. *For every $R \subseteq V_\kappa$, there is an inaccessible reflection point of R.*
4. *For every $R \subseteq V_\kappa$, there is a regular reflection point of R.*

[2]Implicit in this is that the cofinality of κ is uncountable.

Proof. (1) → (2): $C = \{\alpha < \kappa \mid \alpha \text{ is a strong limit cardinal}\}$ is a club, and therefore

$$C \cap \{\alpha < \kappa \mid \alpha = \mathrm{cf}(\alpha)\} = \{\alpha < \kappa \mid \alpha \text{ is inaccessible}\}$$

is stationary in κ.

(2) → (3): Since κ is inaccessible, given any $R \subseteq V_\kappa$, there is a club of reflection points of R, so there is an inaccessible cardinal reflecting R.

(3) → (4): Trivial.

(4) → (1): Let C be a club, then there is some regular α which reflects C, but since C is unbounded in κ, $C \cap V_\alpha = C \cap \alpha$ is unbounded in α. Since C is closed, $\alpha \in C$. Therefore, there is a regular cardinal in every club, so $\{\alpha < \kappa \mid \alpha = \mathrm{cf}(\alpha)\}$ is stationary, so κ is Mahlo. □

Exercise 1.22. If κ is a Mahlo cardinal, then κ is κ-inaccessible. (Hint: $\mathrm{acc}^\alpha(I)$, where I is the set of inaccessible cardinals is a club.)

Exercise 1.23. If κ is the least such that κ is κ-inaccessible, then κ is not Mahlo.

This allows us to define the iterated Mahlo-hierarchy.

Definition 1.24. We say that a strong limit cardinal κ is an *α-Mahlo cardinal* if

$$\{\xi < \kappa \mid \xi \text{ is a } \beta\text{-Mahlo cardinal}\}$$

is stationary in κ for all $\beta < \alpha$. If κ is κ-Mahlo say that it is a *hyper-Mahlo*.

So, κ is a 0-Mahlo cardinal if it is inaccessible. Next, 1-Mahlo implies the set of inaccessible cardinals is stationary, and 2-Mahlo cardinals are those where the accumulation points of this set reflect at a regular cardinal. So, α-Mahlo is one where every $R \subseteq V_\kappa$ reflects at a cardinal whose Mahlo-ness is as close to α as we want it to be. How can we go beyond a hyper-Mahlo, then?

The easy solution is to apply diagonal intersections to the stationary sets. While the diagonal intersection of κ-many clubs is still a club, there is no guarantees for stationary sets. If, however, we are lucky enough, the result may still be stationary. Moreover, up to a non-stationary set this diagonal intersection will be unique.[3]

We can formalise this using the *Mahlo operation* which is

$$M(X) = \{\alpha < \kappa \mid \alpha \cap X \text{ is stationary in } \alpha\}.^4$$

It is not hard to check that if $X \equiv_{\mathrm{NS}} Y$, then $M(X) \equiv_{\mathrm{NS}} M(Y)$. We can therefore iterate M to define $M^\alpha(X)$, but this can be extended beyond κ itself by using diagonal intersections.[5]

And so, a strong limit cardinal, κ, is α-Mahlo if $M^\beta(\{\lambda < \kappa \mid \lambda = \mathrm{cf}(\lambda)\})$ is stationary for all $\beta < \alpha$, and it is *greatly Mahlo* if it is κ^+-Mahlo.

Exercise 1.25. κ is greatly Mahlo if and only if there exists a normal κ-complete filter on $\mathcal{P}(\kappa)$ concentrating on the set of regular cardinals and closed under the Mahlo operation.

[3] Recall that the diagonal intersection is the magical infimum in $\mathcal{P}(\kappa)/\mathrm{NS}_\kappa$.
[4] Again, implicit here is that these ordinals are all of uncountable cofinality.
[5] This is why it is important that the Mahlo operation is invariant up to a non-stationary set.

1.3 "Weakly" large cardinal axioms

Sometimes we want to dispense with the strong limit notions. We can certainly ask about regular limit cardinals, which may or may not be strong limit cardinals. Similarly, we can certainly ask that $\{\alpha < \kappa \mid \alpha = \mathrm{cf}(\alpha)\}$ is stationary without the strong limit requirement.

This leads us to two common large cardinal notions.

Definition 1.26. We say that κ is a *weakly inaccessible cardinal* if it is a regular limit cardinal. We say that κ is a *weakly Mahlo cardinal* if it $\{\alpha < \kappa \mid \alpha = \mathrm{cf}(\alpha)\}$ is stationary in κ. Both of these extend to weak α-inaccessible and weakly α-Mahlo.

Theorem 1.27. *If κ is weakly inaccessible, then κ is inaccessible in L. Therefore the consistency strength of weak and strong inaccessible cardinals is the same.*

Proof. Being a regular limit cardinal is a Π_1 formula, and so it is downwards absolute. Since $L \models \mathsf{GCH}$, it means that κ is a strong limit as well. $\square$

Theorem 1.28. *If κ is weakly Mahlo, then κ is Mahlo in L. Therefore the consistency strength of weak and strong Mahlo-ness is the same.*

Proof. Let $S = \{\alpha < \kappa \mid \alpha = \mathrm{cf}(\alpha)\}^L$, since every regular cardinal in V is regular in L, $\{\alpha < \kappa \mid \alpha = \mathrm{cf}(\alpha)\} \subseteq S$, and therefore S is stationary in V. Given any $C \in L$ which is a club in κ, C is a club in V, so $C \cap S$ is non-empty, and therefore $L \models S$ is stationary, so in L we have that κ is weakly Mahlo, as GCH holds, κ is Mahlo. $\square$

1.4 Inaccessibility for reals

Definition 1.29. We say that ω_1 is *inaccessible to reals* if $\omega_1^{L[x]} < \omega_1$ for all $x \subseteq \omega$.

Exercise 1.30. If ω_1 is inaccessible to reals, then $L \models \omega_1^V$ is a limit cardinal. Consequently, if ω_1 is regular, it is inaccessible in L.

Definition 1.31. We say that $X \subseteq \mathbb{R}$ has the *Perfect Set Property* if it is either countable or contains a closed copy of the Cantor set.

Specker had shown that in ZF, "Every set of reals has the Perfect Set Property" implies that ω_1 is inaccessible to reals. In particular, if $\mathsf{ZF} + \mathsf{DC}$ holds, it must be the case that ω_1 is inaccessible in L. Solovay had later shown that given an inaccessible cardinal, we can use forcing and symmetric extensions to construct a model in which $\mathsf{ZF} + \mathsf{DC}$ holds and every set of reals has the Perfect Set Property. Truss showed that starting with a singular cardinal will also produce a model where every set of reals has the Perfect Set Property, although in that model, DC must fail as ω_1 must be singular.

Chapter 2

Weak compactness and other things we cannot describe

2.1 Compactness

Definition 2.1. $\mathcal{L}_{\kappa,\lambda}$ is the logic obtained by closing first-order logic under fewer than κ disjunctions and quantifying, in a single block, over fewer than λ free variables.[6]

We say that $\mathcal{L}_{\kappa,\lambda}$ is *weakly compact* if whenever T is an $\mathcal{L}_{\kappa,\lambda}$-theory in a language whose size is $\leq \kappa$, and every $T_0 \in [T]^{<\kappa}$ has a model, then T has a model.[7] We also say in this case that T is a *κ-satisfiable* theory.

Exercise 2.2 (*) .

1. If κ is inaccessible and φ is a satisfiable sentence in $\mathcal{L}_{\kappa,\kappa}$, then it has a model of size $< \kappa$.
2. If T is a satisfiable theory of size $\leq \kappa$, then it has a model of size $\leq \kappa$.

Note that for an inaccessible cardinal κ and an $\mathcal{L}_{\kappa,\kappa}$ theory, T, of size at most κ, we can code the Tarskian semantics for T in V_κ in a definable way, and we will make heavy implicit use of this fact throughout the following proofs.

Definition 2.3. We say that κ is a *weakly compact cardinal* if $\mathcal{L}_{\kappa,\kappa}$ is weakly compact.

This is a generalisation of ω, since $\mathcal{L}_{\omega,\omega}$ is just the usual first-order logic and Gödel's compactness theorem tells us that ω is weakly compact.[8] So we will assume that in addition to the weak compactness κ is uncountable.

Theorem 2.4. *If κ is weakly compact, then κ is inaccessible.*

This is a significant jump from the previous examples we have seen, where the notions of inaccessible and Mahlo cardinals have "weakly" version to them. Weak compactness is already strong enough so that it proves inaccessibility.

[6] First-order logic often assumes only countably many free variable symbols, of course we allow as many as needed.

[7] The weakness is the size of the language, and we will discuss the unrestricted version later on when we discuss *strongly* compact cardinals.

[8] And in fact, it tells us much more.

Proof. If κ is singular, take any small unbounded $X \subseteq \kappa$ and let $\{c_\alpha \mid \alpha < \kappa\} \cup \{c\}$ be constant symbols. Consider the theory

$$T = \{c \neq c_\alpha \mid \alpha < \kappa\} \cup \left\{ \bigvee_{\alpha \in X} \bigvee_{\beta < \alpha} c = c_\beta \right\}.$$

The theory is κ-satisfiable, since every $T_0 \in [T]^{<\kappa}$ can only mention a small number of constants, which we can then interpret as some initial segment of κ. But clearly, the theory does not have a model. So, κ must be regular.

To see that κ is a strong limit cardinal, for any $\alpha < \kappa$ and consider the language with constant symbols $\{c_\gamma \mid \gamma < \alpha\} \cup \{d^i_\gamma \mid \gamma < \alpha, i < 2\}$. Let T be the theory

$$\left\{ \bigwedge_{\gamma < \alpha} (c_\gamma = d^0_\gamma \vee c_\gamma = d^1_\gamma \wedge d^0_\gamma \neq d^1_\gamma) \right\} \cup \left\{ \bigvee_{\gamma < \alpha} c_\gamma \neq d^{g(\gamma)}_\gamma \;\middle|\; g \colon \alpha \to 2 \right\}.$$

The theory cannot have a model, since the interpretation of the c_γ defines $c \colon \alpha \to 2$ which is not in 2^α. However, if $2^\alpha \geq \kappa$, then the theory is κ-satisfiable, as for any fewer than κ sentences we can find f which is not mentioned in any of them and define the c_γ according to f. So, κ must be a strong limit as well. $\square$

Remark. If we only require that $|T| \leq \kappa$ in the definition of weak compactness, the above proof only shows that there is not $\alpha < \kappa$ such that $2^\alpha = \kappa$.

Definition 2.5. Let κ be an infinite cardinal, we say that a tree T is a κ-*tree* if it has height κ and every level has size $< \kappa$. We say that $b \subseteq T$ is a *branch* if it is a maximal chain which has elements in every level of the tree.[9] We say that κ has the *tree property* if every κ-tree has a branch.

It T is a κ-tree without a branch we call it a κ-*Aronszajn* tree, and we omit κ in the case of ω_1. The tree property, therefore, says that there are no κ-Aronszajn trees.

Proposition 2.6. *ω has the tree property.*

Proof. Let T be an ω-tree, then T is countable, write it as $\{t_n \mid n < \omega\}$ and without loss of generality if $t < t'$, then the index of t is before the index of t'. Define a branch by recursion: $b_0 = t_0$ is the root of T; if b_n was chosen, let b_{n+1} be t_m where m is minimal such that $\{t \in T \mid t_m < t\}$ is infinite. Since each level is finite and T is infinite, there must be such m. $\square$

Exercise 2.7. If κ is singular, then κ does not have the tree property.

Exercise 2.8 (**) . Suppose that $\kappa^{<\kappa} = \kappa$, then κ^+ does not have the tree property. In particular, ω_1 does not have the tree property.

Definition 2.9. Let M be an algebra of subsets of X, that is a collection of subsets of X closed under finite intersections and complements. We say that $\mathcal{F}$ is an M-*filter* if for every $A, B \in M$, if $A \subseteq B$ and $A \in \mathcal{F}$, $B \in \mathcal{F}$; if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$; and $\varnothing \notin \mathcal{F}$. We say $\mathcal{F}$ is an M-*ultrafilter* if for every $A \in M$, $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$. Finally, we say that $\mathcal{F}$ is κ-complete if for every $\alpha < \kappa$ and $\{M_\xi \mid \xi < \alpha\} \subseteq F$, $\bigcap_{\xi < \alpha} M_\xi \in \mathcal{F}$.

Often we care not about the algebra itself, but rather some structure, M, and the algebra $M \cap \mathcal{P}(X)$. In that case we will write M-ultrafilter to mean $(M \cap \mathcal{P}(X))$-ultrafilter.

[9] Often this would be called a *cofinal* branch instead.

Definition 2.10. We say that κ has the *weak filter property* if whenever $M \subseteq \mathcal{P}(\kappa)$ is an algebra of sets of size κ, then there is a κ-complete free M-ultrafilter.

Exercise 2.11 (*) . If κ has the weak filter property, then κ is inaccessible.

Definition 2.12. We say that κ has the *extension property* if for every $A \subseteq V_\kappa$, there is a transitive set M and some $B \subseteq M$ such that $\langle V_\kappa, \in, A\rangle \prec \langle M, \in, B\rangle$.

Exercise 2.13. If κ has the extension property, then κ is inaccessible.

Definition 2.14. Let M be a transitive set and let $j\colon M \to N$ be an elementary embedding. The *critical point* of j is the least ordinal α such that $\alpha < j(\alpha)$, we denote this ordinal by $\operatorname{crit}(j)$. If there is no critical point, we say that j is trivial.

Exercise 2.15. Suppose that M is a transitive model of ZFC and $j\colon M \to N$ is trivial, then j is the identity.

Definition 2.16. We say that κ is a *weakly critical cardinal* if for every $A \subseteq V_\kappa$ there are transitive sets M and N such that $A, \kappa, V_\kappa \in M \cap N$, and an elementary embedding $j\colon M \to N$ with critical point κ.

Proposition 2.17. *If κ is a weakly critical cardinal, then κ is inaccessible.*

Proof. If κ is not inaccessible, let $f\colon x \to \kappa$ be a function for some $x \in V_\kappa$. Considering f as a subset of V_κ, let M be a transitive set such that $f, \kappa, V_\kappa \in M$ and $j\colon M \to N$ be an elementary embedding with critical point κ. Then $j(f)\colon j(x) \to j(\kappa)$, but since $j \restriction x \cup \{x\} = \mathrm{id}$, $f = j(f)$. So $N \models \sup \operatorname{rng} j(f) < j(\kappa)$, so $M \models \sup \operatorname{rng} f < \kappa$, so κ is inaccessible. □

Theorem 2.18. *Let κ be an uncountable cardinal, then the following are equivalent:*

1. *κ is weakly compact.*
2. *κ is inaccessible and has the tree property.*
3. *κ has the weak filter property.*
4. *κ has the extension property.*
5. *κ is weakly critical.*

Proof. (1) $\to$ (2): By Theorem 2.4 we know that κ is inaccessible. Let T be a κ-tree, and consider the language $\{P_t \mid t \in T\}$ of 0-ary predicate symbols. Consider the theory

$$\left\{ \bigvee_{t \in T_\alpha} P_t \,\middle|\, \alpha < \kappa \right\} \cup \{\neg P_t \wedge P_s \mid t \perp s\}.$$

The theory is κ-consistent, since any small collection has a model in some $T \restriction \alpha$. Let M be a model of the theory, then $\{t \mid M \models P_t\}$ is a branch in T.

(2) $\to$ (3): Suppose that κ is inaccessible and has the tree property and let $M \subseteq \mathcal{P}(\kappa)$ be an algebra of sets of size κ, so we can write it as $\{A_\alpha \mid \alpha < \kappa\}$. We may also assume that M contains all the finite subsets of κ. For $\alpha < \kappa$ and $s \in 2^\alpha$ we define

$$A_s = \bigcap \{A_\xi \mid s(\xi) = 1\} \cup \bigcap \{\kappa \setminus A_\xi \mid s(\xi) = 0\}.$$

Let $T = \{s \in 2^{<\kappa} \mid |A_s| = \kappa\}$, ordered by extension. Note that for each $\alpha < \kappa$, and for each $\gamma < \kappa$, we can find $s \in 2^\alpha$ such that $\gamma \in A_s$, and therefore $\bigcup\{A_s \mid s \in 2^\alpha\} = \kappa$. As κ is regular and $2^\alpha < \kappa$, there must be at least one $s \in T_\alpha$, so T is indeed a κ-tree.

By the tree property, there is a branch $B \subseteq T$, so letting $\mathcal{F} = \{A \in M \mid \exists s \in B, A_s \subseteq A\}$, then it is easy to see that $\mathcal{F}$ is a κ-complete M-ultrafilter, and it is non-principal since for all $\beta < \kappa$, if $\{\beta\} = A_\alpha$, then it must be that $s(\alpha) = 0$ for any $s \in B$ of sufficient height, otherwise the intersection is small.

(3) $\to$ (5): Let $A \subseteq V_\kappa$ and let $M \prec H(\kappa^+)$ be an elementary submodel such that $|M| = \kappa$, $A, \kappa, V_\kappa \in M$, and $M^{<\kappa} \subseteq M$. Note that we can find such M since κ is inaccessible, moreover, since $\kappa \in M$ we get that M is transitive. The algebra $\mathcal{P}(\kappa) \cap M$ has size κ, so there is a κ-complete M-ultrafilter, U. Consider the definable ultrapower, M^κ/U, where we only consider $f\colon \kappa \to M$ which is definable in M. It follows that if f, g are two such functions, $\{\alpha < \kappa \mid f(\alpha) = g(\alpha)\}$ and $\{\alpha < \kappa \mid f(\alpha) \in g(\alpha)\}$ are both definable as well, and so in M. Therefore we get a Łoś's theorem for this ultrapower. Moreover, by κ-completeness we this ultrapower is well-founded and it has critical point κ. If it was not well-founded, we would have a sequence $\{[f_n] \mid n < \omega\}$ such that $A_n = \{\alpha < \kappa \mid f_{n+1}(\alpha) \in f_n(\alpha)\} \in U$, and so by κ-completeness, $\bigcap A_n \in U$ and therefore non-empty. This would mean that there is some α such that $\{f_n(\alpha) \mid n < \omega\}$ is a descending $\in$-sequence in M which is transitive.

So we can apply the transitive collapse to the ultrapower and let N be the resulting model and $j(x) = \pi([c_x])$, where c_x is the constant function returning x and π is the transitive collapse. By κ-completeness, if $\alpha < \kappa$, $j(\alpha) = \alpha$. In particular, $\operatorname{crit}(j) \geq \kappa$, and it is not hard to verify that since U is non-principal, $j(\kappa) > \kappa$.

(5) $\to$ (4): Suppose that $A \subseteq V_\kappa$ and let $j\colon M \to N$ be transitive sets and an elementary embedding witnessing that κ is weakly critical. Then $N \models$ "$j(V_\kappa)$ is transitive", and therefore it is in fact transitive, and since $\operatorname{crit}(j) = \kappa$, we have that $V_\kappa \subsetneq j(V_\kappa)$. Moreover, letting $j(A) = B$, we immediately get that $\langle V_\kappa, \in, A\rangle \prec \langle j(V_\kappa), \in, B\rangle$ as wanted.

(4) $\to$ (1): Let T be a κ-satisfiable theory of size κ, we may assume if $T_0 \in [T]^{<\kappa}$, then T_0 has a model in V_κ. Let $\langle V_\kappa, \in, T\rangle \prec \langle M, \in, T'\rangle$, then V_κ satisfies that every "small" subset of T has a model, and therefore M satisfies the same. But since $T' \cap V_\kappa = T$ is small, from the point of view of M, T has a model. ☐

Remark. The embedding properties can be modified, extended, and refined to require all kind of properties from M or N or the embedding itself. In some of these cases, however, we will need to require explicitly that κ is inaccessible, or at the very least that $\kappa^{<\kappa} = \kappa$.

Theorem 2.19. *If κ is weakly compact, then κ is Mahlo.*

Proof. Suppose that κ is weakly compact, let $C \subseteq \kappa$ be a club, and let $\langle V_\kappa, \in, C\rangle \prec \langle M, \in, C'\rangle$, then $M \models \kappa \in C'$, and therefore in M it holds that "there is a regular cardinal in C'", so in V_κ it must be that there is a regular cardinal in C. ☐

Exercise 2.20. If κ is weakly compact, then κ is greatly Mahlo.

Exercise 2.21. If κ is weakly compact, and $S \subseteq \kappa$ is a stationary set, then there is some $\alpha < \kappa$ such that $S \cap \alpha$ is stationary in α.

One natural question is "how deep does weak compactness go"?

Lemma 2.22. *Let κ be a weakly compact cardinal, $A \subseteq V_\kappa$ and suppose that for all $\alpha < \kappa$, $A \cap V_\alpha \in L$. Then $A \in L$. Consequently, if $V_\kappa \models V = L$, then $V_{\kappa+1} = (V_{\kappa+1})^L$.*

Proof. Let $\langle V_\kappa, \in, A\rangle \prec \langle M, \in, B\rangle$, then V_κ satisfies that for all α there is some β such that $A \cap V_\alpha \in L_\beta$. Therefore M satisfies the same, so there is some β such that $B \cap V_\kappa \in L_\beta$, and so $A \in L_\beta$. If $V_\kappa \models V = L$, then this holds for all $A \in V_{\kappa+1}$, and the conclusion follows. □

Theorem 2.23. *Suppose that κ is weakly compact, then $L \models \kappa$ is weakly compact.*

Proof. Let $T \subseteq L_\kappa$ be such that $L \models$ "T is a κ-tree". Then T is a κ-tree in V, since κ is a cardinal in V. If $b \in V$ is a branch, then $b \upharpoonright \alpha \in L$ for all $\alpha < \kappa$, since for successor nodes $b \upharpoonright \alpha + 1$ is fully determined by the maximal node. By Lemma 2.22, $b \in L$ as well. Since κ is inaccessible, it is inaccessible in L, and so L satisfies that κ is weakly compact. □

Remark. Unlike inaccessibility and Mahlo-ness, it is possible for κ to be weakly compact in V, but not weakly compact in an inner model M. The above proof shows, however, that for $M = L$ this is not the case, and we can extend this proof to models that are sufficiently robust.

Remark. It is consistent that ω_2 has the tree property (in which case $2^{\aleph_0} \geq \aleph_2$). But we can show that if κ has the tree property, then κ is weakly compact in L.

2.1.1 Colourful characterisations of weak compactness

Definition 2.24. We write $\kappa \to (\lambda)^n_m$ to mean "For every $c\colon [\kappa]^n \to m$ there is $H \in [\kappa]^\lambda$ such that $c \upharpoonright [H]^n$ is constant." If $m = 2$, we omit it from the notation and write $\kappa \to (\lambda)^n$.

We will refer to such c as a "colouring" and will often think of c as a symmetric function on tuples of ordinals, and we say that H as in the conclusion is a "homogeneous set", often for a given colour.

Exercise 2.25. $\omega \to (\omega)^2$.

Exercise 2.26. If $\kappa \to (\lambda)^2$, then $\kappa' \to (\lambda')^2$ whenever $\kappa' \geq \kappa$ and $\lambda' \leq \lambda$.

Exercise 2.27 (**) . For any κ, $\kappa \not\to (\omega)^\omega$.

Exercise 2.28 (**) . Let κ be an infinite cardinal, then $2^\kappa \not\to (\kappa^+)$.

Proposition 2.29. *If $\kappa > \omega$ and $\kappa \to (\kappa)^2$, then κ is inaccessible.*

Proof. Suppose that κ is singular, then we can partition it into $\{X_\xi \mid \xi < \mathrm{cf}(\kappa)\}$ where $|X_\xi| < \kappa$ for all ξ. The colouring $c(\alpha, \beta) = 0$ if and only if α, β belong to the same X_ξ, and otherwise $c(\alpha, \beta) = 1$. Clearly, every homogeneous set of value 0 has size $< \kappa$, and any homogeneous set with value 1 must have size $\leq \mathrm{cf}(\kappa)$.

If $\lambda < \kappa$ and $2^\lambda \geq \kappa$, then by the above exercises $2^\lambda \to (\lambda^+)^2$, which is impossible. □

Proposition 2.30. *Suppose that $\langle P, <\rangle$ is a linear order of size κ and $\kappa \to (\kappa)^2$, then there is some $X \subseteq P$ such that $\langle X, <\rangle \cong \kappa$ or $\langle X, >\rangle \cong \kappa$.*

Proof. Write P as $\{p_\alpha \mid \alpha < \kappa\}$, then define c on $[\kappa]^2$ by $c(\alpha, \beta) = 0$ if and only if $\alpha < \beta$ and $p_\alpha < p_\beta$. A homogeneous $H \in [\kappa]^\kappa$ defines an embedding of κ into P, or a reverse embedding. □

Proposition 2.31. *Suppose that κ is an inaccessible cardinal such that any linear order of size κ has an embedding or a reverse embedding from κ. Then κ has the tree property.*

Proof. Let $\langle T, <\rangle$ be a κ-tree, and let $\prec$ be an extension of the order on T into a linear ordering. We may assume that the linear ordering satisfies that that if $x \prec y \prec z$ and $w < x$ and $w < z$, then $w < y$. Namely, $T^x = \{t \in T \mid x < t\}$ is a convex set for all $x \in T$. For example, we can do this by embedding T into $\kappa^{<\kappa}$ through choosing a well-ordering of each level, then considering $\kappa^{<\kappa}$ as a linear ordering in the lexicographic order. Let $C \subseteq T$ be such that $\operatorname{otp}(C, \prec) = \kappa$, we will deal with the case where the order type is the reverse order later.

Let $D = \{x \in T \mid \exists y \in C, \forall z \in C, y \prec z \to x < z\}$. Namely, we use the tail-segments filter on C to measure points in the tree, so a node in T gets into D when its successors contain a tail-segment of C. We first claim that D is non-empty, since the root of T must satisfy this property. Next, we claim that D is a chain in T. If $x, x' \in D$, then there are $y, y' \in C$ witnessing that, and without loss of generality $y' \prec y$, then there is some z such that $y' \prec y \prec z$, and therefore z must be a $<$-successor of both x and x', and therefore x and x' are comparable, so D is a chain.

Let us show that D is cofinal. If $x \in D$, then the set of successors of x in T which have rank α, for some $\alpha < \kappa$, has fewer than κ members. Since the tail segments of C have size κ, it must be that one of the elements of rank α satisfies the property and is in D. Therefore, D is a branch, as wanted.

If C was ordered in the reverse well-ordering, define D using initial segments instead. □

Corollary 2.32. *If $\kappa \to (\kappa)^2$, then κ is weakly compact.* □

2.2 Indescribably large cardinals

Definition 2.33. Let $\mathcal{L}$ be a first-order language. We say that φ is a Σ^n_m-formula (Π^n_m-formula) if it has m alternating blocks of $n+1$-order variables starting with existential (universal) quantifiers, and a formula which does not have any $n+k$-order quantifiers for $k > 0$.

We use full semantics for higher-order quantifiers. Namely, we interpret the quantifiers as ranging over $\mathcal{P}^n(M)$, where M is the structure.

Definition 2.34. Let Q be a class of sentences of the form Π^n_m or Σ^n_m, where $n, m < \omega$. We say that κ is *Q-indescribable* if for all $R \subseteq V_\kappa$ and $\varphi \in Q$, if $\langle V_\kappa, \in, R\rangle \models \varphi$, then there is $\alpha < \kappa$ such that $\langle V_\alpha, \in, R \cap V_\alpha\rangle \models \varphi$.

Note that this is equivalent to $\langle V_{\kappa+n}, \in, R\rangle \models \varphi$ where φ is treated now as a first-order sentence.

Proposition 2.35. *For any $n < \omega$, κ is Σ^1_{n+1}-indescribable if and only if it is Π^1_n-indescribable.*

Proof. Every Π^1_n formula is also Σ^1_{n+1}, so it is enough to verify that Π^1_n-indescribability implies Σ^1_{n+1}-indescribability. If $\langle V_\kappa, \in, R\rangle \models \exists X \varphi(X)$ where φ is a Π^1_n-sentence in X, then there is some $S \subseteq V_\kappa$ such that $\langle V_\kappa, \in, R, S\rangle \models \varphi[S]$, where we replace all the free occurrences of X by the predicate S.

By Π^1_n-indescribability, there is some $\alpha < \kappa$ such that $\langle V_\kappa, \in, R \cap V_\alpha, S \cap V_\alpha\rangle \models \varphi[S \cap V_\alpha]$, and therefore $\langle V_\alpha, \in, R \cap V_\alpha\rangle \models \exists X \varphi(X)$ as wanted. □

We saw that inaccessible cardinals reflect any first-order sentence, so κ is inaccessible if and only if it is Π^1_0-indescribable, or equivalently, Σ^1_1-indescribable.

Theorem 2.36. *κ is weakly compact if and only if it is Π^1_1-indescribable.*

Proof. Suppose that κ is weakly compact, let $R \subseteq V_\kappa$ and let $\forall X\varphi(X)$ be a Π^1_1-sentence such that $\langle V_\kappa, \in, R\rangle \models \forall X\varphi(X)$, then we can take an elementary end-extension $\langle M, \in, R'\rangle$, which we can assume has size κ. Since it holds that for every $X \subseteq V_\kappa$, $\langle V_\kappa, \in, R\rangle \models \varphi(X)$, it is certainly true that $M \models (\forall X \subseteq V_\kappa), \langle V_\kappa, \in, R\rangle \models \varphi(X)$. Therefore, in M it is true that there is some α such that $\langle V_\alpha, \in, R' \cap V_\alpha\rangle \models \forall X\varphi(X)$. By elementarity, this holds in V_κ as wanted.

In the other direction, suppose that κ is Π^1_1-indescribable, then it is inaccessible. Let T be a κ-tree, then we may assume $T \subseteq \kappa^{<\kappa}$. Then for all $\alpha < \kappa$,

$$\langle V_\alpha, \in, T \cap V_\alpha\rangle \models \exists B(B \text{ is a branch in } T \cap V_\alpha),$$

as there are nodes which are arbitrarily high below α. Since κ is Π^1_1-indescribable, it must also satisfy this Σ^1_1-sentence, otherwise its negation will reflection, so T has a branch. $\square$

Exercise 2.37 (***) . Find a characterisation of Mahlo cardinals and α-Mahlo cardinals in terms of expanded indescribability. (That is, find a refinement of Π^1_1-indescribability that is stronger than Σ^1_1-indescribability, which captures exactly Mahlo cardinals.)

Exercise 2.38 (**) . Suppose that κ is Π^n_m-indescribable. Then $\{\alpha \mid \alpha \text{ is } Q\text{-indescribable}\}$ is stationary in κ, where $Q = \Pi^{n-1}_k$ for any k when $n > m = 0$ and $Q = \Pi^n_{m-1}$ if $m > 0$.

REMARK. By repeating a proof similar to Theorem 2.23, using the fact that under GCH, $H(\kappa^{+n})$ and $V_{\kappa+n}$ are bi-interpretable, we can then use the fact that in L, $L_\kappa = V_\kappa$, to encode $V^L_{\kappa+n}$ to $H(\kappa^{+n})^L = L_{\kappa^{+n}}$.

Definition 2.39. We say that κ is a *totally indescribable cardinal* if it is Π^n_m-indescribable for all $n, m < \omega$.

Theorem 2.40. *Suppose that M is a transitive model of* ZFC *and $j\colon M \to M$ is a non-trivial elementary embedding, then $\operatorname{crit}(j)$ is totally indescribable.*

Proof. Let $\kappa = \operatorname{crit}(j)$, let $R \subseteq V_\kappa$ in M, and suppose that $\langle V_\kappa, \in, R\rangle \models \varphi$ for some $\varphi \in \Pi^n_m$. By elementarity, $\langle j(V_\kappa), \in, j(R)\rangle \models \varphi$. As we can replace this by the first-order structure $\langle V_{\kappa+n}, \in, R\rangle$, we have that $\langle V_{j(\kappa)+n}, \in, j(R)\rangle \models \varphi$. However, since $j\colon M \to M$, it follows that $M \models \exists\alpha < j(\kappa),$ "$\langle V_{\alpha+n}, \in, j(R) \cap V_\alpha\rangle \models \varphi$", since $\kappa < j(\kappa)$ witnesses that. So, by elementarity, this holds below κ, so κ is Π^n_m-indescribable for all $n, m < \omega$. $\square$

REMARK. Note that an elementary embedding $j\colon M \to M$ is in a sense "quite large", and we will see that such j cannot be definable in M, although surprisingly the consistency strength of this is not as large as one would initially expect. We will discuss this more in the future. We will also discuss variations of this arguments which will be compatible with internally definable embeddings.

Chapter 3

Critical measuring

3.1 Measurable cardinals

Definition 3.1. We say that κ is a *measurable cardinal* if there exists a κ-complete free ultrafilter on κ. We will call such an ultrafilter a *measure (on κ)*.

Remark. If U is a measure on κ, then $[\kappa]^{<\kappa} \cap U = \varnothing$, and so if κ were singular we could find a short sequence of small sets whose union would cover κ, which would violate the completeness of U. In particular, measurable cardinals must be regular.

As was the case so far, note that ω is measurable, since every ultrafilter is ω-complete. So we will implicitly assume that our cardinals are uncountable.

Theorem 3.2. *Suppose that κ is a measurable cardinal, then κ is an inaccessible cardinal.*

Proof. Since there is a measure, U, on κ it must be that κ is regular, so it is enough to check that if $\alpha < \kappa$, then $2^\alpha < \kappa$.

Suppose that α is such that $2^\alpha \geq \kappa$, then there is a family $S \subseteq 2^\alpha$ such that $|S| = \kappa$, so we can write it as $\{f_\beta \mid \beta < \kappa\}$. For each $\xi < \alpha$, let $A_\xi = \{\beta < \kappa \mid f_\beta(\xi) = 0\}$ and $B_\xi = \{\beta < \kappa \mid f_\beta(\xi) = 1\}$, let C_ξ denote whichever one of those two is in U. Since U is κ-complete, $C = \bigcap_{\xi<\alpha} C_\xi \in U$. However, if $\beta, \gamma \in C$, then for all $\xi < \alpha$ it must be that $f_\beta(\xi) = f_\gamma(\xi)$, so C must be a singleton and therefore U is not a free ultrafilter. $\square$

The first question, therefore, is why do we require so much from the measure? What if we were to require mere ω_1-completeness? As the following theorem shows, this will not change much.

Theorem 3.3. *If U is an ultrafilter on κ, then the completeness of U is a measurable cardinal or ω.*

Proof. Let λ be the completeness of U, which means that there is a sequence of pairwise disjoint sets $\{X_\alpha \mid \alpha < \lambda\} \cap U = \varnothing$ such that $\bigcup X_\alpha = \kappa$. We can define a map $f\colon \kappa \to \lambda$ by $f(\xi) = \alpha$ if and only if $\xi \in A_\alpha$.

Define $U_* = \{A \subseteq \lambda \mid f^{-1}(A) \in U\}$, then U_* is a λ-complete ultrafilter on λ, since the pre-image preserves inclusion, arbitrary intersections, and completements. $\square$

Corollary 3.4. *If κ is the least cardinal on which there is an ω_1-complete free ultrafilter, then κ is a measurable cardinal.* $\square$

The following theorem is a consequence of Theorem 3.2, but it is worth proving explicitly.

Theorem 3.5. *For any λ, λ^+ is not a measurable cardinal.*

Proof. We will construct a (λ^+,λ)-Ulam matrix, a family of sets $\{A_{\alpha,\beta}\subseteq\lambda^+\mid\alpha<\lambda^+,\beta<\lambda\}$ such that:

1. If $\alpha\neq\alpha'$, then for all $\beta<\lambda$, $A_{\alpha,\beta}\cap A_{\alpha',\beta}=\varnothing$.
2. For every α, $|\lambda^+\setminus\bigcup_{\beta<\lambda}A_{\alpha,\beta}|\leq\lambda$.

Suppose that we have managed to construct such an Ulam matrix, if U is a λ^+-complete ultrafilter on λ^+, then for each α there is some $\beta<\lambda$ such that $A_{\alpha,\beta}\in U$. Therefore there is some $\alpha\neq\alpha'$ such that $A_{\alpha,\beta}$ and $A_{\alpha',\beta}$ are both in U, just by cardinality arguments of the index sets. However, this is impossible, since $A_{\alpha,\beta}\cap A_{\alpha',\beta}=\varnothing\in U$.

Let us show, then, that such an Ulam matrix exists. For each $\xi<\lambda^+$ let $f_\xi\colon\lambda\to\xi$ be a surjection.[10] We define

$$A_{\alpha,\beta}=\{\xi<\lambda^+\mid f_\xi(\beta)=\alpha\}.$$

If $\beta<\lambda$, then for each $\xi<\lambda^+$ there is a unique α such that $\xi\in A_{\alpha,\beta}$, namely $f_\xi(\beta)$. So the first condition holds. If $\alpha<\lambda^+$, then for every $\xi>\alpha$, $\alpha\in\operatorname{rng}f_\xi$, so $\lambda^+\setminus\bigcup_{\beta<\lambda}A_{\alpha,\beta}\subseteq\alpha+1$ and therefore the second condition holds and this is indeed an Ulam matrix. □

The original application of an Ulam matrix was to show that if there is an ω_1-complete, non-atomic, and total measure (in the measure theoretic sense) on $\mathbb{R}$, then $2^{\aleph_0}\neq\aleph_1$. The argument shows that $2^{\aleph_0}$, in that case, must be weakly inaccessible. This connects to the concept of *real-valued measurable cardinals* which are equiconsistent with measurable cardinals, but they are distinct from them, for example such cardinals must be at most $2^{\aleph_0}$.

Remark. It is consistent with ZF that ω_1 is measurable. The above proofs shows that in that case we cannot have an Ulam matrix, and moreover, there is no sequence $f_\alpha\colon\omega\to\alpha$ of surjections for all $\alpha<\omega_1$.

3.1.1 Embeddings

Definition 3.6. We say that κ is a *critical cardinal* if there is a transitive set N and an elementary embedding $j\colon V_{\kappa+1}\to N$ such that $\operatorname{crit}(j)=\kappa$.

Exercise 3.7. Every critical cardinal is weakly compact. Indeed, it is the limit of weakly compact cardinals.

Exercise 3.8. If $j\colon V_{\kappa+1}\to N$ is witnessing that κ is critical, then $V_{\kappa+1}\subseteq N$.

Theorem 3.9. *κ is a measurable cardinal if and only if it is a critical cardinal.*

Proof. Suppose that κ is a measurable cardinal and let U be a measure on κ. Then the ultrapower $V_{\kappa+1}^\kappa/U$ is well-founded, since U is ω_1-complete, as in the proof of Theorem 2.18 (3) → (5): if $\{f_n\mid n<\omega\}$ is such that $A_n=\{\alpha<\kappa\mid f_{n+1}(\alpha)\in f_n(\alpha)\}\in U$, then picking any $\alpha\in\bigcap_{n<\omega}A_n$ we have that $\{f_n(\alpha)\mid n<\omega\}$ is a counterexample to the Axiom of Foundation. Therefore we can take N to be the transitive collapse of the ultrapower, and j is the ultrapower embedding composed with the collapsing isomorphism.

[10] Pick any function for $\xi=0$.

To see that $\operatorname{crit}(j) = \kappa$, note that for $\delta < \kappa$, if $\{\alpha < \kappa \mid f(\alpha) < \delta\} \in U$, then there is a unique $\gamma < \delta$ such that $\{\alpha < \kappa \mid f(\alpha) = \gamma\} \in U$, and therefore f is equal (in the ultrapower) to the constant function c_γ, and so $j(\delta) = \delta$. On the other hand, it is easy to see that $\mathrm{id}\colon \kappa \to \kappa$ satisfies for any $\alpha < \kappa$, $\alpha = j(\alpha) = [c_\alpha] < [\mathrm{id}] < [c_\kappa] = j(\kappa)$.

In the other direction, let $j\colon V_{\kappa+1} \to N$ be an elementary embedding with $\operatorname{crit}(j) = \kappa$. We define $U = \{A \subseteq \kappa \mid \kappa \in j(A)\}$. Note that this is indeed a well-defined set as $\mathcal{P}(\kappa) \subseteq V_{\kappa+1}$. Let us check that U is a measure on κ.

If $\{A_\alpha \mid \alpha < \gamma\} \subseteq U$ for some $\gamma < \kappa$, then for all $\alpha < \gamma$, $\kappa \in j(A_\alpha)$. In particular, $\kappa \in \bigcap_{\alpha<\gamma} j(A_\alpha)$. Since $j(\gamma) = \gamma$ this can be written as $\bigcap_{\alpha<j(\gamma)} j(A_\alpha)$, which therefore translates to $j(\bigcap_{\alpha<\gamma} A_\alpha)$. So U is κ-complete. It is easy to verify that U is closed under supersets and for any $A \subseteq \kappa$, either $A \in U$ or $\kappa \setminus A \in U$. $\square$

The following exercises provide ways of discussing elementary embeddings between V and a transitive class M in a first-order way.

Exercise 3.10. Using the Scott equivalence class (taking the least ranked sets in an equivalence relation defined on a proper class) show that Theorem 3.9 can be extended so that the embedding is defined from V to an inner model M.

Exercise 3.11. Suppose that $j\colon M \to N$ is an unbounded embedding between transitive classes (or sets) satisfying ZFC. Namely, for all $\alpha \in \mathrm{Ord}^N$ there is some $\beta \in \mathrm{Ord}^M$ such that $j(\beta) > \alpha$. Show that j is elementary if and only if it is Σ_1-elementary.

Definition 3.12. Suppose that $j\colon M \to N$ is an elementary embedding of transitive sets, the *derived measure from* j is $\operatorname{der}^M(j) = \{A \in \mathcal{P}(\operatorname{crit}(j)) \cap M \mid \operatorname{crit}(j) \in j(A)\}$. In the case where $\mathcal{P}(\operatorname{crit}(j)) \subseteq M$, we omit the superscript.

Theorem 3.13. *Let κ be a measurable cardinal and let $j\colon V \to M$ be an embedding with $\operatorname{crit}(j) = \kappa$. Then the derived measure is a* normal *measure. Namely, any of the following equivalent conditions hold:*

1. $\operatorname{der}(j)$ *is closed under diagonal intersections. Namely, if $\{A_\alpha \mid \alpha < \kappa\} \subseteq \operatorname{der}(j)$, then $\triangle_{\alpha<\kappa} A_\alpha = \{\alpha < \kappa \mid \alpha \in \bigcap_{\beta<\alpha} A_\beta\} \in \operatorname{der}(j)$.*
2. $\operatorname{der}(j)$ *satisfies Fodor's lemma. Namely, if $f\colon S \to \kappa$ is a regressive function for some $S \in \operatorname{der}(j)$, then there is some α such that $f^{-1}(\alpha) \in \operatorname{der}(j)$.*
3. id *represents κ in the ultrapower by* $\operatorname{der}(j)$*. Namely, the equivalence class* $[\mathrm{id}]$ *is mapped to κ by the transitive collapse.*

Proof. Let us prove one of the conditions holds, and that the three are equivalent. Suppose that $S \in \operatorname{der}(j)$ and $f\colon S \to \kappa$ is regressive. Then $\kappa \in j(S)$, and $j(f)\colon j(S) \to j(\kappa)$ is regressive, so $j(f)(\kappa) = \alpha$ for some $\alpha < \kappa$. This means that $\kappa \in j(f)^{-1}(\alpha) = j(f^{-1}(\alpha))$, so $f^{-1}(\alpha) \in \operatorname{der}(j)$ as wanted.

The proof that (1) $\to$ (2) is the same as the proof of Fodor's lemma. If f is regressive, letting $A_\alpha = f^{-1}(\alpha)$, then either $A_\alpha \in \operatorname{der}(j)$ for some α, or else taking $\beta \in S \cap \triangle_{\alpha<\kappa}(\kappa \setminus A_\alpha)$ we have that $f(\beta) \neq \alpha$ for all $\alpha < \beta$, so f is not regressive.

For (2) $\to$ (3), let $[f]$ represent an ordinal below $[\mathrm{id}]$, then $S = \{\alpha < \kappa \mid f(\alpha) < \alpha\} \in \operatorname{der}(j)$ which means that f is regressive on S, so by our assumption there is some $S' \in \operatorname{der}(j)$ such that f is constant on S', in which case $[f] = [c_\alpha]$ for some $\alpha < \kappa$.

Finally, (3) $\to$ (1). Given a sequence $A = \langle A_\alpha \in \operatorname{der}(j) \mid \alpha < \kappa\rangle$, $\kappa \in j(\triangle A)$ if and only if $\kappa \in \bigcap_{\alpha<\kappa} j(A)(\alpha) = \bigcap_{\alpha<\kappa} j(A_\alpha)$. However, since $\kappa \in j(A_\alpha)$ for all $\alpha < \kappa$, this holds. $\square$

Let U be an ω_1-complete ultrafilter on some κ, then the ultrapower V^κ/U is well-founded. We denote by $\mathrm{Ult}(V,U)$ the transitive collapse of the ultrapower and by j_U the elementary embedding. We will use $[f]_U$ to denote both the equivalence class of f and the member of $\mathrm{Ult}(V,U)$ it represents, and when clear from context, we will omit the subscripts with impunity.

Exercise 3.14. Let U be a normal measure on κ, then $\mathrm{Ult}(V,U) = \{j(f)(\kappa) \mid f\colon \kappa \to V\}$.

Theorem 3.15. *Suppose that $j\colon V \to M$ and $\mathrm{crit}(j) = \kappa$. Let $U = \mathrm{der}(j)$, then there exists an elementary embedding $k\colon \mathrm{Ult}(V,U) \to M$ such that $j = k \circ j_U$.*

Proof. Define $k([f]) = j(f)(\kappa)$, for $f\colon \kappa \to V$, then k is the wanted elementary embedding. □

Exercise 3.16. Complete the above proof.

Corollary 3.17. *If U is a normal measure on κ, then $\mathrm{der}(j_U) = U$.* □

3.2 Structure of ultrapowers

Two major questions come to mind at this point. Can we have $\mathrm{Ult}(V,U) = V$ somehow? And if $j\colon V \to M$, does that mean that $j = \mathrm{Ult}(V, \mathrm{der}(j))$? Both answer to the negative in quite significant ways.

Proposition 3.18. *Let κ be a measurable, U a measure, and $j\colon V \to M = \mathrm{Ult}(V,U)$ is the ultrapower embedding.*

1. $\mathcal{P}(\kappa) = \mathcal{P}(\kappa)^M$.
2. $\kappa^+ = (\kappa^+)^M$.
3. $2^\kappa < j(\kappa) < (2^\kappa)^+$.
4. $M^\kappa \subseteq M$.

Proof. (1) is immediate from the fact that $\kappa = \mathrm{crit}(j)$, since $A = j(A) \cap \kappa$ for all $A \subseteq \kappa$. In particular, if $\alpha < \kappa^+$, there is some subset of κ which encodes a well-ordering of type α, so $\alpha < (\kappa^+)^M$, and since $M \subseteq V$, it must be that $(\kappa^+)^M \leq \kappa^+$, so (2) holds.

To prove (3), note that $j(\kappa) = [c_\kappa]$, is strictly larger than any $f \in \kappa^\kappa$, and therefore $2^\kappa < j(\kappa)$, on the other hand, $j(\kappa) < (2^\kappa)^+$, since any function representing $\alpha < j(\kappa)$ must be represented by $f\colon \kappa \to \kappa$.

Finally, $M^\kappa \subseteq \kappa$, since if $\{x_\alpha \mid \alpha < \kappa\} \subseteq M$, then there are $f_\alpha\colon \kappa \to V$ which represent the x_α, and fix $h\colon \kappa \to \kappa$ which represents κ in M, we define $f\colon \kappa \to V$ such that $f(\xi)\colon h(\xi) \to V$ and $h(\xi)(\alpha) = f_\alpha(\xi)$. By Łoś's theorem $[f] = \{x_\alpha \mid \alpha < \kappa\} \in M$. □

Theorem 3.19. *Let U be a measure on κ, then $U \notin \mathrm{Ult}(V,U)$.*

Proof. Since $(\kappa^\kappa)^V \in \mathrm{Ult}(V,U)$, if U was in the model, the function $f \mapsto [f]$ would be in the model as well. However, in that case, $\mathrm{Ult}(V,U) \models 2^\kappa < j(\kappa) < (2^\kappa)^+$. But $j(\kappa)$ is inaccessible in $\mathrm{Ult}(V,U)$. □

Corollary 3.20. *If there is a measurable cardinal, then $V \neq L$. In particular, if κ is measurable, then it is not measurable in L.* □

Exercise 3.21 (*) . Find a direct proof that if κ is measurable, then $V \neq L$.

Exercise 3.22. Suppose that U is a normal measure on κ, then there is some $f\colon \kappa \to \kappa$ such that $\{f``A \mid A \in U\}$ is a measure that is not normal.

We actually get much more than just $V \neq L$. We will see that if $\kappa = \omega_1$, then there is an elementary embedding $j\colon L_\kappa \to L_\kappa$, and that if $\lambda = \omega_\omega$, then $L \models \operatorname{cf}(\lambda) = \lambda$.

3.3 Iterations and Gaifman's theorem

At this point, one is left wondering, are there any embeddings that do not come from ultrapowers? The answer to that is in a sense both yes and no.

If U was a measure on κ and $j_U\colon V \to \operatorname{Ult}(V,U)$, then $j_U(U)$ is a measure on $j(\kappa)$ in $\operatorname{Ult}(V,U)$. Therefore, we can define $\operatorname{Ult}(\operatorname{Ult}(V,U), j_U(U))$. This process can be iterated, of course. We need a notion of a direct limit for the generalisation.

Definition 3.23. Let $\langle I, <\rangle$ be a directed set and suppose that $\{M_i, e_{i,j} \mid i,j \in I\}$ is a directed system of models of ZFC such that:

1. $e_{i,k}\colon M_i \to M_k$.
2. $e_{i,i} = \mathrm{id}$.
3. $e_{j,k} \circ e_{i,j} = e_{i,k}$ for all $i < j < k \in I$.

We define an equivalence relation on $\overline{M} = \bigcup_{i\in I}\{i\} \times M_i$ given by $\langle i,a\rangle \equiv \langle j,b\rangle$ if and only if there exists k such that $i,j < k$ and $e_{i,k}(a) = e_{j,k}(b)$. Then $M = \overline{M}/{\equiv}$ is the *direct limit* of the diagram and defining $e_i\colon M_i \to M$ with $e_i(a) = [\langle i,a\rangle]$ is an elementary embedding.

Exercise 3.24. Verify that M is well-defined and that e_i are elementary maps.

Definition 3.25. Let U be a measure on κ, we define $\operatorname{Ult}^\alpha(V,U)$ to be the αth iterated ultrapower, where at limit stages we take the direct limit and at successor steps we take $\operatorname{Ult}(\operatorname{Ult}^\alpha(V,U), j_U^\alpha(U))$.

Exercise 3.26. $\operatorname{Ult}^\alpha(\operatorname{Ult}^\beta, j_U^\beta(U)) = \operatorname{Ult}^{\beta+\alpha}(V,U)$.

Exercise 3.27. Suppose $\alpha + \beta = \gamma$ and let $j_{\alpha,\gamma}$ be the factor embedding between $\operatorname{Ult}^\alpha(V,U)$ and $\operatorname{Ult}^\gamma(V,U)$. Then $j_{\alpha,\gamma} = j^\beta_{j^\alpha_U(U)}$.

Theorem 3.28 (Gaifman). *Let κ be a measurable cardinal and U a measure. Then for all α, $M_\alpha = \operatorname{Ult}^\alpha(V,U)$ is well-founded.*

Proof. For $\alpha = 0$, $M_0 = V$, and for $\alpha + 1$ this follows from the ω_1-completeness of $j_U^\alpha(U)$ in $\operatorname{Ult}^\alpha(V,U)$. Finally, suppose that M_α is ill-founded for a limit ordinal α, and suppose that M_β was well-founded for all $\beta < \alpha$. As M_α is not well-founded, let $\xi \in \mathrm{Ord}$ be the least one such that $j_\alpha(\xi)$ is not well-founded and let $[\langle \beta_n, f_n\rangle]$ be a descending sequence witness the ill-foundedness.

Let $\beta = \beta_0$ and let λ be the ordinal in M_β that f_0 represents and let δ be an ordinal such that $\beta + \delta = \alpha$. Note that $\lambda < j_\beta(\xi)$, since $j_{\beta,\alpha}(\lambda) < j_\alpha(\xi) = j_{\beta,\alpha}(\xi)$.

By our assumption, $V \models (\forall \alpha' \leq \alpha)(\forall \xi' < \xi) M_{\alpha'}$ is well-founded below $j_{\alpha'}(\xi')$. If we apply j_β we get that

$$M_\beta \models (\forall \alpha' \leq j_\beta(\alpha))(\forall \xi' < j_\beta(\xi))\, \operatorname{Ult}^{\alpha'}(M_\beta, j_\beta(U)) \text{ is well-founded below } j_\beta(\xi').$$

But since $\delta \le \alpha \le j_\beta(\alpha)$ and $\lambda < j_\beta(\xi)$, we can apply the factor embedding, $j_{\beta,\alpha}$, which is the δth iterate in M_β. So we get

$$M_\beta \models \mathrm{Ult}^\delta(M_\beta, j_\beta(U)) \text{ is well-founded below } j^\delta_{j_\beta(U)}(\lambda).$$

However, this is a contradiction since $j_{\beta,\alpha}(\lambda)$ is ill-founded in $M_\alpha = \mathrm{Ult}^\delta(M_\beta, j_\beta(U))$. $\square$

Since $\mathrm{Ult}^\alpha(V, U)$ is always well-founded, we can confuse it with its transitive collapse, as we normally do, and the following exercise becomes meaningful.

Exercise 3.29. Let α be a limit ordinal, then $j^\alpha_U(\kappa) = \sup\{j^\beta_U(\kappa) \mid \beta < \alpha\}$.

Under some additional axioms, e.g. that the universe is a canonical enough inner model, all elementary embeddings are obtained by iterating ultrapowers, and in the smallest such model there will be exactly one measurable cardinal with a single normal measure on it, and every embedding will be an iterate of that ultrapower.

Exercise 3.30. Let α be a limit ordinal, then $\mathrm{Ult}^\alpha(V, U)$ is not closed under ω-sequences.

3.4 Covering more structural consequences

Corollary 3.31. *If U is a measure on κ, then there is a closed and unbounded class of ordinals, C, such that $\alpha \in C$ if and only if there is an iterated ultrapower embedding mapping κ to α.* $\square$

Fixing κ and U let us write κ_α to denote $j^\alpha_U(\kappa)$. Let us explore some consequence of the existence of a measurable cardinal.

Proposition 3.32. $L_{\kappa_\alpha} \models \mathsf{ZFC}$.

Proof. $L_\kappa \models \mathsf{ZFC}$, since κ is inaccessible, since for any elementary embedding, j, $j(L_\alpha) = L_{j(\alpha)}$ and $V \models$ "$L_\kappa \models \mathsf{ZFC}$" the conclusion follows. $\square$

Proposition 3.33. *Let $\eta = \omega_1$. $L_\eta \models \mathsf{ZFC}$ and there is a non-trivial elementary embedding $j\colon L_\eta \to L_\eta$.*

Proof. Let M be the elementary submodel of $L_{\kappa_{\omega_1}}$ that is generated by $\{\kappa_\alpha \mid \alpha < \omega_1\}$, then $M \cong L_\eta$, and so $M \models \mathsf{ZFC}$, since $L_{\kappa_{\omega_1}} \models \mathsf{ZFC}$ and there is a non-trivial elementary embedding $j\colon L_\eta \to L_\eta$ given by noting that the map $\kappa_\alpha \mapsto \kappa_{1+\alpha}$ induces such an embedding. $\square$

This embedding from $L_\eta \to L_\eta$ can be extended to an embedding $L \to L$, and in fact if we consider the elementary submodel of L generated by the class C, it must be isomorphic to L again, and so any order preserving embedding $C \to C$ extends to an embedding $L \to L$, this defines a closed and unbounded class of ordinals which are critical points of these sort of embeddings. In particular, we can show that all the cardinals from V are in this class, and much more than that. This leads us to these consequences.

Proposition 3.34. *If η is a cardinal, then $L \models \mathrm{cf}(\eta) = \eta$.*

Proof. If $\eta \le \omega_1$ this is always true. Otherwise, there is an elementary embedding of $L \to L$ such that $j(\omega_1^V) = \eta$. $\square$

Exercise 3.35. If $\eta > \omega$ is a cardinal, then $L \models$ "η is weakly compact".

The important thing to notice is that these elementary embeddings from L to itself, or from L_η to itself, are not themselves in L. This shows that not only $V \neq L$, but in fact V is very far from being L. Embeddings like this are referred to as "sharps" with the case for L called $0^\#$, and more generally if $L(A)$ has an embedding into itself we say that $A^\#$ exists.

We will not prove the following theorem.

Theorem 3.36. *The following are equivalent.*

1. *There exists a non-trivial elementary embedding* $j\colon L \to L$.
2. *There exist a pair of cardinals,* $\lambda < \kappa$ *such that* $L_\lambda \prec L_\kappa$.
3. $\aleph_\omega$ *is regular in* L.
4. $\aleph_\omega$ *is inaccessible in* L.
5. $\aleph_\omega$ *is weakly compact in* L.
6. $(\aleph_\omega^+)^L < \aleph_{\omega+1}$.
7. *There exists an uncountable set* $A \subseteq \mathrm{Ord}$ *such that whenever* $B \in L$ *and* $A \subseteq B$, $|A| < |B|$.

There are many more equivalences we can add to this list. The last equivalent condition is known as "Jensen's covering lemma" which can be stated as "$0^\#$ exists if and only if L does not cover all the uncountable sets of ordinals." This can be generalised to other inner models and have been studied extensively.

3.5 Higher measurability

In Theorem 3.19 we saw that if U is a measure on κ, then $U \notin \mathrm{Ult}(V, U)$. So is κ a measurable cardinal in $\mathrm{Ult}(V, U)$?

Proposition 3.37. *Suppose that* φ *is a first-order property and* $j\colon V \to M$ *is an elementary embedding with* $\mathrm{crit}(j) = \kappa$. *If* $M \models \varphi(\kappa)$, *then* $\{\alpha < \kappa \mid V \models \varphi(\alpha)\} \in \mathrm{der}(j)$.

Proof. Let $A = \{\alpha < \kappa \mid V \models \varphi(\alpha)\}$ and notice that $\kappa \in j(A)$. $\square$

In particular, if $M \models$ "κ is measurable", then there is an unbounded set of measurable cardinals below κ. The following exercise shows that this is not nearly enough.

Exercise 3.38. If κ is the least measurable that is a limit of measurable cardinals, then there is no embedding $j\colon V \to M$ with $\mathrm{crit}(j) = \kappa$ where $M \models$ "κ is measurable".

Since $\mathrm{der}(j)$ is a normal ultrafilter, it must contain the club filter, so the sets in $\mathrm{der}(j)$ are always stationary, to say the least. Therefore there is a large number of measurable cardinals below κ. But we can look at this from a different perspective.

Definition 3.39. Let κ be a measurable cardinal and let U_0, U_1 be two normal measures. We say that U_0 is below U_1 in the *Mitchell order* if $U_0 \in \mathrm{Ult}(V, U_1)$. We denote this by $U_0 \lhd U_1$.

Theorem 3.40. *Show that* $U_0 \lhd U_1$ *if and only if there is a function* $f\colon \kappa \to V_\kappa$ *such that*

$$A \in U_0 \iff \{\alpha \mid A \cap \alpha \in f(\alpha)\} \in U_1.$$

In particular, $f(\alpha)$ *must be a measure on* α *on a* U_1*-large set.*

Proof. Suppose that $U_0 \lhd U_1$, then there is $f\colon \kappa \to V$ such that $j(f)(\kappa) = U_0$, then $A \in j(f)(\kappa)$ if and only if $j(f)(A) \cap \kappa \in j(f)(\kappa)$, so $\{\alpha < \kappa \mid A \cap \alpha \in f(\alpha)\} \in U_1$. The other direction follows from the same argument ran backwards. □

Lemma 3.41. *Suppose that $U_0 \lhd U_1$ are normal measures on κ, then $j_{U_0}(\kappa) < j_{U_1}(\kappa)$.*

Proof. In $M = \mathrm{Ult}(V, U_1)$ we can compute $j_{U_0} \restriction V_{\kappa+1}$, so $M \models 2^\kappa < j_{U_0}(\kappa) < (2^\kappa)^+ < j_{U_1}(\kappa)$. □

So, is κ always measurable in its ultrapowers? The answer is negative.

Theorem 3.42. *The Mitchell order on κ is a well-founded strict partial order.*

Proof. The fact that $\lhd$ is irreflexive is an immediate consequence of Theorem 3.19. Using Theorem 3.40 we see that if $U_0 \lhd U_1 \lhd U_2$, then the function f for which $j_{U_1}(f)(\kappa) = U_0$ is in $\mathrm{Ult}(V, U_2)$, and therefore $U_0 \lhd U_2$. It remains to show that the order is well-founded. Suppose that U_n is a sequence of measures such that $U_{n+1} \lhd U_n$ for all $n < \omega$. By Lemma 3.41, $j_{U_n}(\kappa)$ form a decreasing sequence of ordinals, which is impossible. □

Exercise 3.43. Find a direct argument that $\lhd$ is well-founded by arguing that if κ is the least one for which $\lhd$ is ill-founded we can obtain a contradiction.

Corollary 3.44. *If κ is a measurable cardinal, then there is a normal measure U such that $\{\alpha < \kappa \mid \alpha$ is not measurable$\} \in U$.* □

Definition 3.45. We write $o(U)$ to denote the rank of U in the Mitchell order and $o(\kappa)$ to denote $\sup\{o(U) + 1 \mid U$ is a measure on $\kappa\}$.

Exercise 3.46. Using Theorem 3.40 show that if U is a normal measure on κ, then $o(U) < (2^\kappa)^+$.

It follows from this exercise that $o(\kappa) \leq (2^\kappa)^+$, so under GCH, which is a common assumption in this situation, $o(\kappa) \leq \kappa^{++}$. Since GCH holds in "canonical enough" inner models, the Mitchell order large cardinal axioms are often given in that context. For example.

Fact 3.47. *The following are equiconsistent.*

1. *For each $n < \omega$, $2^{\aleph_n} = \aleph_{n+1}$ and $2^{\aleph_\omega} = \aleph_{\omega+2}$.*
2. *$\exists \kappa (o(\kappa) = \kappa^{++})$.*

Proposition 3.48. *Let U be a normal measure on κ. If $o(U) = \alpha$, then $\mathrm{Ult}(V, U) \models o(\kappa) = \alpha$.*

Proof. If $U' \lhd U$, then $U' \in \mathrm{Ult}(V, U)$. Therefore the measures on κ in $\mathrm{Ult}(V, U)$ are exactly $\lhd \restriction U$. As $\mathrm{Ult}(V, U)$ is transitive, it agrees on well-foundedness and rank with V, so $o(\kappa) = \alpha$. □

If κ is Mahlo, we find this is witnessed by a canonical object: a stationary set; if κ is a measurable cardinal, this is witnessed by a measure on κ. These are, in a sense, sort of local conditions. They have implications on the universe, but these are global. What about higher Mitchell orders, then?

If κ is a measurable cardinal and U is a normal measure on κ, then we can construct the model $L[U]$, which is a model in which $U \cap L[U]$ is the unique normal measure on κ. In this model κ is the only measurable cardinal and any embedding is an iterated ultrapower embedding starting from $U \cap L[U]$. If we want a single object to capture $o(\kappa) > 1$, this requires us to have not only multiple measures on κ, but also measures on many cardinals below κ.

If our goal is to capture this concept in a minimal fashion, the simplest well-founded order which has rank α is the ordinal α itself. So we should expect this to be a sequence of measures of some length. And we want it to be coherent, in the sense that each measure on the sequence which capture that very sequence up to it.

Definition 3.49. We say that $\mathcal{U}$ is a *coherent sequence of measures* if it is a function such that:

1. $\operatorname{dom}(\mathcal{U}) = \{\langle\kappa, \beta\rangle \mid \kappa < \operatorname{len}(\mathcal{U}) \text{ and } \beta < o^{\mathcal{U}}(\kappa)\}$, where $\operatorname{len}(\mathcal{U})$ is a cardinal which is the length of the sequence, and $o^{\mathcal{U}}$ is a function mapping ordinals to ordinals.
2. If $\langle\kappa, \beta\rangle \in \operatorname{dom}(\mathcal{U})$, then $\mathcal{U}(\kappa, \beta)$ is a normal measure on κ.
3. If $U = \mathcal{U}(\kappa, \beta)$, then $o^{j_U(\mathcal{U})}(\kappa) = \beta$ and $j_U(\mathcal{U})(\kappa, \beta') = \mathcal{U}(\kappa, \beta')$ for all $\beta' < \beta$.

In other words, $\mathcal{U}$ is a sequence of measures on many cardinals at once, which have the property that each measure in the sequence "captures" the sequence up to the measure. So, not only this will witness a Mitchell order, but in fact, it will be a well-ordered.

Fact 3.50. *If $\mathcal{U}$ is a coherent sequence of measures, then $L[\mathcal{U}]$ is a model where for any cardinal κ, $L[\mathcal{U}] \models o(\kappa) = o^{\mathcal{U}}(\kappa)$. In particular, κ is measurable in $L[\mathcal{U}]$ if and only if $o^{\mathcal{U}}(\kappa) > 0$.*

Chapter 4

Strength in numbers!

4.1 Strong cardinals

We saw that a measurable cardinal is a critical point of an embedding. If we wanted the critical point to remain measurable in the target model, it turns out that we needed to assume stronger axioms than just a single measurable cardinals hold in the universe. We also saw that if U is a measure on κ, then $\mathrm{Ult}(V, U)$ is not $(2^\kappa)^+$-closed, since U itself is not in the model. So we can ask, what does it take to "know more" about the universe in the target model?

Definition 4.1. We say that a cardinal κ is an *α-strong cardinal* if there exists an elementary embedding $j\colon V \to M$ such that $\mathrm{crit}(j) = \kappa$, $\kappa + \alpha < j(\kappa)$, and $V_{\kappa+\alpha} \subseteq M$, we will often refer to the embedding j as being an *α-strong embedding*. We say that κ is a *strong cardinal* if it is α-strong for all α.

REMARK.

1. In many places we forego the $\kappa + \alpha$ in favour of α itself. Namely, $\kappa + \alpha$-strong would be α-strong in our terminology. If $\alpha > \kappa^2$, then these two coincide, and in any case the term "strong cardinal" is unaffected.
2. In some places the requirement that $\kappa + \alpha < j(\kappa)$ is also removed. We will see later why if this is not the case, there will be a finite iterate of j which does satisfy this property, but for now it is a simplifying assumption that we aim to keep.

Exercise 4.2. Show that κ is a strong cardinal if and only if for every x there exists M and $j\colon V \to M$ elementary with $\mathrm{crit}(j) = \kappa$ and $x \in M$.

Exercise 4.3. κ is measurable if and only if κ is 1-strong.

Exercise 4.4 (*) . If U is a measure on κ, then j_U is not a 2-strong embedding.

If 2-strong cardinals, and therefore strong cardinals, are certainly above measurable, what can we say about their Mitchell order? It comes, perhaps, as no surprise, that the Mitchell order is in fact maximal.

Theorem 4.5. *If κ is 2-strong, then $o(\kappa) = (2^\kappa)^+$.*

Proof. Given any collection of at most 2^κ measures on κ we can encode them as a single subset of $\mathcal{P}(\kappa)$. By Lemma 4.6, there is a normal measure U for which this set of measures in $\mathrm{Ult}(V, U)$. Therefore, by induction we can prove that for any $\alpha < (2^\kappa)^+$, there is a normal measure of Mitchell rank at least α. □

Lemma 4.6. *If κ is 2-strong, then for every $A \subseteq \mathcal{P}(\kappa)$ there is a normal measure on κ, U, such that $A \in \mathrm{Ult}(V, U)$.*

Proof. Towards contradiction assume that "There exists a subset of $\mathcal{P}(\kappa)$ which is not in any ultrapower by a normal measure on κ". Let $j\colon V \to M$ be a 2-strong embedding with $\mathrm{crit}(j) = \kappa$, then $A \in M$. Since M computes ultrapowers of $V_{\kappa+2}$ (by measures on κ) correctly, M agrees on the truth of our assumption.

Let $D = \mathrm{der}(j)$ and let $k\colon \mathrm{Ult}(V, D) \to M$ be the factor elementary embedding. Then $\mathrm{crit}(k) = j_D(\kappa) > \kappa$, and since k is an elementary map, $\mathrm{Ult}(V, D)$ agrees that "There exists a subset of $\mathcal{P}(\kappa)$ which is not in any ultrapower by a normal measure on κ". Let $X \in \mathrm{Ult}(V, D)$ be such witness, then $k(X) = X$, so in M it holds that X cannot be in an ultrapower by a normal measure on κ. However, this is a contradiction, since $D \in M$, and so M can see that X is in $\mathrm{Ult}(V, D)$. $\square$

We saw that measurable cardinals corresponded to ultrapower embeddings, so a natural question is if there is a nice combinatorial objects that will capture the notion of a strong embedding. As we saw above, ultrapower embeddings will not let us even capture 2-strong embeddings. One can ask, therefore, if there is a canonical combinatorial object which can capture these notions?

4.2 Extenders

4.2.1 Deriving an extender

For the purpose of this section, let us fix an elementary embedding $j\colon V \to M$ with $\mathrm{crit}(j) = \kappa$. We do note require that j is strong or that M is closed. Let us also fix $\kappa \leq \lambda \leq j(\kappa)$. We want to approximate j with ultrapowers, but this may very well be impossible, as $j_{\mathrm{der}(j)}(\kappa)$ could be significantly smaller than $j(\kappa)$. Indeed, much worse could happen, it can be that $\mathrm{der}(j) \in M$, in which case even iterating the ultrapower by $\mathrm{der}(j)$ will not be a good approximation.

Instead, we will derive many different measures and use them to define a directed system of models. So, for some $\lambda \leq j(\kappa)$ we want to find some $N(\lambda)$ and elementary embedding $i_\lambda\colon V \to N(\lambda)$ which factors j, so that $j = k_\lambda \circ i_\lambda$, and $\mathrm{crit}(k_\lambda) \geq \lambda$. In other words, we will approximate j up to λ. If j was an α-strong embedding, setting $\lambda = \kappa + \alpha$ we get that i_λ and $N(\lambda)$ are as close as we can get to an ultrapower approximation of j and M.

For $a \in [\lambda]^{<\omega}$ we define the measure on $[\kappa]^{|a|}$,

$$E_a = \{X \subseteq [\kappa]^{|a|} \mid a \in j(X)\}.$$

Exercise 4.7. E_a is a κ-complete ultrafilter on $[\kappa]^{|a|}$.

If $a \subseteq b$ are two finite subsets of λ, $b = \{\alpha_0 < \cdots < \alpha_n\}$ and $a = \{\alpha_{i_0} < \cdots < \alpha_{i_m}\}$, then there is a projection map $\pi_{b,a}\colon [\kappa]^{|b|} \to [\kappa]^{|a|}$ given by $\pi_{b,a}(\{\xi_0 < \cdots < \xi_n\}) = \{\xi_{i_0} < \cdots < \xi_{i_m}\}$.

Exercise 4.8. $X \in E_a \iff \pi_{b,a}^{-1}(X) \in E_b$.

Exercise 4.9. The map $j_{a,b}\colon \mathrm{Ult}(V, E_a) \to \mathrm{Ult}(V, U_b)$ given by $j_{a,b}([f]) = [f \circ \pi_{b,a}]$ is the factor embedding between j_{E_a} and j_{E_b}.

Exercise 4.10 (*) . The system $\{\mathrm{Ult}(V, E_a), j_{a,b} \mid a, b \in [\lambda]^{<\omega}\}$ is directed and has a well-founded direct limit.

Definition 4.11. Let $j\colon V \to M$ be an elementary embedding and let $\kappa = \mathrm{crit}(j)$, $\lambda \in [\kappa, j(\kappa)]$. We say that $E = \{E_a \mid a \in [\lambda]^{<\omega}\}$ is a (κ, λ)-*extender*. We say that λ is the *length* of the extender. We write $\mathrm{Ult}(V, E)$ to denote the direct limit of the system, and j_E is the associated embedding.

We can represent E in many different ways, e.g.

$$E = \{\langle a, X\rangle \mid a \in [\lambda]^{<\omega}, X \subseteq [\kappa]^{<\omega}, a \in j(X)\}.$$

In that case, $E_a = \{X \in [\kappa]^{|a|} \mid \langle a, X\rangle \in E\}$ is the same ultrafilter we defined before.

Exercise 4.12. If E is a (κ, λ)-extender, then $\mathrm{Ult}(V, E) = \{j_E(f)(a) \mid a \in [\lambda]^{<\omega}, f\colon [\kappa]^{<\omega} \to V\}$.

4.2.2 Extender ultrapowers

Theorem 4.13. *Let $j\colon V \to M$ be an elementary embedding with $\mathrm{crit}(j) = \kappa$, $\kappa \leq \lambda \leq j(\kappa)$ and let E be the derived (κ, λ)-extender. Then $k\colon \mathrm{Ult}(V, E) \to M$ given by $k([a, f]) = j(f)(a)$ satisfies $k \circ j_E = j$. In particular, k is an elementary embedding with $\mathrm{crit}(k) = j_E(\kappa) \geq \lambda$.*

Proof. Suppose that $\alpha < \kappa$, then for each a, E_a is a κ-complete ultrafilter, it is not hard to check that $\mathrm{crit}(j_{E_a}) \geq \kappa$, so $\mathrm{crit}(j_E) \geq \kappa$. Since $E_{\{\kappa\}} \cong \mathrm{der}(j)$ under the bijection $\alpha \mapsto \{\alpha\}$, we get that $\mathrm{crit}(j_{E_{\{\kappa\}}}) = \kappa$, and so $j_E(\kappa) > \kappa$ as wanted.

To see the elementarity of k holds note that

$$\begin{aligned} &\mathrm{Ult}(V, E) \models \varphi([a, f]) && \text{if and only if} \\ &\mathrm{Ult}(V, E_a) \models \varphi(j_{E_a}(f)(a)) && \text{if and only if} \\ &M \models \varphi(j(f)(a)). \end{aligned}$$

In particular, since $V \models \mathrm{crit}(j) = \kappa$, $\mathrm{Ult}(V, E) \models \mathrm{crit}(k) = j_E(\kappa)$. Finally, if $\alpha < \lambda$, let $[a, f]$ be a pair representing α in $\mathrm{Ult}(V, E)$, then $j_{E_a}(f)(a) = \alpha$, therefore $j(f)(a) = j_{E_a}(f)(a) = \alpha$, and so $k(\alpha) = \alpha$, so $\mathrm{crit}(k) \geq \lambda$ as wanted. $\square$

Theorem 4.14. *$E \subseteq [\lambda]^{<\omega} \times \mathcal{P}([\kappa]^{<\omega})$ is a (κ, λ)-extender if and only if for each $a \in [\lambda]^{<\omega}$ setting $E_a = \{X \mid \langle a, X\rangle \in E\}$, satisfies:*

1. *E_a is a κ-complete ultrafilter on $[\kappa]^{|a|}$ for all a.*
2. *E_a is not κ^+-complete for each least one a.*
3. *For each $\alpha < \kappa$ there is a such that $\{s \in [\kappa]^{|a|} \mid \alpha \in s\} \in E_a$.*
4. *The projections $\pi_{b,a}$ defined previously satisfy Exercise 4.8.*
5. *If $\{s \in [\kappa]^{|a|} \mid f(s) < \max s\} \in E_a$, then there is some β such that*

$$\{t \in [\kappa]^{|a \cup \{\beta\}|} \mid (f \circ \pi_{a \cup \{\beta\}, a})(t) \in t\} \in E_{a \cup \{\beta\}}.$$

6. *The direct limit of the ultrapower models is well-founded.*

Definition 4.15. We say that an extender, E, is α-strong if α is the largest ordinal such that $V_{\kappa+\alpha} \subseteq \mathrm{Ult}(V, E)$. We will also say, in this case, that α is the *strength* of E.

Exercise 4.16 (**) . If κ is a strong cardinal, then for all $\lambda > \kappa$ there is a (κ, λ)-extender, E, and $f\colon \kappa \to \kappa$ such that $j_E(f)(\kappa) = \lambda$.

Exercise 4.17. The direct limit of a (κ, λ)-extender is well-founded if and only if for every sequence $a_n \in [\lambda]^{<\omega}$ and $X_n \in E_{a_n}$, there is a function $f\colon \bigcup_{n<\omega} a_n \to \kappa$ such that $f``a_n \in X_n$ for all $n < \omega$.

If an extender satisfies the condition in the above exercise we say that it is *countably closed.*

Theorem 4.18. *κ is an α-strong cardinal if and only if there exists an α-strong $(\kappa, |V_{\kappa+\alpha}|^{+})$-extender.* □

Remark. 1. It is worth noting that we say that we can derive these extenders from every embedding. It is also true that we can use any ordinal λ. If $\lambda > j(\kappa)$, however, we might need to have the filters defined on larger ordinals. We can therefore define extenders in general as having a sequence of critical points, $\langle \kappa_a \mid a \in [\lambda]^{<\omega}\rangle$, and have the ultrafilters concentrate on $[\kappa_a]^{|a|}$ instead. If there is more than one critical point, the extender is called *long* and otherwise it is called *short.*

2. Extenders will often be defined on mice, which are usually taken as models of a fragment of ZFC which have a largest cardinal, and a predicate for a measure or an extender which allows us to iterate various embeddings and "stretch" the mouse to become taller and taller.

3. We can define a coherent sequence of extenders in a way that is similar to coherent sequence of measures, which allows us to define "extender models" which are of the form $L[\mathcal{E}]$ for some coherent sequence of extenders, $\mathcal{E}$. Here the story can get much more complicated as the coherence becomes more difficult to maintain as we have more extenders. One condition of interest is "overlapping" (or "non-overlapping") sequence of extenders which tells us that the extenders can be applied in a "more or less independent way", so their "reach" (in both critical points and strength) is or is not overlapping.

4.3 Structural consequences

Since strong cardinals capture a significant amount of information in various embeddings, one can ask if there can be $j\colon V \to V$ which is non-trivial. We will spoil the answer now and reveal that *Kunen's inconsistency theorem* shows that such embedding contradicts the Axiom of Choice when it is definable. In fact, if $j\colon V_{\lambda+2} \to V_{\lambda+2}$ is a non-trivial elementary embedding, then the Axiom of Choice must fail. We will see a proof of this theorem later, but the theorem has an interesting consequence.

Theorem 4.19. *Suppose that a strong cardinal exists, then $V \neq L[A]$ for all A.*

Proof. Let κ be a strong cardinal and let A be a set of ordinals such that $V = L[A]$. Then there is some $j\colon V \to M$ with $\operatorname{crit}(j) = \kappa$ and $A \in M$. Since $A \in M$, $L[A] \subseteq M \subseteq V = L[A]$, so $M = L[A]$ as well, and therefore $j\colon L[A] \to L[A]$ allows us to derive an extender and have a definable elementary embedding $j\colon V \to V$. □

Theorem 4.20. *If κ is a strong cardinal, then $V_\kappa \prec_{\Sigma_2} V$.*

Proof. Let $\varphi(x, y, z)$ be a Δ_0-formula and suppose that $a \in V_\kappa$ such that $\exists y \forall z \varphi(a, y, z)$ holds in V. Let b be such that $\forall z \varphi(a, b, z)$ holds, then there is a strong embedding $j\colon V \to M$ such that

$b \in M$ and $\operatorname{crit}(j) = \kappa$ and $j(\kappa) > \operatorname{rank}(b)$. Therefore in M it holds that $(\exists y \in V_{j(\kappa)})\forall z\varphi(a, y, z)$, so by elementarity, we can assume that $b \in V_\kappa$. Therefore $\forall z\varphi(a, b, z)$ holds, but since V_κ is transitive and $\forall z\varphi(a, b, z)$ is absolute, the conclusion follows.

In the other direction, suppose that $V_\kappa \models \exists y\forall z\varphi(a, y, z)$, let $b \in V_\kappa$ be a witness, so that $V_\kappa \models \forall z\varphi(a, b, z)$. For any $c \in V$ there is some strong embedding with critical point κ such that $c \in M$ and $j(\kappa) > \operatorname{rank}(c)$. In this M it holds that $V_{j(\kappa)} \models \forall z\varphi(a, b, z)$, so $V_{j(\kappa)} \models \varphi(a, b, c)$ in M. But since $V^M_{j(\kappa)}$ is a transitive set, $\varphi(a, b, c)$ holds in V. □

Exercise 4.21. If κ is inaccessible, then $V_\kappa \prec_{\Sigma_1} V$.

4.4 Superstrong cardinals

Definition 4.22. A cardinal κ is called a *superstrong cardinal* if there exists an elementary embedding $j\colon V \to M$ with $\operatorname{crit}(j) = \kappa$ and $V_{j(\kappa)} \subseteq M$.

As before, we will refer to such an embedding as a superstrong embedding.

Theorem 4.23. *If κ is a superstrong cardinal, then there exists a transitive model of* ZFC *with a strong cardinal.*

Proof. Let $j\colon V \to M$ be a superstrong embedding with $\operatorname{crit}(j) = \kappa$. Therefore $V_{j(\kappa)}$ is an elementary end-extension of V_κ, and so in particular $j(\kappa)$ is a strong limit cardinal. We can derive an extender from j such that for all $\lambda < j(\kappa)$, the restriction of the extender to λ is in $V_{j(\kappa)}$. Therefore $V_{j(\kappa)} \models \kappa$ is a strong cardinal. □

We will see later on that the consistency strength of a superstrong is much more than just that of a strong cardinal, but even the above theorem tells us that it is a stronger axiom. However, this is where we start seeing a stark difference between the "largeness hierarchy" and the "consistency hierarchy".

Theorem 4.24. *Suppose that there exist both a superstrong cardinal and a strong cardinal. Then the least superstrong cardinal is below the least strong cardinal.*

Proof. Fix a superstrong embedding $j\colon V \to M$, namely $V_{j(\operatorname{crit} j)} \subseteq M$. We can derive an extender from j which will witness this embedding, so the statement that there exists a superstrong cardinal is equivalent to "There exists κ and β and a (κ, β)-extender E such that $j_E(\kappa) = \beta$ and $V_\beta \subseteq M$", let κ, β, and E be such objects. Let $\lambda < \kappa$ be a strong cardinal, then there is a strong embedding, $i\colon V \to N$, which captures $V_{j_E(\kappa)+1}$ as well as E, etc. with $i(\lambda) > \beta$. Therefore in N we have that κ is a superstrong cardinal, as the extender is in N and $V_\beta \subseteq N$. In particular, N thinks that there is a superstrong cardinal below $i(\lambda)$, so in V there is a superstrong cardinal below λ. □

Chapter 5

Woodin cardinals

Definition 5.1. We say that κ is a λ-A-*strong cardinal*, for some class A, if there is a λ-strong embedding j such that $\operatorname{crit}(j) = \kappa$ and $j(A) \cap V_\lambda = A \cap V_\lambda$. If κ is λ-A-strong for all λ we say that it is A-*strong*.

Remark. As we are going to be concerned with sufficiently strong embeddings, we will make the tacit assumption that $\kappa + \lambda = \lambda$, as to never worry about $V_\lambda \neq V_{\kappa+\lambda}$.

Proposition 5.2. *Suppose that δ is an inaccessible cardinal and $A \subseteq V_\delta$, if $\kappa < \delta$ is a γ-A-strong cardinal for some $\gamma < \delta$, then there is an extender witnessing this in V_δ.*

Proof. Let $j\colon V \to M$ be a γ-A-strong embedding with $\operatorname{crit}(j) = \kappa$. Taking some $\lambda < \delta$ such that $\gamma < \lambda = |V_\lambda|$, we derive E, a (κ, λ)-extender from j, then $E \in V_\delta$. Let us argue that j_E is a γ-A-strong embedding. Let $k\colon \operatorname{Ult}(V, E) \to M$ be the factor embedding, then $\operatorname{crit}(k) \geq \lambda$, so $\operatorname{crit}(k) \geq \gamma$, and therefore $k(V_\gamma^{\operatorname{Ult}(V,E)}) = V_\gamma^M = V_\gamma$, so k is γ-strong. Moreover, as $\operatorname{crit}(k) > \gamma$ we get that

$$A \cap V_\gamma = j(A) \cap V_\gamma = k(j_E(A)) \cap V_\gamma = k(j_E(A) \cap V_\gamma) = j_E(A) \cap V_\gamma. \qquad \square$$

Exercise 5.3. Let δ be an inaccessible cardinal and $E \in V_\delta$ is an extender, then $j_E(\delta) = \delta$.

Definition 5.4. We say that δ is a *Woodin cardinal* if for every $A \subseteq V_\delta$,

$$V_\delta \models \text{There is a proper class of } A\text{-strong cardinals.}$$

Equivalently, there are unboundedly many $\kappa < \delta$ such that for all $\gamma < \delta$, κ is γ-A-strong.

Proposition 5.5. *If δ is a Woodin cardinal, then δ is inaccessible.*

Proof. If δ is Woodin, let $\alpha < \delta$ and let $f\colon V_\alpha \to \delta$ be any function. Then $f \subseteq V_\delta$, find $\kappa > \alpha$ which is f-strong in V_δ, pick some $x \in V_\alpha$, and with $\lambda = \operatorname{rank}(f(x)) + 1$ let $j\colon V \to M$ be a λ-f-strong embedding with $\operatorname{crit}(j) = \kappa$. Then $j(f) \cap V_\lambda = f \cap V_\lambda$, so $j(f) = f$ as $\operatorname{dom} f = V_\alpha \in V_\kappa$. However, since $j(f)(x) = f(x) \in V_\lambda \subseteq V_{j(\kappa)}$, by elementarity, $f(x) \in V_\kappa$. Therefore $\operatorname{rng} f \subseteq \kappa < \delta$. $\square$

Exercise 5.6. If δ is Woodin, then δ is a Mahlo cardinal.

Exercise 5.7. The least Woodin cardinal is not weakly compact. But every Woodin cardinal is the limit of measurable cardinals.

Lemma 5.8. *Suppose that δ is a cardinal such that for every $f\colon \delta \to \delta$ there is some $\kappa < \delta$ such that $f``\kappa \subseteq \kappa$ and an embedding $j\colon V \to M$ such that $\operatorname{crit}(j) = \kappa$ and j is $j(f)(\kappa)$-strong, then δ is inaccessible.*

Proof. If δ is a singular cardinal, let $f\colon \operatorname{cf}(\delta) \to \delta$ be a cofinal function with $\min \operatorname{rng} f > \operatorname{cf}(\delta)$ and extend it so that $f(\alpha) = 0$ for $\alpha \geq \operatorname{cf}(\delta)$. Then there is no $\kappa < \delta$ which is closed under f. So δ must be regular. Next, note that if $\alpha < \delta$, then taking the function $f(\xi) = \alpha$, it must be that if $f``\kappa \subseteq \kappa$, then $\alpha < \kappa$. Therefore, δ is the limit of measurable cardinals, so it must be a strong limit, and therefore inaccessible. $\square$

Theorem 5.9. *The following are equivalent for a cardinal δ:*

1. *For any $A \subseteq V_\delta$, the set $\{\kappa < \delta \mid V_\delta \models \kappa \text{ is } A\text{-strong}\}$ is stationary in δ.*
2. *δ is Woodin.*
3. *For any $A \subseteq V_\delta$, there exists $\kappa < \delta$ such that $V_\delta \models \kappa$ is A-strong.*
4. *For every $f\colon \delta \to \delta$, there is $\kappa < \delta$ such that $f``\kappa \subseteq \kappa$ and an embedding $j\colon V \to M$ such that $\operatorname{crit}(j) = \kappa$ and $V_{j(f)(\kappa)} \subseteq M$.*

Proof. (1)→(2) and (2)→(3) are trivial. Let us show that (3) implies (4). Let $f\colon \delta \to \delta$ be a function and let $\kappa < \delta$ be the f-strong cardinal (in V_δ). Let $\gamma = \sup f``\kappa + 42$ and let $j\colon V \to M$ be a γ-f-strong embedding, then for $\alpha < \kappa$,

$$j(f(\alpha)) = j(f)(j(\alpha)) = j(f)(\alpha) = f(\alpha) < \gamma < j(\kappa),$$

and so $f(\alpha) < \kappa$, so $\gamma \leq \kappa$, and therefore $f``\kappa \subseteq \kappa$. Moreover, for $\alpha = \kappa$, $\langle \kappa, f(\kappa)\rangle \in V_\gamma$, therefore $j(f)(\kappa) = f(\kappa)$, and in particular $V_{j(f)(\kappa)} \subseteq V_\gamma \subseteq M$, as wanted.

Finally, (4)→(1), let $A \subseteq V_\delta$ and let C be a club. Define $g\colon \delta \to \delta$ given by

$$g(\alpha) = \begin{cases} 0 & \text{if } V_\delta \models \alpha \text{ is } A\text{-strong} \\ \gamma & \gamma \text{ is the least such that } \alpha \text{ is not } \gamma\text{-}A\text{-strong in } V_\delta. \end{cases}$$

Next define $f(\alpha) = \max\{g(\alpha) + 5, \min(C \setminus \alpha)\}$. By the assumption, there is some $\kappa < \delta$ such that $f``\kappa \subseteq \kappa$ and there is $j\colon V \to M$ with $\operatorname{crit}(j) = \kappa$ and $V_{j(f)(\kappa)} \subseteq M$, moreover since δ is inaccessible, we may assume that j is given by an extender in V_δ, so in particular $j(\delta) = \delta$. We will show that $\kappa \in j(C)$ and that in M, κ is γ-$j(A)$-strong for all $\gamma < \delta$. Since $f``\kappa \subseteq \kappa$, it must be that $C \cap \kappa$ is a club in κ, so $\kappa \in j(C)$.

If $j(g)(\kappa) = 0$, then the proof is complete, since in M it means that κ is γ-$j(A)$-strong for all $\gamma < \delta$. Suppose this is not the case. Since $j(f)(\kappa) > j(g)(\kappa) + 1$, we can derive from j a $(\kappa, j(g)(\kappa)+1)$-extender, E. Since j was $j(f)(\kappa)$-strong, $E \in M$, so we can compute $\operatorname{Ult}(M, E)$ inside M. However, the factor embedding $k\colon \operatorname{Ult}(V, E) \to M$ has a critical point $j_E(\kappa)$, so $V^{\operatorname{Ult}(V,E)}_{j(g)(\kappa)} = V^{\operatorname{Ult}(M,E)}_{j(g)(\kappa)}$, and so j^M_E is a $j(g)(\kappa)$-$j(A)$-strong embedding in M. That is,

$$M \models E \text{ is } j(g)(\kappa)\text{-}j(A)\text{-strong},$$

but this implies that in V we have that κ is at least $g(\kappa)$-A-strong, using the same extender, which is a contradiction to the assumption that $g(\kappa) > 0$. $\square$

We can now complete the proof that superstrong cardinals are in fact significantly stronger, consistency-wise, than strong cardinals. Indeed, superstrong cardinals are significantly stronger than Woodin cardinals.

Theorem 5.10. *If κ is superstrong, then there is a normal measure on κ concentrating on Woodin cardinals.*

Proof. Let $j\colon V \to M$ be a superstrong embedding with $\mathrm{crit}(j) = \kappa$, then for every $f\colon \kappa \to \kappa$, $j(f)\colon j(\kappa) \to j(\kappa)$ and $j(f) \upharpoonright \kappa = f$, so $j(f)``\kappa \subseteq \kappa$. Let E be the $(\kappa, |V_{j(f)(\kappa)} + 1|^M)$-extender derived from j, it is not hard to see that $E \in V_{j(\kappa)}$, computing $\mathrm{Ult}(M, E)$ we have that j_E^M is a $j(f)(\kappa)$-strong embedding. Therefore, by elementarity,

$$\{\alpha < \kappa \mid f``\alpha \subseteq \alpha \text{ and there is an extender } E \text{ which is } j_E(f)(\alpha)\text{-strong}\} \in \mathrm{der}(j). \qquad \square$$

Definition 5.11. We say that κ is a *Shelah cardinal* if for every $f\colon \kappa \to \kappa$ there is an elementary embedding $j\colon V \to M$ with $\mathrm{crit}(j) = \kappa$ and $V_{j(f)(\kappa)} \subseteq M$.

Exercise 5.12. If κ is Shelah, then it is Woodin.

Chapter 6

Higher compactness

6.1 Strongly compact cardinals

Definition 6.1. We say that κ is a *strongly compact cardinal* if $\mathcal{L}_{\kappa,\kappa}$ is strongly compact. Namely, whenever T is a κ-satisfiable $\mathcal{L}_{\kappa,\kappa}$-theory, then T is satisfiable.

Theorem 6.2. *Let κ be an uncountable cardinal, then the following are equivalent:*

1. *κ is strongly compact.*
2. *Every κ-complete filter can be extended to a κ-complete ultrafilter.*

Proof. (1)→(2): Suppose that κ is a strongly compact cardinal and $\mathcal{F}$ is a κ-complete filter over some set S. Consider the $\mathcal{L}_{\kappa,\kappa}$-theory, T', of the structure $\langle S \cup \mathcal{P}(S), \in, X\rangle_{X\subseteq S}$, where X is a constant symbol for each subset of S. Adding a new constant symbol c and letting T denote T' along with the sentence $c \in X$ for every $X \in \mathcal{F}$. Then every $T_0 \in [T]^{<\kappa}$ has a model, since $\mathcal{F}$ is κ-complete. Let $M \models T$, and let $\mathcal{U} = \{X \subseteq S \mid M \models c \in X\}$. Trivially, $\mathcal{U}$ is an ultrafilter extending $\mathcal{F}$, to see that $\mathcal{U}$ is κ-complete, if $\{X_\alpha \mid \alpha < \gamma\} \subseteq \mathcal{U}$, then $M \models \bigwedge_{\alpha<\gamma} c \in X_\alpha$, so $\bigcap_{\alpha<\kappa} X_\alpha \in \mathcal{U}$ as well.

(2)→(1): Suppose that T is a κ-satisfiable $\mathcal{L}_{\kappa,\kappa}$ theory. Then for each $s \in [T]^{<\kappa}$ there is a model, M_s. Consider the filter on $[T]^{<\kappa}$ generated by $\{\{s \in [T]^{<\kappa} \mid \varphi \in s\} \mid \varphi \in T\}$, it is not hard to check that this filter is κ-complete, so it extends to an ultrafilter U. Then $\prod_{s\in[T]^{<\kappa}} M_s/U$ is a model of T, since for each $\varphi \in T$, the set $\{s \mid M_s \models \varphi\} \supseteq \{s \mid \varphi \in s\} \in U$. □

We denote by $\mathcal{P}_\kappa(X)$ the set $[X]^{<\kappa}$.[11]

Definition 6.3. A filter $\mathcal{U}$ over $\mathcal{P}_\kappa(X)$, is *fine* if for every $x \in X$, $\{s \in \mathcal{P}_\kappa(X) \mid x \in s\} \in \mathcal{U}$.

Definition 6.4. We say that κ is λ-*compact* if there is a fine κ-complete ultrafilter over $\mathcal{P}_\kappa(\lambda)$.

Theorem 6.5. *κ is strongly compact if and only if it is λ-compact for all $\lambda \geq \kappa$.*

Proof. To prove that κ is strongly compact, let T be an $\mathcal{L}_{\kappa,\kappa}$-theory of size λ and let U be a fine κ-complete ultrafilter on $\mathcal{P}_\kappa(T)$, then for each $s \in \mathcal{P}_\kappa(T)$ fix some $M_s \models T_0$, and it is easy to check that $\prod_{s\in\mathcal{P}_\kappa(T)} M_s/U \models T$. In the other direction, note that if $\lambda \geq \kappa$, then

$$\{\{s \in \mathcal{P}_\kappa(\lambda) \mid \alpha \in s\} \mid \alpha < \lambda\}$$

is a filter base for a κ-complete filter, so by (2) in Theorem 6.2, κ is λ-compact. □

[11] Why yes, *it is* unfortunate that we have two ways to denote the same set.

Theorem 6.6. *κ is strongly compact if and only if for every λ there is an elementary $j\colon V \to M$ such that* $\operatorname{crit}(j) = \kappa$*, $M^\kappa \subseteq M$, and there is some $X \in M$ such that $M \models |X| < j(\kappa)$ and $j``\lambda \subseteq Y$.*

Proof. Suppose that κ is strongly compact and $\lambda \geq \kappa$, otherwise there is nothing to check, let $M = \operatorname{Ult}(V, U)$ where U is a fine measure on $\mathcal{P}_\kappa(\lambda)$. It is not hard to check that $M^\kappa \subseteq M$ as in Proposition 3.18 and $\operatorname{crit}(j) = \kappa$. Let $X = [\mathrm{id}]$, then for all $\alpha < \lambda$, since U is fine, $\{s \in \mathcal{P}_\kappa(\lambda) \mid \alpha \in s\} \in U$, so $\alpha \in X$. However, since $\{x \in \mathcal{P}_\kappa(\lambda) \mid |x| < \kappa\} = \mathcal{P}_\kappa(\lambda) \in U$, it means that $M \models |X| < j(\kappa)$.

In the other direction, suppose that $j\colon V \to M$ is such an embedding, let

$$U = \{A \subseteq \mathcal{P}_\kappa(\lambda) \mid X \cap \mathcal{P}_{j(\kappa)}(j(\lambda))^M \in j(A)\},$$

it is not hard to verify that U is a fine κ-complete measure on $\mathcal{P}_\kappa(\lambda)$. $\square$

Exercise 6.7. If j is as above, then $\lambda < j(\kappa)$.

We say that j as in the theorem is a λ-*strongly compact embedding.*

Exercise 6.8. $j\colon V \to M$ is a λ-strongly compact embedding if and only if for every $Z \subseteq M$ such that $|Z| = \lambda$, there is some $Y \in M$ for which $M \models |Y| < j(\operatorname{crit}(j))$ and $Z \subseteq Y$.

One is tempted to expect that strongly compact are superstrong, or at least Woodin, or at least strong. This, however, is not provable.

Fact 6.9. *It is consistent, relative to suitable large cardinal axioms, that the least strongly compact cardinal is the least measurable cardinal. In particular, it is consistent that $o(\kappa) = 1$ while κ is strongly compact. However, if a strongly compact cardinal exists, then for every λ, there is an inner model M such that $\{\alpha \mid M \models \alpha \text{ is measurable}\}$ maps onto λ. In fact, much more is true. There is an inner models with infinitely many Woodin cardinals.*

6.1.1 Structural results of strongly compact cardinals

Theorem 6.10. *Suppose that κ is measurable and it is the limit of strongly compact cardinals, then κ is strongly compact.*

Proof. Let D be a measure on κ concentrating on $S = \{\alpha < \kappa \mid \alpha \text{ is strongly compact}\}$.[12] Let $\lambda \geq \kappa$ be some cardinal and U_α a fine measure on $\mathcal{P}_\alpha(\lambda)$ for $\alpha \in S$. We define a fine measure, U, on $\mathcal{P}_\kappa(\lambda)$ by

$$U = \{X \subseteq \mathcal{P}_\kappa(\lambda) \mid \{\alpha \in S \mid X \cap \mathcal{P}_\alpha(\lambda) \in U_\alpha\} \in D\}.$$

It is not hard to check that U is a κ-complete ultrafilter on $\mathcal{P}_\kappa(\lambda)$. It is fine, since for every $\xi < \lambda$ and $\alpha \in S$, $\{s \in \mathcal{P}_\alpha(\lambda) \mid \xi \in s\} \in U_\alpha$, so $\{s \in \mathcal{P}_\kappa(\lambda) \mid \xi \in s\} \in U$. $\square$

The following theorem is due to Magidor and Shelah and shows that even at singular limits, strongly compact cardinals have an effect on the universe.

Theorem 6.11. *Suppose that λ is a singular limit of strongly compact cardinals. Then λ^+ has the tree property.* $\square$

The following theorem is due to Solovay which shows how the continuum function is affected by strongly compact cardinals.

Theorem 6.12. *Suppose that κ is a strongly compact cardinal. Then* SCH *holds above κ. Namely, if $\lambda > \kappa$ is a singular cardinal, then $\lambda^{\operatorname{cf}(\lambda)} = 2^\lambda$.* $\square$

[12] Note that D might not be a normal measure.

6.2 Supercompact cardinals

Definition 6.13. We say that a measure U on $\mathcal{P}_\kappa(\lambda)$ is *normal* if whenever $f\colon \mathcal{P}_\kappa(\lambda) \to \lambda$ is a choice function, then there is $A \in U$ such that f is constant on A.

Exercise 6.14. U is normal on $\mathcal{P}_\kappa(\lambda)$ if and only if whenever $\{X_\alpha \mid \alpha < \lambda\} \subseteq U$,

$$\mathop{\triangle}_{\alpha<\lambda} X_\alpha = \left\{x \in \mathcal{P}_\kappa(\lambda) \;\middle|\; x \in \bigcap_{\alpha \in x} X_\alpha\right\} \in U.$$

Definition 6.15. We say that κ is a λ-*supercompact cardinal* if and only if there is a fine and normal κ-complete ultrafilter on $\mathcal{P}_\kappa(\lambda)$. If κ is λ-supercompact for all $\lambda \geq \kappa$, we say that κ is *supercompact*. We will refer to such ultrafilters as *(λ-)supercompact measures*.

Easily, if κ is supercompact, then κ is strongly compact. Let us study a few structural consequences of ultrapowers by supercompact measures.

Proposition 6.16. *Let U be a supercompact measure on $\mathcal{P}_\kappa(\lambda)$, let $M = \mathrm{Ult}(V, U)$ and $j = j_U$. Then the following hold:*

1. $[\mathrm{id}]$ *represents $j``\lambda$ in M.*
2. *$X \in U$ if and only if $j``\lambda \in j(X)$.*
3. *$M^\gamma \subseteq M$ if and only if $j``\gamma \in M$.*

Proof. For (1), suppose that $\gamma < \lambda$, since U is a fine measure, $j(\gamma) \in [\mathrm{id}]$, since it is a choice function from $\{s \in \mathcal{P}_\kappa(\lambda) \mid \gamma \in s\}$. On the other direction, if $[f] \in [\mathrm{id}]$, then f must be a choice function by normality, so it is a constant function into λ, so $[f] = j(\gamma)$ for some $\gamma < \lambda$.

Note that (2) follows immediately from (1) and the definition of ultrapowers.

To see (3) holds, note that one direction is trivial, so it is enough to show that if $j``\gamma \in M$, then $M^\gamma \subseteq M$. For this, let $h$ be a function representing $j``\gamma$, and let $\{[f_\alpha] \mid \alpha < \gamma\} \subseteq M$. We define f such that $f(x)$ is a function, $\operatorname{dom} f(x) = h(x)$ and $f(x)(i) = f_{h(x)}(i)$. Then f represents a function with doing $[h] = j``\gamma$ and $[h](\alpha) = [f_\alpha]$. That is, $\{[f_\alpha] \mid \alpha < \gamma\}$. $\square$

Exercise 6.17. Let U be a supercompact measure on $\mathcal{P}_\kappa(\lambda)$, then $x \mapsto \operatorname{otp} x$ represents λ and $x \mapsto x \cap \kappa$ represents κ.

Exercise 6.18. If U is a supercompact measure on $\mathcal{P}_\kappa(\lambda)$, then $[f] = j(f)(j``\lambda)$ in $\mathrm{Ult}(V, U)$.

Corollary 6.19. *For $\kappa \leq \lambda$ the following are equivalent:*

1. *κ is λ-supercompact.*
2. *There exists $j\colon V \to M$ such that $\mathrm{crit}(j) = \kappa$, $M^\lambda \subseteq M$, and $j(\kappa) \geq \lambda$.* $\square$

In particular if κ is $|V_{\kappa+\alpha}|$-supercompact, then κ is α-strong. Therefore supercompact cardinals are strong, and in particular $V_\kappa \prec_{\Sigma_2} V$.

Remark. When working in ZF the two most common definitions of supercompactness are either the existence of supercompact measures, or a formulation based on embeddings: κ is supercompact if for all α there is some $\beta \geq \alpha$ and an elementary embedding $j\colon V_\beta \to N$ with N transitive, such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \alpha$, and $N^{V_\alpha} \subseteq N$. It is consistent (relative to significantly weaker assumptions than supercompactness) that ω_1 is supercompact if we use the measures, but critical points (of these sort of embeddings) are never successor cardinals, and so in that sense the definition via measures in a deep sense "the wrong definition" in a choiceless context.

Theorem 6.20. *Suppose that κ is 2^κ-supercompact, then it is not the first measurable cardinal.*

Proof. Let $j: V \to M$ be a 2^κ-supercompact embedding and let $U = \mathrm{der}(j)$, then $U \in M$, and therefore $M \models \kappa$ is measurable. In particular, $\{\alpha < \kappa \mid \alpha \text{ is measurable}\} \in \mathrm{der}(j)$. □

Exercise 6.21. Suppose that κ is a supercompact cardinal, then κ is a limit of Woodin cardinals.

Exercise 6.22. Suppose that κ is 2^κ-supercompact, then there is a normal measure on κ concentrating on superstrong cardinals.

Fact 6.23. *It is consistent, relative to the existence of a supercompact cardinal, that the least supercompact cardinal is also the least strongly compact cardinal. This is the essence of Magidor's identity crisis of strong compactness. It is still open whether or not the two large cardinal axioms are equiconsistent or not.*

Magidor proved the following interesting equivalence to supercompactness.

Theorem 6.24. *The following are equivalent:*

1. *κ is supercompact.*
2. *For every $\beta > \kappa$ there is $\alpha < \kappa$ and an elementary $j: V_\alpha \to V_\beta$ with $j(\mathrm{crit}(j)) = \kappa$.*

Proof. Suppose that κ is supercompact and $\beta > \kappa$. Let $j: V \to M$ be a $|V_\beta|$-supercompact embedding, then $j \upharpoonright V_\beta : V_\beta \to V^M_{j(\beta)}$ is an elementary embedding inside M, so

$$M \models \exists \alpha < j(\kappa), i: V_\alpha \to V_j(\beta), i(\mathrm{crit}(i)) = j(\kappa),$$

so by elementarity,

$$V \models \exists \alpha < \kappa, i: V_\alpha \to V_\beta, i(\mathrm{crit}(i)) = \kappa.$$

In the other direction, letting $\beta > \kappa$ and letting $\alpha' < \kappa$ such that $j: V_{\alpha'} \to V_{\beta+\omega}$ with $\mathrm{crit}(j) = \delta$ and $j(\delta) = \kappa$. By elementarity, $\alpha' = \alpha + \omega$ with $j(\alpha) = \beta$. Since $\mathcal{P}(\mathcal{P}_\delta(\alpha)) \in V_{\alpha'}$ and $j``\alpha \in \mathcal{P}_\kappa(\beta)$, we can define

$$U = \{X \subseteq \mathcal{P}_\delta(\alpha) \mid j``\alpha \in j(X)\}.$$

This is a supercompact measure on $\mathcal{P}_\delta(\alpha)$, therefore $j(U)$ is a supercompact measure on $\mathcal{P}_{j(\delta)}(j(\alpha))$, that is, a supercompact measure on $\mathcal{P}_\kappa(\beta)$. □

Theorem 6.25. *Let κ be a supercompact cardinal. There exists $f: \kappa \to V_\kappa$ such that for every x and $\lambda \geq \kappa, |\mathrm{tcl}(x)|$, there is a supercompact measure on $\mathcal{P}_\kappa(\lambda)$, U, such that $j_U(f)(\kappa) = x$.*

We call such f a *Laver function* (or Laver diamond).

Proof. Suppose this is false. Then for every $f: \kappa \to V_\kappa$ there is a least $\lambda_f \geq \kappa$ and x such that $|\mathrm{tcl}(x)| \leq \lambda_f$ and there is no supercompact measure, U, on $\mathcal{P}_\kappa(\lambda_f)$ such that $j_U(f)(\kappa) = x$. Let $\mu > \sup\{\lambda_f \mid f: \kappa \to V_\kappa\}$ and let $j: V \to M$ be a μ-supercompact embedding.

Letting $\varphi(g,\delta)$ denote the statement that for some cardinal α, $g\colon \alpha \to V_\alpha$ and δ is the least for which there is some x such that $|\operatorname{tcl}(x)| \leq \delta$ and no supercompact measure, U, on $\mathcal{P}_\alpha(\delta)$ satisfies that $j_U(g)(\alpha) = x$. Namely, φ is the statement that g is not a Laver function and that δ is the smallest witness for that.

As M is the target of a μ-supercompact embedding, $M^\mu \subseteq M$ and so $M \models \varphi(f, \lambda_f)$ for all $f\colon \kappa \to V_\kappa$. Define $A = \{\alpha < \kappa \mid \forall f\colon \alpha \to V_\alpha \exists \lambda_f\, \varphi(f,\lambda_f)\}$, then $\kappa \in j(A)$, and in particular $|A| = \kappa$. Define $f\colon \kappa \to V_\kappa$ to be $f(\alpha) = x_\alpha$ if x_α is a witness for $\varphi(f \restriction \alpha, \lambda_{f\restriction\alpha})$ when $\alpha \in A$ and otherwise $f(\alpha) = \varnothing$.

In M, letting $x = j(f)(\kappa)$, then by elementarity x must be a witness that $\varphi(f,\lambda_f)$ holds in M. However, letting $U = \{X \in \mathcal{P}_\kappa(\lambda_f) \mid j``\lambda_f \in j(X)\}$ be the derived supercompact measure. Then j factors through $\operatorname{Ult}(V,U)$. However, $j_U(f)(\kappa) = j(f)(\kappa) = x$, which is a contradiction to the assumption that x was a witness for $\varphi(f,\lambda_f)$. $\square$

Remark. Laver diamonds are very useful when using supercompactness in forcing. We can use them to "predict" possible larger objects, so that the iteration is done "only" up to κ, but the supercompactness allows us to capture much larger objects.

6.3 Extendibility

Definition 6.26. We say that κ is an *η-extendible cardinal* if there is some β and an elementary embedding $j\colon V_{\kappa+\eta} \to V_\beta$ with $\operatorname{crit}(j) = \kappa$ and $\eta < j(\kappa)$. If κ is η-extendible for all η, then we say that it is an *extendible cardinal*.

Exercise 6.27. If κ is 1-extendible, then $o(\kappa) > 1$.

Theorem 6.28. *If κ is extendible, then κ is supercompact.*

Proof. Suppose that κ is extendible and let λ be any cardinal, then there is an embedding $j\colon V_\lambda \to V_\beta$ witnessing that κ is $\lambda+1$-extendible, so in particular $j(\kappa) > \lambda$, and therefore we can define an ultrafilter over $\mathcal{P}_\kappa(\lambda)$ by $U = \{X \subseteq \mathcal{P}_\kappa(\lambda) \mid j``\lambda \in j(X)\}$.

Since $\operatorname{crit}(j) = \kappa$ this ultrafilter is κ-complete, and it is fine as for each $\alpha < \kappa$, $j(\alpha) \in j``\lambda$, and therefore $j``\lambda \in j(\{s \in \mathcal{P}_\kappa(\lambda) \mid \alpha \in s\})$. To see that U is a normal measure, let $F\colon \mathcal{P}_\kappa(\lambda) \to \lambda$ be a choice function and let α be such that $j(F)(j``\lambda) = j(\alpha)$, then $\{s \in \mathcal{P}_\kappa \mid F(s) = \alpha\} \in U$, and so F is constant on a large set, and so U is a supercompact measure. $\square$

Exercise 6.29 (*) . Show that if κ is supercompact and 1-extendible, then κ is a limit of supercompact cardinals.

Definition 6.30. We write $\mathcal{L}^n_{\kappa,\lambda}$ to denote the infinitary nth order logic which allows quantification over variables up to order n and in blocks of $< \kappa$ variables at a time, as well as conjunctions of length $< \lambda$.

Theorem 6.31. *The following are equivalent:*

1. *κ is extendible.*
2. *$\mathcal{L}^n_{\kappa,\kappa}$ is strongly compact for all $n \geq 1$.*
3. *$\mathcal{L}^2_{\kappa,\kappa}$ is strongly compact.*

Proof. Suppose that κ is extendible and let T be a κ-satisfiable $\mathcal{L}^n_{\kappa,\kappa}$-theory, and we may assume that $\mathcal{L}^n_{\kappa,\kappa}$ is coded entirely within V_κ, and so it is not changed by j with critical point κ.

Let η be large enough limit ordinal such that $V_\eta \models$ "T is κ-satisfiable". And let $j\colon V_\eta \to V_\beta$ be an embedding witnessing that κ is extendible. Then by elementarity $j(T)$ is $j(\kappa)$-satisfiable in V_β, so $j``T$ must have a model in $V_\beta$, since $|j``T| < \eta < j(\kappa)$. Let $M \in V_\beta$ be a model for $j``T$, since $\beta$ is a limit ordinal, $\mathcal{P}^n(M)^{V_\beta} = \mathcal{P}^n(M)$, and therefore the $n$th order truth is the same in $V$ and in $V_\beta$. Since $j$ induces a translation between $T$ and $j``T$, $M \models T$, as wanted.

The next implication is trivial, so it remains to show that if $\mathcal{L}^2_{\kappa,\kappa}$ is strongly compact, then κ is extendible. Towards that, let η be an ordinal and let T be the $\mathcal{L}^2_{\kappa,\kappa}$ theory of the structure $\langle V_{\kappa+\eta}, \in, x\rangle_{x\in V_{\kappa+\eta}}$, adjoined with constant symbols c_α for $\alpha \leq \kappa+\eta$ and the sentences $c_\alpha < \kappa$ and $c_\alpha < c_\beta$ for $\alpha < \beta \leq \kappa+\eta$. It is easy to see that T is κ-satisfable, by simply interpreting the relevant c_α inside $V_{\kappa+\eta}$.

Let M be a model of T, then M is well-founded, as that is a second-order sentence which is true in $V_{\kappa+\eta}$ (or an $\mathcal{L}_{\omega_1,\omega_1}$-sentence which is true there). Let σ be the Π_1 sentence that for any transitive set X, $X \models \sigma$ if and only if $X = V_\alpha$ for some α. Then $V_{\kappa+\eta}$ satisfies the second-order sentence saying that there are cofinally many transitive sets satisfying σ or else there is a largest one and every set is a subset of it, depending on whether or not η is a limit ordinal.

Therefore the transitive collapse of M is of the form V_β for some ordinal β. The map $j\colon V_{\kappa+\eta} \to V_\beta$ given by mapping each $x \in V_{\kappa+\eta}$ to the interpretation of its constant is elementary, and for all $\alpha < \kappa$ it is easy to see that $j(\alpha) = \alpha$ by an $\mathcal{L}_{\kappa,\kappa}$ formula which defines α. However, $j(\kappa) > \kappa+\eta$, since the constants c_α must be interpreted as distinct ordinals below $j(\kappa)$. $\square$

Exercise 6.32. If κ is extendible, then there is a proper class of inaccessible cardinals. In fact, $\{\lambda \mid V_\kappa \prec V_\lambda\}$ contains unboundedly many inaccessible cardinals. (Hint: If $j\colon V_\eta \to V_\beta$ witnessing the extendibility of κ, what can you say about $j(\kappa)$?)

Theorem 6.33. *If κ is extendible, then $V_\kappa \prec_{\Sigma_3} V$.*

Proof. Let $\psi(u,x)$ be a Π_2 formula, that is $\forall y\exists z\varphi(u,x,y,z)$ where φ is a Δ_0-formula. Let $a \in V_\kappa$. If $V_\kappa \models \exists x\psi(a,x)$, then there is some $b \in V_\kappa$ such that $V_\kappa \models \psi(a,b)$. Since κ is extendible, and therefore supercompact, by Corollary 6.19, $V_\kappa \prec_{\Sigma_2} V$, so $V \models \psi(a,b)$ and therefore $V \models \exists x\psi(a,x)$ as wanted.

In the other direction, suppose that $V \models \exists x\psi(a,x)$ and let b be a witness for that, then we can find large enough inaccessible η such that $b \in V_\eta$ and $V_\kappa \prec V_\eta$. Since ψ is a Π_2 formula and η is inaccessible, $V_\eta \prec_{\Sigma_1} V$ and so ψ is downwards absolute to V_η. In particular, since $V_\eta \models \psi(a,b)$, it holds that $V_\eta \models \exists x\psi(a,x)$, and so $V_\kappa \models \exists x\psi(a,x)$ as wanted. $\square$

Definition 6.34. We say that an cardinal λ is Σ_n*-correct* if $V_\lambda \prec_{\Sigma_n} V$.

We saw that inaccessible cardinals are Σ_1-correct, strong cardinals are Σ_2-correct, and extendible are Σ_3-correct.

Exercise 6.35. Show that ZFC proves that there is a proper class of cardinals which are Σ_n-correct for all $n < \omega$.

Exercise 6.36. If λ is a Σ_1-correct cardinal, then $\beth_\lambda = \lambda$, i.e. $|V_\lambda| = \lambda$.

Exercise 6.37. The smallest Σ_2-correct cardinal is larger than the smallest measurable (if it exists).

Exercise 6.38. The statement "$j\colon V_\alpha \to V_\beta$ is an elementary embedding with $\mathrm{crit}(j) = \kappa$" can be expressed as a Σ_2-formula with α, β, κ as parameters.

While both strong cardinals and supercompact cardinals were, in principle, defined by requiring unboundedly many embeddings from $V \to M$ with certainly properties, we found out that these can be restated in terms of ultrafilters and embeddings, which made them into combinatorial statements. Extendible cardinals already talk about embeddings between sets, rather than embeddings between proper classes, so their existence is easily seen as formulated by a first-order statement. However, for a very long time it was open as to whether or not we can formulate extendibility in terms of measures or extenders. Quite recently, Goldberg and Bagaria gave (independently) a positive answer.

Theorem 6.39. *κ is extendible if and only if there is a proper class of Σ_2-correct cardinals, λ, such that there is a supercompact measure concentrating on $\{s \in \mathcal{P}_\kappa(\lambda) \mid \mathrm{otp}(s)$ is Σ_2-correct$\}$.*

Proof. Suppose that κ is extendible and λ is a Σ_2-correct cardinal, let $j\colon V_{\lambda+1} \to V_{\lambda'+1}$ be an elementary embedding with $\mathrm{crit}(j) = \kappa$. Since λ is Σ_2-correct, it is also Σ_2-correct inside $V_{j(\kappa)}$, since $j(\kappa)$ is Σ_2-correct. So $j``\lambda \in \{s \in \mathcal{P}_{j(\kappa)}(j(\lambda)) \mid \mathrm{otp}(s)$ is $\Sigma_2$-correct$\}$. Therefore the ultrafilter on $\mathcal{P}_\kappa(\lambda)$ derived from $j``\lambda$ is a supercompact measure with the wanted property.

In the other direction, if $\mathcal{P}_\kappa(\lambda)$ carries a supercompact measure as in the hypothesis, we show that κ is γ-extendible for all $\gamma < \lambda$. Therefore, if there are unboundedly many such λ, κ is extendible. Fix such measure U and let $j\colon V \to M = \mathrm{Ult}(V, U)$ be the ultrapower embedding. Since $M^\lambda \subseteq M$ and λ is Σ_2-correct, $|V_\lambda| = \lambda$, and therefore $j \restriction V_\lambda \in M$.

Moreover, as U concentrates on things with Σ_2-correct order type and $x \mapsto \mathrm{otp}\, x$ is representing λ, $M \models$ "λ is Σ_2-correct". So in $V^M_{j(\kappa)}$ it holds that κ is α-extendible for all $\alpha < \lambda$, which is something that is expressible entirely in $V_\lambda = V^M_\lambda$. Since λ was originally Σ_2-correct, this is true in V as wanted. $\square$

Fact 6.40 (The HOD Dichotomy). *Suppose that κ is an extendible cardinal. Then exactly one of the two following holds.*

1. *If $\delta > \kappa$ is singular, then δ is singular in* HOD *and $\delta^+ = (\delta^+)^{\mathrm{HOD}}$.*
2. *If $\delta > \kappa$ is a regular cardinal, then it is measurable in* HOD*.*

This, in a deep sense, is a covering lemma for HOD. If $0^\#$ exists, then every regular cardinal is inaccessible in L; if $0^\#$ does not exist, then every singular cardinal is singular in L and the successor is computed correctly. Woodin proved the HOD Dichotomy and stated the *HOD Hypothesis* as saying that the first option holds, and the *HOD Conjecture* as the statement that the HOD Hypothesis is in fact provable from ZFC.

Chapter 7

Huge parallels of strength

7.1 Huge cardinals

There is a deep parallel between supercompact cardinals and strong cardinals. If we look at λ such that $|V_\lambda| = \lambda$, then being a λ-strong embedding is a closure under a very specific sequence of length λ, whereas being a λ-supercompact embedding is a closure under *all* λ-sequences. There are other deep relationship between the two notions being somewhat dual, and we will discuss them later in this chapter.

But, if supercompactness is a "stronger version" of strongness, what would be the equivalent of superstrong cardinals, then?

7.1.1 Huge cardinals

Definition 7.1. We say that κ is a *huge cardinal* if there exists an elementary $j\colon V \to M$ such that $\operatorname{crit}(j) = \kappa$ and $M^{j(\kappa)} \subseteq M$. We call such an embedding a *huge embedding*.

This definition is in fact local. To be a strong or a supercompact cardinal, many embeddings are needed, but to be a superstrong or a huge cardinal you only need the one embedding.

Exercise 7.2. If κ is a huge cardinal and j is a witnessing embedding, then j is a superstrong embedding and $j(\kappa)$ is inaccessible.

Exercise 7.3 (*) . If κ is a huge cardinal and j is a witnessing embedding, then $j(\kappa)$ is measurable.

Exercise 7.4. Suppose that $j\colon V \to M$ is elementary with $\operatorname{crit}(j) = \kappa$ and $M^\lambda \subseteq M$, then $\{X \subseteq \mathcal{P}_\kappa(\lambda) \mid j``\lambda \in j(X)\}$ is a supercompact measure.

Proposition 7.5. *If κ is huge and j is a huge embedding, then κ is $<j(\kappa)$-supercompact. Namely, for all $\lambda < j(\kappa)$, $\mathcal{P}_\kappa(\lambda)$ carries a supercompact measure.*

Proof. Apply Exercise 7.4 to derive supercompact measures on $\mathcal{P}_\kappa(\lambda)$ and note that these are all in $V_{j(\kappa)}$. □

Theorem 7.6. *If κ is huge, then in V_κ there is a proper class of supercompact cardinals.*

Proof. Let $j\colon V \to M$ be a huge embedding, then $V_{j(\kappa)} \models \kappa$ is supercompact, since j is also a superstrong embedding, $M \models$ "$V_{j(\kappa)} \models$ There is a supercompact cardinal". Therefore, the set $\{\alpha < \kappa \mid V_\kappa \models \alpha \text{ is supercompact}\} \in \operatorname{der}(j)$. □

In the case of superstrong cardinals, we derived a superstrong extender and showed that by Σ_2-correctness, the least superstrong is below the least strong cardinal. If we can derive a "local object" from the huge embedding, this will allow us to conclude that the least huge is below the least supercompact.

Definition 7.7. We say that an ultrafilter U on $\mathcal{P}(\lambda)$ is a *huge measure (on λ)* if it satisfies:

(Fine) For all $\alpha < \lambda$, $\{x \in \mathcal{P}(\lambda) \mid \alpha \in x\} \in U$.

(Normal) For all $\{X_\alpha \mid \alpha < \lambda\}$, $\triangle_{\alpha<\lambda} X_\alpha = \{x \in \mathcal{P}(\lambda) \mid x \in \bigcap_{\alpha\in x} X_\alpha\} \in U$.

Theorem 7.8. *For an uncountable cardinal κ, κ is a huge cardinal if and only if there is some λ and a κ-complete huge mesaure on $\mathcal{P}(\lambda)$ concentrating on $\{x \in \mathcal{P}(\lambda) \mid \operatorname{otp}(x) = \kappa\}$.*

Proof. Suppose that κ is a huge cardinal and let j be a huge embedding, setting $\lambda = j(\kappa)$, then $U = \{X \in \mathcal{P}(\lambda) \mid j``\lambda \in X\}$. Just like the case of deriving strong and supercompact measures, $U$ is a huge measure, and $j(\{x \in \mathcal{P}(\lambda) \operatorname{otp}(x) = \kappa\}) = \{x \in \mathcal{P}(j(\lambda)^M \mid \operatorname{otp}(x) = j(\kappa) = \lambda\}$, since $j``\lambda$ has order type λ, the measure is as wanted.

In the other direction, let $M = \operatorname{Ult}(V, U)$ and let $j = j_U$. By normality, $[\mathrm{id}] = j``\lambda$,[13] and so $M^\lambda \subseteq M$ much like in the proof of Proposition 6.16. Moreover, $\lambda = \operatorname{otp}(j``\lambda \cap j(\lambda))$, therefore λ is represented by $x \mapsto \operatorname{otp} x$, and since U concentrates on $\{x \in \mathcal{P}(\lambda) \mid \operatorname{otp}(x) = \kappa\}$, the function is equivalent to $c_\kappa(x) = \kappa$. That is to say, $j(\kappa) = \lambda$, so j is a huge embedding. □

Corollary 7.9. *The statement "κ is a huge cardinal" is a Σ_2 statement. Therefore "There exists a huge cardinal" is a Σ_2 sentence, and so the least huge cardinal is smaller than the least supercompact and least strong cardinal.* □

7.1.2 Weaker and stronger variations

Let us visit some definitions related to huge cardinals. In all cases we will define the cardinal, and without mention, the embedding that witnesses property "x" will be called x-embedding.

Definition 7.10. We say that κ is an *almost huge cardinal* if there is an elementary embedding $j\colon V \to M$ with $\operatorname{crit}(j) = \kappa$ and $M^{<j(\kappa)} \subseteq M$.

Exercise 7.11. If κ is almost huge, then κ is superstrong and $j(\kappa)$ is inaccessible.

We want to show that huge cardinals are stronger than almost huge cardinals, which sounds like a reasonable thing. However, huge measures will not be of any particular help, since they capture hugeness, and we are just on the precipice of hugeness, but not quite there.

Definition 7.12. Suppose that $\kappa \le \lambda$ and for each $\kappa \le \gamma < \lambda$, U_γ is a supercompact measure on $\mathcal{P}_\kappa(\gamma)$. We say that $\langle U_\gamma \mid \kappa \le \gamma < \lambda\rangle$ is a *coherent sequence* if and only if for every $\kappa \le \gamma \le \delta < \lambda$,

$$U_\delta \restriction \gamma = \{\{x \cap \gamma \mid x \in X\} \mid X \in U_\delta\} = U_\gamma.$$[14]

To simplify the notation, we will write j_γ to denote j_{U_γ} in the context of such a coherent sequence of measures.

[13] Note that here $\mathrm{id}\colon \mathcal{P}(\lambda) \to \mathcal{P}(\lambda)$.

[14] We say that U_γ is the projection of U_δ to γ in this case.

Exercise 7.13. If $\langle U_\gamma \mid \kappa \leq \gamma < \lambda\rangle$ is a coherent sequence of supercompact measures, then there are embeddings $k_{\gamma,\delta}\colon \operatorname{Ult}(V, U_\gamma) \to \operatorname{Ult}(V, U_\delta)$ such that $j_\delta = k_{\gamma,\delta} \circ j_\gamma$. Moreover, these factor embeddings are given by $k_{\gamma,\delta}([f]_{U_\gamma}) = [f \circ (x \mapsto x \cap \gamma)]_{U_\delta}$ and $j_\gamma(\kappa) > \operatorname{crit}(k_{\gamma,\delta}) \geq \gamma$.

Theorem 7.14. *κ is almost huge if and only if there exists an inaccessible cardinal $\lambda > \kappa$ and a coherent sequence of supercompact measures $\langle U_\gamma \mid \kappa \leq \gamma < \lambda\rangle$ such that whenever $\gamma < \lambda$ and $\gamma \leq \alpha < j_\gamma(\kappa)$, there is some δ such that $\gamma \leq \delta < \lambda$ such that $k_{\gamma,\delta}(\alpha) = \delta$.*

Proof. If κ is almost huge, let $j\colon V \to M$ witness this, then $\lambda = j(\kappa)$ is inaccessible and setting $U_\gamma = \{X \subseteq \mathcal{P}_\kappa(\gamma) \mid j``\gamma \in j(X)\}$ is a coherent sequence of supercompact measures. Suppose that $\gamma < \lambda$ and $f\colon \mathcal{P}_\kappa(\gamma) \to \kappa$ represents $\alpha$ as in the condition. Since $\operatorname{otp}([\mathrm{id}]_{U_\gamma}) = \gamma$, then $\operatorname{otp}(j(f)(j``\gamma)) = \delta \geq \gamma$ and $\delta < j(\kappa) = \lambda$. Now we have that $\operatorname{otp}(j(f)(j``\delta \cap j(\gamma))) = \delta$, so $k_{\gamma,\delta}([f]_{U_\gamma}) = \delta$ by the definition of the factor embedding of ultrapowers.

In the other direction, suppose that $\langle U_\gamma \mid \kappa \leq \gamma < \lambda\rangle$ is a coherent sequence of supercompact measures with the wanted property. As λ is inaccessible, the direct limit of the system is well-founded, M, and let $j\colon V \to M$ be the limit embedding with $k_\gamma\colon \operatorname{Ult}(V, U_\gamma) \to M$ the factor embedding. It is not hard to check that κ is the critical point of j and that $\operatorname{crit}(k_\gamma) > \gamma$. Let us verify that j is an almost huge embedding.

Since $\gamma < |j_\gamma(\kappa)| < (2^{\gamma^{<\kappa}})^+ < \lambda$, and $j(\kappa) = k_\gamma(j_\gamma(\kappa))$ it follows immediately that $j(\kappa) \geq \lambda$. In the other direction, suppose that $\beta < j(\kappa)$, then there is some γ such that $k_\gamma(\alpha) = \beta$. Since $\beta < k_\gamma(j_\gamma(\kappa)) = j(\kappa)$, it must be that $\alpha < j_\gamma(\kappa)$. If $\alpha \leq \gamma$, then $k_\gamma(\alpha) = \alpha = \beta < \lambda$. Otherwise, there is some $\delta > \gamma$ such that $k_{\gamma,\delta}(\alpha) = \delta$, but

$$j(\kappa) = k_\delta(k_{\gamma,\delta}(j_\gamma(\kappa))) > k_\delta(k_{\gamma,\delta}(\alpha)) = k_\delta(\delta) = \delta.$$

Therefore $\alpha \leq \delta < j_\delta(\kappa) < |(2^{\delta^{<\kappa}})^+|^V < \lambda$.

Finally, let us check that $M^{<\lambda} \subseteq M$. If $\beta < \lambda$ and $\{x_\alpha \mid \alpha < \beta\} \subseteq M$ then there is some $\gamma > \beta$ and $y_\alpha \in \operatorname{Ult}(V, U_\beta)$ such that $k_\beta(y_\beta) = x_\beta$. However, $\operatorname{Ult}(V, U_\beta)$ is closed under γ-sequences, so in particular $\{y_\alpha \mid \alpha < \beta\} \in \operatorname{Ult}(V, U_\beta)$ and since $\beta < \operatorname{crit}(k_\gamma)$, we have that

$$k_\gamma(\{y_\alpha \mid \alpha < \beta\}) = \{k_\gamma(y_\alpha) \mid \alpha < \beta\} = \{x_\alpha \mid \alpha < \beta\} \in M. \qquad \square$$

The obvious question, now, is whether or not the notion of almost huge is actually equivalent to that of a huge cardinal. Is it possible, perhaps, that there is some continuity at play?

Theorem 7.15. *If κ is huge, then there is a normal measure on κ concentrating on almost huge cardinals.*

Proof. Let $j\colon V \to M$ be a huge embedding and let $\lambda = j(\kappa)$. We can derive a coherent sequence of supercompact measures from j, as in Theorem 7.14, then this sequence is in M, and therefore $M \models$ "κ is almost huge". In particular, $\{\alpha < \kappa \mid \alpha \text{ is almost huge}\} \in \operatorname{der}(j)$. $\square$

Definition 7.16. We say that κ is a *superhuge cardinal* if for every $\lambda > \kappa$ there is a huge embedding j with $\operatorname{crit}(j) = \kappa$ and $j(\kappa) > \lambda$.

Clearly, a superhuge cardinal is supercompact. But we can say much more than that.

Theorem 7.17. *If κ is superhuge, then κ is extendible and there is a normal measure on κ which concentrates on extendible cardinals.*

Proof. Let $\eta > \kappa$ be an ordinal and let $j\colon V \to M$ be a huge embedding with $j(\kappa) > \eta$. Since $j(\kappa)$ is inaccessible, $|V_\eta| < j(\kappa)$. Therefore $j \restriction V_\eta \in M$, moreover, $j \restriction V_\eta$ is an embedding $V_\eta = V_\eta^M \to V_{j(\eta)}^M$. Therefore,

$$M \models \exists\beta\exists i\colon V_\eta \to V_\beta, \operatorname{crit}(i) = \kappa, i(\kappa) > \eta.$$

Since this statement is a Σ_2 statement in η and κ, and since $j(\kappa)$ is supercompact in M, and therefore Σ_2-correct, $V_{j(\kappa)}^M$ satisfies the statement. However, since $V_{j(\kappa)}^M = V_{j(\kappa)}$, this is true in V, and therefore κ is η-extendible in V. Therefore κ is extendible in V.

The argument above shows that κ is η-extendible for all $\eta < j(\kappa)$, $V_{j(\kappa)} \models$ "κ is extendible". In particular, $\{\delta < \kappa \mid V_\kappa \models \delta \text{ is extendible}\} \in \operatorname{der}(j)$. Let δ be such cardinal, then $j(\delta) = \delta$ and $V_{j(\kappa)} \models$ "δ is extendible". To complete the proof we need to show that δ is extendible in V, so for any η take a huge embedding i such that $i(\kappa) > \eta$, then $V_{i(\kappa)} \models$ "δ is η-extendible", but since i is a huge embedding, $V_{i(\kappa)}$ is computed the same in both models. Therefore δ must be extendible, so indeed $\operatorname{der}(j)$ concentrates on extendible cardinals. □

Definition 7.18. We say that κ is a *n-huge cardinal* if there is an elementary embedding $j\colon V \to M$ such that $\operatorname{crit}(j) = \kappa$ and $M^{j^n(\kappa)} \subseteq M$. We define *almost n-huge* and *super-n-huge* cardinals in the analogous way to almost huge and superhuge cardinals.

All of these definitions make sense in the case where $n = 0$, where we simply get measurable cardinals.

Theorem 7.19. *If κ is an almost 2-huge cardinal, then in V_κ there is a proper class of superhuge cardinals.*

Proof. Let $j\colon V \to M$ be an almost 2-huge embedding. In M we can still derive the huge measure on $j(\kappa)$ to show that κ is huge and has a huge embedding mapping it to $j(\kappa)$. Therefore, in M it is true that $j(\kappa)$ has a huge measure. So in V not only κ has a huge measure, there is a set $H \in \operatorname{der}(j)$ such that if $\eta \in H$, then there is a huge measure on κ concentrating on sets of order type η. Pick any such η, then the aforementioned huge measure on κ is also in M. So in M it is true that κ is in the set

$$\{\lambda < j(\kappa) \mid \text{ There is a huge measure on } \lambda \text{ concentrating on sets of order type } \eta\}.$$

In particular, there is a set in $\operatorname{der}(j)$ of possible targets for huge embeddings with critical point η. So, $V_\kappa \models \eta$ is superhuge. □

Exercise 7.20. κ is $(n+1)$-huge with an embedding $j\colon V \to M$ with $j^{n+1}(\kappa) = \lambda$ if and only if there are $\kappa = \lambda_0 < \cdots < \lambda_n = \lambda$ and there is a huge measure on λ concentrating on the set $\{x \in \mathcal{P}(\lambda) \mid \operatorname{otp}(x \cap \lambda_{i+1}) = \lambda_i\}$.

Exercise 7.21. κ is almost $(n+1)$-huge, then in V_κ there is a proper class of super-n-huge cardinals.

Definition 7.22. We say that κ is *n-superstrong* if there is an elementary embedding $j\colon V \to M$ with $\operatorname{crit}(j) = \kappa$ and $V_{j^n(\kappa)} \subseteq M$.

Exercise 7.23. The consistency of a $(n+1)$-superstrong cardinal implies the consistency of many n-huge cardinals. And an almost $(n+1)$-huge cardinal is also $(n+1)$-superstrong.

7.2 Vopěnka's Principle

Definition 7.24. We say that *Vopěnka's Principle* (VP) holds if for every first-order language $\mathcal{L}$, if $\mathcal{C}$ is a proper class of $\mathcal{L}$-structures, then there is one that elementarily embeds into another.

Exercise 7.25 (*) . VP implies that there is a proper class of extendible cardinals.

Theorem 7.26. *If κ is almost huge, then $V_\kappa \models \mathsf{VP}$.*

The problem with the statement of VP is that it quantifies over classes. It is a second-order statement. So when we write $V_\kappa \models \mathsf{VP}$, we need to explain which classes are concerned here. Luckily for us, we do not need to worry and we can just consider $V_{\kappa+1}$ as the collection of classes in this case.[15]

Proof. Let $j\colon V \to M$ be an almost huge embedding with $\operatorname{crit}(j) = \kappa$ and suppose that $A = \langle X_\alpha \mid \alpha < \kappa\rangle$ is a proper class in V_κ of $\mathcal{L}$-structures.

In M, we have $j(A) = \langle X_\alpha \mid \alpha < \kappa\rangle \cup \langle X_\alpha \mid \kappa \leq \alpha < j(\kappa)\rangle$ as a sequence in $V_{j(\kappa)}$. Then $j(j(A))$ is now a sequence of length $j^2(\kappa)$, but $j \upharpoonright X_\kappa$ is an elementary embedding into $X_{j(\kappa)}$. Since $X_\kappa \in V_{j(\kappa)}$, we have that $|X_\kappa| < j(\kappa)$ and therefore $j \upharpoonright X_\kappa \in M$.

Therefore, letting S be the set of $\alpha < \kappa$ such that X_α embeds elementarily into some $X_\beta \in j(A)$ for $\beta < j(\kappa)$, we get that $\kappa \in j(S)$, so in particular S is non-empty. Taking any $\alpha \in S$, since in V, there is some $\beta < j(\kappa)$ such that $X_\alpha \prec X_\beta$, and $X_\beta \in V_{j(\kappa)}$, it must be that M satisfies the same. Namely, M satisfies that there is some $\beta < j(\kappa)$ and $X_\alpha \in j(A)$ such that $X_\alpha \prec X_\beta$. By elementarity, there is such $\beta < \kappa$. $\square$

Definition 7.27. We say that κ is a *Vopěnka cardinal* if $V_\kappa \models \mathsf{VP}$.

We can therefore rewrite the statement of Theorem 7.26 as "If κ is almost huge, then κ is Vopěnka." How do Vopěnka cardinals compare to the cardinals we have seen so far? We know that they lie below the hugeness hierarchy, and above the extendible cardinals. But can we say more? Yes. It turns out that Vopěnka cardinals are exactly "Woodin for supercompactness" and "Woodin for extendible" cardinals.

Definition 7.28. We say that κ is an *η-A-extendible cardinal* if there is some β and an elementary embedding $j\colon V_{\kappa+\eta} \to V_\beta$ with $\operatorname{crit}(j) = \kappa$, $j(\kappa) > \eta$, and $j(A \cap V_{\kappa+\eta}) = A \cap V_\beta$.

We say that κ is an *η-A-supercompact cardinal* if there is $\alpha < \kappa$ and an elementary embedding $j\colon V_\alpha \to V_\eta$ such that $j(\operatorname{crit}(j)) = \kappa$ and $j(A \cap V_\alpha) = A \cap V_\eta$.

Exercise 7.29 (**) . The following are equivalent.

1. κ is Vopěnka.
2. For every $A \subseteq V_\kappa$ there is some $\alpha < \kappa$ which is η-A-extendible for all $\eta < \kappa$.
3. For every $A \subseteq V_\kappa$ there is some $\alpha < \kappa$ which is η-A-supercompact for all $\eta < \kappa$.
4. For every increasing $f\colon \kappa \to \kappa$ and $R\colon \kappa \to V_\kappa$ such that $R(\alpha) \subseteq V_{f(\alpha)}$, there are $\alpha < \beta < \kappa$ such that $\langle V_{f(\alpha)}, \in, \{\alpha\}, R(\alpha)\rangle$ embeds elementarily into $\langle V_{f(\beta)}, \in, \{\beta\}, R(\beta)\rangle$.

As a corollary of this exercise, of course, we can replace "there is some" by "there are stationarily many" or "there are unboundedly many". An interesting fact is that this characterisation of Vopěnka cardinals pre-dates Woodin cardinals by a number of years.[16]

[15] That means that we are allowing full second-order quantification here.

[16] Why yes, it *does* mean that Woodin cardinals are "Vopěnka for strongness" rather than the other way around.

Definition 7.30. We say that κ is an *η-$C^{(n)}$-extendible cardinal* if there is an elementary embedding $j\colon V_{\kappa+\eta} \to V_\beta$ with $\mathrm{crit}(j) = \kappa$ and $j(\kappa)$ is Σ_n-correct. We say that κ is $C^{(n)}$-extendible if it is η-$C^{(n)}$-extendible for all η. We define $C^{(n)}$-supercompact, -strong, -measurable, -superstrong, -huge, etc. in a similar way.

Exercise 7.31. If κ is a measurable (strong) cardinal, then it is $C^{(n)}$-measurable ($C^{(n)}$-strong) for all n.

We will say that $\mathsf{VP}(\Gamma)$ holds, where Γ is a class of formulas (e.g. Σ_n or $\boldsymbol{\Sigma_n}$ when allowing parameters), if whenever $\mathcal{C}$ is a class of structures in the same language which is definable by a formula in Γ, then there is a structure in $\mathcal{C}$ which embeds elementarily into the other.

Fact 7.32. *In the following list, each list enumerates equivalent statements.*

1. (a) *There exists an infinite ordinal.*
   (b) $\mathsf{VP}(\Pi_0)$.
   (c) $\mathsf{VP}(\Sigma_1)$.
2. (a) *There exists a proper class of infinite ordinals.*
   (b) $\mathsf{VP}(\boldsymbol{\Pi_0})$.
   (c) $\mathsf{VP}(\boldsymbol{\Sigma_1})$.
3. (a) *There exists a supercompact cardinal.*
   (b) $\mathsf{VP}(\Pi_1)$.
   (c) $\mathsf{VP}(\Sigma_2)$.
4. (a) *There exists a proper class of supercompact cardinals.*
   (b) $\mathsf{VP}(\boldsymbol{\Pi_1})$.
   (c) $\mathsf{VP}(\boldsymbol{\Sigma_2})$.
5. (a) *There exists a $C^{(n)}$-extendible cardinal.*
   (b) $\mathsf{VP}(\Pi_{n+1})$.
   (c) $\mathsf{VP}(\Sigma_{n+2})$.
6. (a) *There exists a proper class of $C^{(n)}$-extendible cardinals.*
   (b) $\mathsf{VP}(\boldsymbol{\Pi_{n+1}})$.
   (c) $\mathsf{VP}(\boldsymbol{\Sigma_{n+2}})$.

The pattern in the fact above is due to the fact that a class is Σ_{n+1}-definable if and only if it is Π_n-definable.

Many structures can be interpreted as graphs in a natural way. So we can restate VP as "There is no full embedding of Ord into **Graph**". Namely, there is no sequence of graphs $\langle G_\alpha \mid \alpha \in \mathrm{Ord}\rangle$ such that $\alpha < \beta$ if and only if there is a homomorphism $G_\alpha \to G_\beta$, and any two graphs on the sequence have at most one homomorphism between them.

The Weak Vopěnka Principle (WVP) states the dual statement. Namely, $\mathrm{Ord}^{\mathrm{OP}}$ does not embed, in the same sense as above, into **Graph**. Wilson proved recently that WVP is equivalent to its statement without the requirement that homomorphisms are unique, and it turns out to be equivalent to the statement "Ord is Woodin". Namely, every definable class A has arbitrarily large A-strong cardinals. Or, if we want to think about this in terms of a second-order statement.

Fact 7.33. *δ is a Woodin cardinal if and only if $V_\delta \models \mathsf{WVP}$.*

Chapter 8

Pushing it? Touching inconsistency

8.1 Kunen's Inconsistency Theorem

Theorem 8.1 (Kunen's Inconsistency Theorem). *There is no non-trivial elementary embedding $j\colon V \to V$.*

Before proving Kunen's theorem, let us prove a weaker version due to Suzuki.

Proposition 8.2 (Suzuki). *There is no definable non-trivial elementary embedding $j\colon V \to V$.*

Proof. Suppose that $\varphi(x, y, z)$ defines an elementary embedding, j_z, where z is a parameter.[17] Let κ_z be the critical point of the embedding when z is used as a parameter, and let κ be the smallest κ_z for $z \in V$. Then in V we have that κ is the minimal critical point given by all the embeddings definable using φ. Picking z for which $\kappa_z = \kappa$, we have $j_z(\kappa) > \kappa$ and by elementarity, $j_z(\kappa)$ is the minimal critical point given by all the embeddings definable using φ, which is a contradiction. $\square$

Suzuki's theorem gives us, immediately, that measurable, strong, supercompact, huge, and any such cardinals, will always have to define embeddings into inner models. So there is no hope to try and get some class-length sequence of extenders and use it to try and define an embedding $V \to V$. But Kunen's theorem is stronger still. It is formalised in the expanded language $\mathsf{ZFC}(j)$, where we add j as a symbol, add the axioms that j is an elementary embedding, and add all the Replacement axioms for the expanded language.[18]

Proof of Kunen's Inconsistency Theorem. Let $j\colon V \to M$ be a non-trivial elementary embedding. We will prove that $V \neq M$. Let $\lambda = \sup\{j^n(\mathrm{crit}(j)) \mid n < \omega\}$, then we will show that $j``\lambda \notin M$. Assume towards a contradiction that this is not the case. Firstly, note that $j(\lambda) = \lambda$, and so $j``\lambda \subseteq \lambda$.

Let η be the least ordinal such that for some $F\colon \eta \to \mathcal{P}(\lambda)$ we have that $j``\lambda \in \mathrm{rng}(j(F))$. Note that η must be a cardinal and that $\eta \leq 2^\lambda$, since if $f\colon 2^\lambda \to \mathcal{P}(\lambda)$ is a bijection, then $j(f)$ is a bijection from $j(2^\lambda)$ to $j(\mathcal{P}(\lambda)) = \mathcal{P}(j(\lambda)) = \mathcal{P}(\lambda)$, and so $j``\lambda$ must be in $\mathrm{rng}(j(f))$ in that case.

[17] Recall that we can express full elementarity by Σ_1-elementarity and a cofinal image, the latter is trivial in the case of $V \to V$.

[18] Alternatively, it can be seen as a theorem in a second-order set theory such as Kelley–Morse or von Neumann–Gödel–Bernays.

Fix F to be an injection from η to $\mathcal{P}(\lambda)$ as above, and let $S = \operatorname{rng}(F) \subseteq \mathcal{P}(\lambda)$. We define a measure on S given by

$$U = \{X \subseteq S \mid j``\lambda \in j(X)\}.$$

Note that this is in fact a measure on η which we translated to S. Easily, U is an ω_1-complete measure, so we can take $i = j_U$ to be the ultrapower embedding $i\colon V \to \operatorname{Ult}(V, U) = N$.

Note the following facts are true:

1. $i``\lambda = [\mathrm{id}]_U$.
2. $i``\lambda \in \operatorname{rng}(i(F))$.
3. For all $\alpha \leq \lambda$, α is represented by $[A \mapsto \operatorname{otp}(A \cap \alpha)]_U$.
4. $i \restriction \lambda + 1 = j \restriction \lambda + 1$.
5. $i``\eta^+ \notin N$.

Since $i``\lambda \in N$, it follows that $N^\lambda \subseteq N$, and so we have that $\mathcal{P}(\lambda)^N = \mathcal{P}(\lambda)^M = \mathcal{P}(\lambda)$ and that $i$ is a $\lambda$-strong embedding. We will show that $\eta = 2^\lambda$, and that $i(\eta) = \sup i``\eta$. From that a contradiction will follow: $i``\lambda \in \operatorname{rng}(i(F))$, so for some $\alpha < \eta$, $i``\lambda \in \operatorname{rng}(i(F) \restriction i(\alpha)) = \operatorname{rng}(i(F \restriction \alpha))$, and since i and j are the same up to λ, this is a contradiction to the minimality to η.

To prove that $\eta = 2^\lambda$, note that we already know that $\eta \leq 2^\lambda$, so if we can show that $i``2^\lambda \in N$, then by the final fact, $2^\lambda < \eta^+$ and equality ensues. Note that if $A \subseteq \lambda$, then $A = \bigcup_{\alpha<\lambda} A \cap \alpha$, but since $i(\lambda) = j(\lambda) = \lambda$, we have that

$$i(A) = i\left(\bigcup_{\alpha<\lambda} A \cap \alpha\right) = \bigcup_{\alpha<j(\lambda)} i(A \cap \alpha) = \bigcup_{\alpha<\lambda} i(A \cap \alpha).$$

In particular, $i``\mathcal{P}(\lambda) = \{\bigcup_{\alpha<\lambda} i(A \cap \alpha) \mid A \in \mathcal{P}(\lambda)\}$, but since $\mathcal{P}(\lambda)^N = \mathcal{P}(\lambda)$, this set must be in N as well.

Finally, to show that $i(\eta) = \sup i``\eta$, the fact that $i``\mathcal{P}(\lambda) \in N$ means that $N^{2^\lambda} \subseteq N$, and therefore all enumerations of $\mathcal{P}(\lambda)$ exists in N. And therefore the cardinal 2^λ is computed correctly in N. In particular, $i(\eta) = \sup i``\eta$. This completes the proof by contradiction, so $M \neq V$. □

Exercise 8.3. Prove the facts from the above theorem.

There are a number of different proofs. Kunen's original proof used ω-Jónsson algebras; Woodin's proof utilises a partition of $\{\xi < \lambda^+ \mid \operatorname{cf}(\xi) = \omega\}$ into $\operatorname{crit}(j)$-many stationary sets, and uses that to derive a contradiction. An additional proof by Zapletal utilises Shelah's PCF theory. These proofs are all distinct and provide us with various corollaries.

Corollary 8.4. *If λ is an ordinal, then there is no elementary embedding from $V_{\lambda+2}$ to itself.*

Corollary 8.5. *If $j\colon V_\eta \to M$, where η is a limit, then $j`` \sup_{n<\omega}(j^n(\operatorname{crit}(j)) \notin M$.*

Corollary 8.6. *ω-huge cardinals are inconsistent.*[19]

[19] There is a historical quirk where "ω-huge" meant "ω-superstrong" for a short period of time in the 1980s. But we strictly rely on the definition of n-huge cardinals

One must ask, then, how can there is an elementary embedding $j: L \to L$, as is the case if a measurable cardinal exist? Have we been deceived all of this time and large cardinals are thoroughly inconsistent? The answer, of course, is that $0^\#$ is incredibly undefinable. We cannot compute it in L itself, and in more general terms it really tells us that our universe is *significantly* richer than the constructible universe.

One final corollary must be mentioned.

Corollary 8.7. *In the definition of α-supercompact*[20] *cardinal we do not need to require that $j(\kappa) > \alpha$.*

Proof. If $j: V \to M$ is a α-supercompact embedding and $j(\kappa) < \alpha$, then either there is some $n < \omega$ for which $j^n(\kappa) > \alpha$ and j^n is also α-supercompact, or else this is never the case and $\sup j^n(\kappa) = \lambda \leq \alpha$, which means that $j``\lambda \in M$. $\square$

8.2 The end of the road*

As the corollary above tells us, there cannot be any ordinal λ for which there is a non-trivial $j: V_{\lambda+2} \to V_{\lambda+2}$. At least not when the Axiom of Choice is assumed. But what about something weaker?

Definition 8.8. We say that λ is an I3 *cardinal* if there exists a non-trivial elementary embedding $j: V_\lambda \to V_\lambda$.

Remark. The name "I" comes from "inconsistency" as these were, allegedly, axioms that were going to be inconsistent.

Proposition 8.9. *Suppose that λ is an* I3 *cardinal and that j is a witnessing embedding, then $\lambda = \sup_{n<\omega} j^n(\mathrm{crit}(j))$.*

Proof. If this is not the case, then $\lambda' = \sup_{n<\omega} j^n(\mathrm{crit}(j)) < \lambda$, in which case $j``\lambda' \in V_\lambda$. $\square$

Exercise 8.10 (*) . If λ is an I3 cardinal, then λ is worldly.

There is an argument to be made that the large cardinal here is not λ, but rather $\mathrm{crit}(j)$. Since really it is the critical point of an embedding with very strong closure properties, and in the other cases the critical point of the embedding was the large cardinal. However, this is slightly awkward to our theory, since this would imply that if $j: V_\lambda \to V_\lambda$ is an I3 embedding, then all of $j^n(\mathrm{crit}(j))$ are also critical points of I3 embeddings.

Definition 8.11. Suppose that $j: V \to M$ is an elementary embedding, the *critical sequence* of j is $\kappa_0(j) = \mathrm{crit}(j)$, $\kappa_{n+1}(j) = j(\kappa_n)$, and $\lambda(j) = \sup_{n<\omega} \kappa_n(j)$. When the embedding is clear from context we will omit it from the notation.

With this language we can recast the Kunen's Inconsistency Theorem to say that $j``\lambda(j) \notin M$ and that I3 is the limit of its critical sequence. Moreover, since I3 is $\lambda(j)$ for any of its I3 embeddings, we can comfortably talk about I3 embeddings without needing to discuss the cardinals in question.

Exercise 8.12. If j is an I3 embedding and $\kappa = \kappa_0(j)$, then $\mathrm{der}(j)$ concentrates on n-huge cardinals. In particular $\{\alpha < \kappa \mid \forall n < \omega, \alpha \text{ is } n\text{-huge}\} \in \mathrm{der}(j)$.

Exercise 8.13. If λ is an I3 cardinal, then in V_λ there is a proper class of cardinals which are super-n-huge cardinals for all $n < \omega$.

[20] As well as strong, extendible, etc.

* Assuming the Axiom of Choice, anyway.

8.2.1 LD algebras

Definition 8.14. Let $j\colon V_\lambda \to V_\lambda$ be an embedding and $X \subseteq V_\lambda$, define

$$j^+(X) = \bigcup_{\alpha<\lambda} j(X \cap V_\alpha).$$

Definition 8.15. If λ is an I3 cardinal, $\mathcal{E}_\lambda = \{j \mid j\colon V_\lambda \to V_\lambda \text{ is an elementary embedding}\}$.

Exercise 8.16. If $j, k\colon V_\lambda \to V_\lambda$ are elementary embedding, then $j^+(k)$ and $j \circ k$ are elementary embeddings $V_\lambda \to V_\lambda$. Moreover, show that $j^+(j) \neq j \circ j$ whenever j is non-trivial.

We write $j \cdot k$ to denote the embedding $j^+(k)$. Then $\langle \mathcal{E}_\lambda, \cdot \rangle$ satisfies the following identity:

$$j \cdot (k \cdot h) = (j \cdot k) \cdot (j \cdot h).$$

Structures which satisfy this property are called *left-distributive algebras*, or LD-algebras for short. We can also define this in a more abstract way. Let T_n be the set of words generated by $x_1, \dots, x_n$ and a binary operator $\cdot$, and let $\sim$ be the LD equivalence relation:

$$t_1 \cdot (t_2 \cdot t_3) \sim (t_1 \cdot t_2) \cdot (t_1 \cdot t_3),$$

then $T_n/{\sim}$ is called a *free* LD-algebra with n generators and we denote it by F_n.

Fact 8.17 (Laver). *Let λ be an* I3 *cardinal and $j \in \mathcal{E}_\lambda$ a non-trivial embedding, then the subalgebra generated by j is isomorphic to F_1.*

8.2.2 Even more inconsistent?

Definition 8.18. Let λ be a cardinal. We say that it is...

...an I2 *cardinal* if there is an elementary embedding $j\colon V \to M$ and $V_\lambda = V_{\lambda(j)} \subseteq M$.[21]

...an I1 *cardinal* if there is an elementary embedding $j\colon V_{\lambda+1} \to V_{\lambda+1}$.

...an I0 *cardinal* if there is an elementary embedding $j\colon L(V_{\lambda+1}) \to L(V_{\lambda+1})$ with $\lambda = \lambda(j)$.

A natural question at this point is to ask, is I2 actually stronger than I3, or can we somehow extend an I3 embedding into the whole universe by deriving some extenders?

Theorem 8.19. *If λ is an* I2 *cardinal and j is a witnessing embedding, then*

$$\{\alpha < \mathrm{crit}(j) \mid \exists i\colon V_\lambda \to V_\lambda, \mathrm{crit}(i) = \alpha\} \in \mathrm{der}(j).$$

Proof. Let $j\colon V \to M$ be an I2 embedding with $\lambda(j) = \lambda$. Note that $j' = j \restriction V_\lambda$ satisfies $j'\colon V_\lambda \to V_\lambda$, so it is an I3 embedding. However, $j' \subseteq V_\lambda$, and so $j(j') \cap V_\lambda = j' \cdot j'$, in the sense of $\mathcal{E}_\lambda$. In particular, $M \models \lambda$ is I3. It remains to show that κ is $\mathrm{crit}(i)$ for some $i\colon V_\lambda \to V_\lambda$ such that $i \in M$. Let us denote by κ_n the cardinal $\kappa_n(j) = j^n(\kappa)$.

Let $I = \{i \mid i\colon V_\alpha \to V_\beta \text{ for some } \alpha < \beta < \lambda, \mathrm{crit}(i) = \kappa\}$. We define a partial order on I, $<$, as the transitive closure of the relation

$$i < i' \iff i' \subseteq i \land \exists n < \omega, \kappa_n \in \mathrm{dom}(i \setminus i').$$

[21] This is, essentially, ω-superstrong, in terms of what we had seen before.

Since $\langle \kappa_n \mid n < \omega \rangle \in M$ and since $V_\lambda \in M$, both I and $<$ are correctly defined in M.[22] Since M is transitive, both V and M agree on whether or not $\langle I, < \rangle$ is a well-founded set. However, in V, $\langle j \upharpoonright V_{\kappa_{n+1}} \mid n < \omega \rangle$ is a witness that $<$ is ill-founded, so it must also be ill-founded in M. Taking any such descending sequence $\langle i_n \mid n < \omega \rangle \in M$, and letting $i = \bigcup_{n<\omega} i_i$, we have that $i \colon V_\lambda \to V_\lambda$ is an I3 embedding with $\mathrm{crit}(i) = \kappa$, so the conclusion follows. □

Theorem 8.20. *The following are equivalent for any pair $\kappa < \lambda$.*

1. *There is an* I2 *embedding, $j \colon V \to M$, with $\kappa = \kappa_0(j)$ and $\lambda = \lambda(j)$.*
2. *There is an* I3 *embedding, $j \colon V_\lambda \to V_\lambda$ with $\mathrm{crit}(j) = \kappa$ and whenever $R \subseteq V_\lambda$ is a well-founded relation, $j^+(R)$ is a well-founded relation.*

Sketch of Proof. Assuming (1) the proof is straightforward, since we can take $k \colon V_\lambda \to V_\lambda$ to be $j \upharpoonright V_\lambda$, in which case $k^+(R) = j(R)$, and so well-foundedness is preserved. In the other direction, let j be an I3 embedding with the additional property. We define, for each $n < \omega$, a measure on $\mathcal{P}(\kappa_n)$:

$$U_n = \{X \subseteq \mathcal{P}(\kappa_n) \mid j``\kappa_n \in j(X)\}.$$

Letting $M_n = \mathrm{Ult}(V, U_n)$, then for $n \leq m$, there is a natural embedding $k_{n,m} \colon M_n \to M_m$ given by

$$k_{n,m}([f]_{U_n}) = [f \circ (x \mapsto x \cap \kappa_n)]_{U_m}.$$

Moreover, $\mathrm{crit}(k_{n,m}) \geq \kappa_n$ and the embeddings satisfy that $k_{n,m} \circ k_{\ell,n} = k_{\ell,m}$. Finally, letting M be the direct limit of $\{M_n, k_{n,m} \mid n, m < \omega\}$ and let i be the limit embedding. It is not hard to check that i is λ-strong with $\mathrm{crit}(i) = \kappa$ and $\kappa_n(i) = \kappa_n$. If M is ill-founded, however, then there is a descending sequence of ordinals, each represented by some $\langle \kappa_{n_x}, f_{n_x} \rangle$ in M. Since $|\mathrm{rng}\, f_{n_x}| < \kappa_{n_x+1} \leq \kappa_{n_{x+1}}$, we can then define a relation $R \subseteq V_\lambda$ which encodes this well-ordering. By the fact that M_n is an ultrapower, and $[f_{n_x}]_{U_{n_x}} = j_{U_{n_x}}(j``\kappa_{n_x})$, we get that $j^+(R)$ must be ill-founded as well. As j was chosen with j^+ preserving well-foundedness, this is not the case, so M is transitive and i is I2. □

Corollary 8.21. I2 *cardinals can be characterised in first-order logic.*

Exercise 8.22. If $j \colon V_{\lambda+1} \to V_{\lambda+1}$ is an I1 embedding, then λ is an I2 cardinal and $\mathrm{der}(j)$ concentrates on critical points of I2 embeddings.

Finally, If $j \colon L(V_{\lambda+1}) \to L(V_{\lambda+1})$ is an I0 embedding, then $j \upharpoonright V_{\lambda+1}$ is very clearly and I1 embedding. But we can actually say more.

Fact 8.23. *If there exists a non-trivial $j \colon L_1(V_{\lambda+1}) \to L_1(V_{\lambda+1})$,*[23] *then there exists many* I1 *cardinals.*

Remark. It is worth pointing that under I0 axioms, the model $L(V_{\lambda+1})$ looks quite similar to $L(\mathbb{R})$ under strong determinacy assumptions. This lends itself to a very interesting and rich structure in both set theoretic and topological fashions.

[22]Note that this is not $j(I)$, but rather I^M, where we reinterpret the definition with the same parameters, just in M.

[23]Here $L_1(X)$ is simply $\mathrm{Def}(X)$, the set of definable subsets of X.

Chapter 9

Beyond the inconsistency

9.1 Dropping the Axiom of Choice

We want to understand how large cardinals can extend and grow, even further, when the Axiom of Choice is removed. Specifically, how much of Kunen's Inconsistency Theorem holds if the Axiom of Choice is removed? Clearly, Suzuki's theorem still holds, and so any self-embedding is still not going to be definable. But can we do more?

The problem, of course, is that we tend to measure consistency by comparing things to large cardinal axioms. We started with, say, strongly compact cardinals, and decided that these are "reasonable axioms", and so we can prove the consistency of measurable cardinals. If we decided that I2 is a reasonable large cardinal axiom, then we can prove the consistency of I3 cardinals. But if we accept Kunen's theorem as an upper limit, what would be the philosophical argument towards accepting large cardinal axioms which refute Kunen's theorem?

9.1.1 Ultrapowers without the Axiom of Choice

Theorem 9.1. *Suppose that U is an ultrafilter on a set X. The following are equivalent.*

1. $\operatorname{Ult}(V, U)$ *satisfies the Axiom of Extensionality.*
2. $\operatorname{Ult}(V, U)$ *has a unique empty set.*
3. *If f is a surjective function onto X, then there is some $A \in U$ and a function g with* $\operatorname{dom} g = A$ *such that* $g \circ f = \mathrm{id}$.
4. *If $\{S_x \mid x \in X\}$ is a family of non-empty sets, then there is some $A \in U$ such that* $\prod_{x \in A} S_x \neq \varnothing$.
5. *Łoś's theorem for* $\operatorname{Ult}(V, U)$.
6. *j_U is elementary.*

Proof. The implications (1)→(2) and (6)→(1) are trivial. The equivalence between each of the pairs (3)–(4) and (5)–(6) are the standard proofs and do not rely on the Axiom of Choice.

(4) implies (5): the only non-trivial part of the proof is when $\operatorname{Ult}(V, U) \models \exists x \varphi([f], x)$ in producing g such that $\{i \in X \mid V \models \varphi(f(i), g(i))\} \in U$. For each $i \in X$, let S_i be the set of least-ranked witnesses, namely $\{t \mid \varphi(f(i), t)\} \cap V_{\alpha_i}$, where α_i is the least for which this is

non-empty. Then there is some set $A \in U$ for which $\prod_{i \in A} S_i \neq \varnothing$, pick some g in this product and extend it to some function on X.

To complete the proof we need to establish the implication from (2) to any of (3), (4), (5), or (6). Let us show that (2) implies (4). Let $S\colon X \to V$ be a function such that $S(x) \neq \varnothing$ for all $x \in X$. Note that in $\mathrm{Ult}(V, U)$ it must be that $[S] \neq [\varnothing]$, and so $\mathrm{Ult}(V, U) \models \exists x(x \in [S])$. Therefore, there is some g such that $\mathrm{Ult}(V, U) \models [g] \in [S]$, and so by the very definition of the ultrapower structure, $A = \{x \in X \mid g(x) \in S(x)\} \in U$, and therefore $g \restriction A \in \prod_{x \in A} S(x)$. □

Remark. It is consistent that a critical cardinal does not have any ultrapower embeddings witnessing its criticality. It is also consistent that no embedding goes into a countably closed.[24]

It is also consistent for ω_1 to be a measurable cardinal, and for a measurable cardinal to have no normal measures, but it is easy to see that in either case these are not critical cardinals. In fact, the least measurable cardinal can be the least Mahlo cardinal, or even the least inaccessible cardinal.[25]

9.2 Beyond the inconsistency

Definition 9.2. We say that λ is a *Kunen cardinal* if there is a non-trivial $j\colon V_{\lambda+2} \to V_{\lambda+2}$ such that $\lambda = \lambda(j)$.[26] We say that j is a *Kunen embedding* if it witnesses that $\lambda(j)$ is a Kunen cardinal.

Easily now, if λ is a Kunen cardinal, then the Axiom of Choice must fail in $V_{\lambda+2}$. Quite recently, Schlutzenberg proved the following fact.

Fact 9.3. *If* $\mathsf{ZFC} + \mathrm{I0}$ *is consistent, then* ZF *is consistent with a Kunen cardinal. And if* ZF *is consistent with a Kunen cardinal, then* $\mathsf{ZF} + \mathrm{I0}$ *is consistent as well.*

This is an incredible breakthrough, and the first in many years in the study of choiceless large cardinals. An interesting thing to note here is that the proof is not sufficient to conclude that a Kunen cardinal implies the consistency of $\mathsf{ZFC} + \mathrm{I0}$.

Definition 9.4. We say that κ is a *Reinhardt cardinal* if it is the critical point of an elementary embedding $j\colon V \to V$.

Exercise 9.5. Proposition 8.2 is provable in ZF. In particular Reinhardt cardinals are not first-order definable in ZF.

The consequence of this exercise is that when we talk about Reinhardt cardinals, we are must fallback to one of the settings for Kunen's Inconsistency Theorem. We will do that implicitly from this point on. So if we say that $V_\eta \models$ "There exists a Reinhardt cardinal", we will mean that in the context of ZF_2, so the structure in question will be $\langle V_\eta, \in, V_{\eta+1}\rangle$.

Exercise 9.6. Let κ be a Reinhardt cardinal, and let $j\colon V \to V$ be a witnessing embedding, then $\kappa_n(j)$ is Reinhardt for all $n < \omega$.

We will say, therefore, that κ and κ' are distinct Reinhardt cardinals if they are critical points of j and j' respectively, with $\lambda(j) \neq \lambda(j')$.

[24] Compare to Proposition 3.18.

[25] Interestingly, "ω_1 is measurable" and "the least Mahlo cardinal is the least measurable" both are equiconsistent with a single measurable cardinal. In contrast, if we require "the least measurable is the least inaccessible", the consistency strength goes up and is at least $o(\kappa) = \kappa$.

[26] This is not a standard term, and in some cases you will see it refer to the critical point of the embedding.

Proposition 9.7. *Let κ be a Reinhardt cardinal and let j be a witnessing embedding, then* $\operatorname{der}(j)$ *concentrates on critical points of Kunen embeddings.*

Proof. Let $\lambda = \lambda(j)$, then $j(\lambda) = \lambda$, and therefore $j(V_{\lambda+2}) = V_{j(\lambda)+2} = V_{\lambda+2}$, and in particular $j \restriction V_{\lambda+2}$ is a Kunen embedding. Since V satisfies that κ is a critical point of a Kunen embedding, $\operatorname{der}(j)$ must concentrate of the set of such points. Note that this only tells us that λ has various Kunen embeddings with critical points below κ. □

If j is a Reinhardt embedding with minimal critical point and minimal λ, letting $\eta > \lambda(j)$ be an inaccessible cardinal,[27] then $j(\eta) = \eta$ and so in V_η it is still true that $j \restriction V_\eta$ is a Reinhardt embedding, but by the choice and minimality, it must be that no Reinhardt embeddings go beyond $\lambda(j)$. Otherwise, there would be some embedding i such that $i(\operatorname{crit}(i)) > \lambda(j)$, but as a Reinhardt embedding, $i(\operatorname{crit}(i))$ must be inaccessible and there are no inaccessible cardinals in V_η above $\lambda(j)$.

It is tempting to think that Reinhardt cardinals, therefore, have a limit power over the structure of the universe. This is not true. The following two facts, due to Gabe Goldberg, show just how much power is hiding behind a Reinhardt cardinal

Fact 9.8. *If there is a Reinhardt cardinal, then there is a proper class regular cardinals,*[28] *moreover there is a proper class of measurable cardinals.*[29]

Fact 9.9. *If there is a Reinhardt cardinal, then* SVC *fails.*[30]

Definition 9.10. We say that κ is a *super-Reinhardt cardinal* if for every α there is an embedding $j\colon V \to V$ with $\operatorname{crit}(j) = \kappa$ and $j(\kappa) > \alpha$.

Exercise 9.11. If there exists a super-Reinhardt cardinal, then there is a proper class of inaccessible cardinals.

Theorem 9.12. *If there is a super-Reinhardt cardinal, then there is a transitive model of* ZF_2 *with a Reinhardt cardinal.*

Proof. Let κ be a super-Reinhardt cardinal and let $j\colon V \to V$ be an embedding with $\operatorname{crit}(j) = \kappa$, and let δ be the least inaccessible cardinal above $\lambda(j)$. Then in V_δ, $j \restriction V_\delta\colon V_\delta \to V_\delta$ is an elementary embedding with critical point κ, so κ is Reinhardt cardinal. □

Exercise 9.13. Find a different proof the previous theorem. Specifically, if κ is a super-Reinhardt cardinal, then there is some $\delta < \kappa$ such that V_δ has a Reinhardt cardinal.

Exercise 9.14 (*) . Show that if κ is a Reinhardt cardinal, then there is some j witnessing that such that $j \restriction V_{\lambda(j)}$ is not a Reinhardt embedding.

Definition 9.15. We say that a cardinal κ is an *A-super-Reinhardt cardinal* for a class A, if there is are elementary embedding $j\colon V \to V$ with $\operatorname{crit}(j) = \kappa$ and $j^+(A) = A$ such that $j(\kappa)$ is arbitrarily large.

Definition 9.16. We say that κ is a *totally Reinhardt cardinal* if for every $A \subseteq V_\kappa$, $V_\kappa \models$ "There exists an A-super-Reinhardt cardinal".

Exercise 9.17. If κ is totally Reinhardt, then κ is regular.

[27] There are many definitions of inaccessible cardinals, and they are not all equivalent. The standard one is "$V_\kappa \models \mathsf{ZF}_2$", or equivalently, "if $x \in V_\kappa$, then for all $f\colon x \to \kappa$, $\sup \operatorname{rng} f < \kappa$".

[28] Recall that ZF cannot prove the existence of uncountable regular cardinals, at least assuming a proper class of strongly compact cardinals is consistent.

[29] In the sense that there exists a complete measure.

[30] Small Violations of Choice (SVC) states that there is a complete Boolean algebra which forces the Axiom of Choice. In a sense, this implies that the universe is somehow "close" to being a model of ZFC.

9.3 Berkeley cardinals

Definition 9.18. We say that an ordinal δ is *proto-Berkeley* if for every transitive set M such that $\delta \in M$, there exists an elementary embedding $j\colon M \to M$ with $\operatorname{crit}(j) < \delta$.

The problem with this definition is that if δ is a proto-Berkeley cardinal, then every ordinal above δ is proto-Berkeley. Even restricting to cardinals will not make a difference. So, in a sense, we need to capture the essence of the definition.

Theorem 9.19. *Suppose that δ is the least proto-Berkeley cardinal, then for every transitive set M such that $\delta \in M$ and every $\eta < \delta$, there is some $j\colon M \to M$ such that $\eta < \operatorname{crit}(j) < \delta$.*

Proof. Suppose that this is not the case. Let η be the least ordinal such that there is a transitive set M with $\delta \in M$ and there is no $j\colon M \to M$ with $\eta < \operatorname{crit}(j) < \delta$. We will show that η is in a proto-Berkeley cardinal, contradicting the minimality of δ.

Let N be a transitive set with $\eta \in N$ and let X be $\operatorname{tcl}(\{\langle N, M, \eta\rangle\})$. Then N, M, and η are definable in X, and so if $j\colon X \to X$ is an elementary embedding, $j(N) = N$ and $j(M) = M$. In particular, $j \upharpoonright M\colon M \to M$, and therefore $\operatorname{crit}(j) < \eta$. But since $j \upharpoonright N\colon N \to N$, this means that $j \upharpoonright N$ is non-trivial on N. Therefore η is proto-Berkeley, and therefore we have a contradiction. $\square$

Definition 9.20. We say that δ is a *Berkeley cardinal* if for every transitive set M such that $\delta \in M$ and every $\eta < \delta$, there exists an elementary embedding $j\colon M \to M$ such that $\eta < \operatorname{crit}(j) < \delta$.

Remark. We write δ_0 to denote the least proto-Berkeley cardinal and so Theorem 9.19 shows that it is in fact a Berkeley cardinal. More generally, δ_α denotes the least proto-Berkeley cardinal where we are guaranteed that there is an embedding whose critical point is above α. The same argument shows that δ_α is a Berkeley cardinal as well.

Note that this definition is a Π_2 definition. Therefore the least Berkeley cardinal lies below the least Σ_3-correct cardinal, and in particular the least Berkeley cardinal is below the least extendible cardinal.

Exercise 9.21. If δ is a Berkeley cardinal, then δ is the limit of inaccessible cardinals. If δ is a limit of Berkeley cardinals, then δ is a Berkeley cardinal.

Exercise 9.22. If δ is a Berkeley cardinal and $\lambda > \delta$ is a limit ordinal, then $V_\lambda \models$"δ is a Berkeley cardinal".

Exercise 9.23. If λ is a Σ_2-correct cardinal, $\delta < \lambda$, and $V_\delta \models$"δ is a Berkeley cardinal", then δ is a Berkeley cardinal.

Proposition 9.24. *δ is a Berkeley cardinal if and only if for every transitive set M such that $\delta \in M$, for every $\eta < \delta$ and $x \in M$, there is $j\colon M \to M$ such that $\eta < \operatorname{crit}(j) < \delta$ and $j(x) = x$.*

Proof. One direction is trivial. In the other direction, assume that δ is a Berkeley cardinal, let M be a transitive set and let η, x as in the assumption. Then $N = M \cup \{M, \{M, x\}\}$ is a transitive set such that $\delta \in N$, therefore, there is some $j\colon N \to N$ such that $\eta < \operatorname{crit}(j) < \delta$. However, $\{M, x\}$ is the only set of maximal rank, so $j(\{M, x\}) = \{M, x\}$, and therefore $j(M) = M$ and $j(x) = x$. So, $j \upharpoonright M$ is as wanted. $\square$

The same arguments above hold for proto-Berkeley ordinals. They are also Π_2-definable, and so Σ_2-correct ordinals will identify those correctly.

Theorem 9.25. *If δ is the least Berkeley cardinal, then for all sufficiently large limit ordinals λ, if $j\colon V_\lambda \to V_\lambda$ is an embedding with $\operatorname{crit}(j) < \delta$, then $j(\delta) = \delta$ and $\sup\{\alpha < \delta \mid j(\alpha) = \alpha\} = \delta$.*

Proof. Let $\lambda_0 > \delta$ be the least Σ_2-correct cardinal, and let $\lambda > \lambda_0$ be a limit ordinal. Then V_λ knows that δ is the least Berkeley cardinal. In particular, $j(\delta) = \delta$ for all $j\colon V_\lambda \to V_\lambda$.

Towards contradiction, assume that $j\colon V_\lambda \to V_\lambda$ is an elementary embedding with $\operatorname{crit}(j) < \delta$ and $\sup\{\alpha < \delta \mid j(\alpha) = \alpha\} = \eta_0 < \delta$. Let $\eta_{n+1} = j(\eta_n)$, then for $\eta = \sup\{\eta_n \mid n < \omega\}$ we have that $j(\eta) = \eta$, and therefore it must be that $\eta = \delta$.

Let $M_0 \in V_{\lambda_0}$ be a transitive set of minimal rank witnessing that η_0 is not proto-Berkeley. Letting $M_{n+1} = j(M_n)$, we have that M_{n+1} is a witness that η_n is not proto-Berkeley. Since we chose M_0 to have minimal rank, it must be that M_{n+1} also has minimal rank, and therefore $M_{n+1} \in V_{\lambda_0}$ as well, since V_{λ_0} can detect the failure of Berkeley-ness of each η_n.

In particular, the sequence $\mathcal{M} = \langle M_n \mid n < \omega\rangle \in V_{\lambda_0+1} \subseteq V_\lambda$. Letting $i\colon V_\lambda \to V_\lambda$ be an embedding with $\operatorname{crit}(i) < \delta$ and $i(\mathcal{M}) = \mathcal{M}$, we take $n < \omega$ to be the least such that $\operatorname{crit}(i) < \eta_n$. However, $i(M_n) = M_{i(n)} = M_n$, and therefore M_n was a witness that η_n is not a proto-Berkeley ordinal, in contradiction to the fact that $i \restriction M_n\colon M_n \to M_n$ has critical point below η_n. □

Theorem 9.26. *Suppose that δ is a Berkeley cardinal, then there is an inaccessible cardinal $\eta < \delta$ such that V_η has a Reinhardt cardinal.*

Proof. We can assume that δ is the least Berkeley cardinal, and so we can find some sufficiently large λ such that $j\colon V_\lambda \to V_\lambda$ has $\operatorname{crit}(j) < \delta$ and $\delta = \sup\{\alpha < \delta \mid j(\alpha) = \alpha\}$.

Fix such j and let $\kappa = \operatorname{crit}(j)$. We let α be such that $\kappa < \alpha = j(\alpha) < \delta$ and let η be the least inaccessible cardinal above α. Since η is definable in V_λ from α, $j(\eta) = \eta$, so $j \restriction V_\eta\colon V_\eta \to V_\eta$. □

Exercise 9.27. If δ is a Berkeley cardinal, then there is an inaccessible cardinal $\eta < \delta$ such that V_η has two distinct Reinhardt cardinals.

Definition 9.28. We say that δ is a *club Berkeley cardinal* if it is a regular cardinal and for every transitive set M with $\delta \in M$, $\{\operatorname{crit}(j) \mid j\colon M \to M, \operatorname{crit}(j) < \delta\}$ is stationary in δ.

Theorem 9.29. *If δ is a club Berkeley cardinal, then δ is totally Reinhardt.*

Proof. Let $A \subseteq V_\delta$, we want to show there is an A-super-Reinhardt in V_δ. Letting M be a transitive set such that $V_{\delta+1} \in M$ and A is definable in M, then there is $\kappa < \delta$ such that for any $\alpha < \delta$ there is an elementary embedding $j\colon M \to M$ such that $\operatorname{crit}(j) = \kappa$, $j(\kappa) > \alpha$, and $j(A) = A$. To prove this first fix M, and suppose this is not the case. Then for each κ, there is a least ordinal, $f(\kappa)$ for which there is no such j with $\operatorname{crit}(j) = \kappa$. Since δ is regular, $C = \{\alpha < \delta \mid f``\alpha \subseteq \alpha\}$ is a closed set. If C is a club, then there is some $j\colon M \to M$ such that $\operatorname{crit}(j) = \kappa$ is in this club, this means that $j(\kappa) \in j(C)$, but using the same trick as Proposition 9.24 we may assume $j(C) = C$. However, $\kappa < j(\kappa)$ and $f(\kappa) > j(\kappa)$, which is a contradiction to the definition of f. Therefore, taking any M where A is definable, and letting κ be as above, we get that $j \restriction V_\delta$ satisfies that $(j \restriction V_\delta)^+(A) = j(A) = A$ as wanted. □

Definition 9.30. We say that δ is a *rank Berkeley cardinal* if for all $\lambda \geq \delta > \eta$ there is an elementary embedding $j\colon V_\lambda \to V_\lambda$ with $\eta < \operatorname{crit}(j) < \delta$.

Exercise 9.31 (*) . If $j\colon V \to V$ is a Reinhardt embedding, then $\lambda(j)$ is a rank Berkeley cardinal.

Chapter 10

Coda: Exacting cardinals and the HOD Conjecture

In this chapter, which assumes ZFC, we discuss a preprint of Aguilera, Bagaria, and Lücke from November 2024 ("Large cardinals, structural reflection, and the HOD Conjecture"). These concepts and results may still change as we progress forward.

10.1 Exacting cardinals

Definition 10.1. We say that λ is an *exacting cardinal* if for all $\alpha < \lambda < \zeta$, there exists $X \prec V_\zeta$ with $V_\lambda \cup \{\lambda\} \subseteq X$ and an elementary embedding $j\colon X \to V_\zeta$ with $\alpha \leq \operatorname{crit}(j) < \lambda = j(\lambda)$.

Exercise 10.2. If λ is an exacting cardinal, then λ is an I3 cardinal.

Exercise 10.3. If λ is the least exacting cardinal, then there are no extendible cardinals below λ.

Exercise 10.4 (*) . λ is an exacting cardinal if and only if for all $\zeta > \lambda$, there exists $X \prec V_\zeta$ with $V_\lambda \cup \{\lambda\} \subseteq X$ and an elementary embedding $j\colon X \to V_\zeta$ with $\operatorname{crit}(j) < \lambda = j(\lambda)$.

Fact 10.5. *If $j\colon L(V_{\lambda+1}) \to L(V_{\lambda+1})$ is an* I0 *embedding, then there is a transitive set M satisfying* ZFC *in which λ is an exacting cardinal.*

We will not prove this. But the model, however, is relatively easy to describe. We identify a set Γ in $L(V_{\lambda+1})$ and we take $M = L_{\lambda^+}(\Gamma)$.

Exercise 10.6. If λ is an I3 cardinal, then for every $j\colon V_\lambda \to V_\lambda$ there is a well-ordering on V_λ, $\lhd$, of order type λ, such that $j^+(\lhd) = \lhd$.

Fix $\lhd$ as in the exercise, then $\Gamma = V_\lambda \cup \{\vec{\lambda}, \lhd\}$. And so, we take $M = L_{\lambda^+}(\Gamma)$.

Exercise 10.7. $M \in L(V_{\lambda+1})$ and $j(M) = M$. Therefore $j \restriction M\colon M \to M$ is an elementary.

Theorem 10.8. *Suppose that λ is an exacting cardinal. Then $\operatorname{cf}(\lambda)^{\mathrm{HOD}} = \lambda$.*

Proof. Towards a contradiction, assume that $\operatorname{cf}(\lambda)^{\mathrm{HOD}} < \lambda$. Let ζ be a Σ_3-correct cardinal such that $\lambda < \zeta$. Then there is an $X \prec V_\zeta$ such that $V_\lambda \cup \{\lambda\} \subseteq X$ and $j\colon X \to V_\zeta$ satisfies $j(\lambda) = \lambda > \operatorname{crit}(j) > \operatorname{cf}(\lambda)^{\mathrm{HOD}}$.

Let $c\colon \operatorname{cf}(\lambda)^{\mathrm{HOD}} \to \lambda$ be the HOD-least cofinal function. Since the well-ordering of HOD is Σ_2-definable, c is Σ_3-definable in V using λ as a parameter. In particular, V_ζ agrees that c is the least such function, and in particular, $c \in X$. As $j(\lambda) = \lambda$, it must be that $j(c) = c$ as well, and since $\operatorname{crit}(j) > \operatorname{dom} c$ it must be that $j(c) = j``c$.

However, $\lambda = \lambda(j)$, since $j \restriction V_\lambda$ is an I3 embedding, λ is the least fixed point of j. In particular, taking ξ such that $c(\xi) > \operatorname{crit}(j)$, it must be that $c(\xi) < j(c(\xi)) = j(c)(j(\xi)) = c(\xi)$. □

Exercise 10.9. If λ is an exacting cardinal, then $\operatorname{cf}(\lambda)^{\mathrm{HOD}^{V_\lambda}} = \lambda$.

Corollary 10.10. *If there is an exacting cardinal, then $V \neq \mathrm{HOD}$. If there is a proper class of exacting cardinals, then for all x, $V \neq \mathrm{HOD}_x$.*

Remark. Most large cardinal axioms that are not known to be inconsistent with ZFC are consistent with $V = \mathrm{HOD}$. Even those without "canonical" inner models. This includes axioms such as I0. So in a somewhat surprising twist, we have an axiom that is incompatible with $V = \mathrm{HOD}$. As a corollary we get that while I0 proves the existence of a transitive model of ZFC with an exacting cardinal, it does not prove the existence of an exacting cardinal.

Definition 10.11. We say that λ is an *ultraexacting cardinal* if for all $\alpha < \lambda < \zeta$ there exists $X \prec V_\zeta$ such that $V_\lambda \cup \{\lambda\} \subseteq X$ and an elementary embedding $j\colon X \to V_\zeta$ such that $\alpha \leq \operatorname{crit}(j) < \lambda$, $\lambda = j(\lambda)$, and $j \restriction V_\lambda \in X$.

Exercise 10.12. If λ is an ultraexacting cardinal and $j\colon X \to V_\zeta$ is a witnessing embedding, then $j \restriction V_\lambda$ is an I2 embedding.

Theorem 10.13. *If λ is an ultraexacting cardinal and $j\colon X \to V_\zeta$ is a witnessing embedding with $\zeta \in C^{(2)}$, then $(j \restriction V_\lambda)^+\colon V_{\lambda+1} \to V_{\lambda+1}$ is an* I1 *embedding.*

Proof. Let $i = (j \restriction V_\lambda)^+$, since $j \restriction V_\lambda \in X$, we have that $i \in X$ as well. Moreover, if $x \in V_{\lambda+1} \cap X$, then $i(x) = j(x)$. In particular,

$$X \models \text{“}V_{\lambda+1} \models \varphi(x)\text{”} \iff V_\zeta \models \text{“}V_{\lambda+1} \models \varphi(i(x))\text{”} \iff X \models \text{“}V_{\lambda+1} \models \varphi(i(x))\text{”}.$$

Therefore,

$$X \models \text{“}i \text{ is an I1 embedding”}.$$

Therefore V_ζ satisfies the same, and since it is Σ_2-correct, i is an I1 embedding. □

Definition 10.14. For a set a we say that $a^\#$ exists if there is an elementary embedding $j\colon L(a) \to L(a)$ such that $\operatorname{crit}(j) > \operatorname{rank}(X)$.

Exercise 10.15. If κ is measurable, then $a^\#$ exists for all $a \in V_\kappa$. If there exists a strong cardinal, then every set has a sharp.

Fact 10.16. *Suppose that λ is an ultraexacting cardinal and $V_{\lambda+1}^\#$ exists, then λ is the limit of* I0 *cardinals. In particular, there is a set model of* ZFC *with a proper class of* I0 *cardinals.*

Fact 10.17. *Suppose that λ is an* I0 *cardinal, then there is an inner model of a generic extension in which* ZFC *holds and λ is an ultraexacting cardinal.*

This is somewhat surprising. Normally, adding a sharp on top of a weaker assumption is not going to cause a very significant jump in the consistency strength. But whereas a single I0 gives us a class model with an ultraexacting cardinal, adding a single measurable cardinal above gives us the consistency of a proper class of I0 cardinals.

Exercise 10.18 (*) . Suppose that δ is an extendible cardinal and there exists (ultra)exacting cardinal above δ. Then δ is the limit of (ultra)exacting cardinals and there is a proper class of (ultra)exacting cardinals.

10.2 Conjectures, conjectures, conjectures...

Fact 6.40 (The HOD Dichotomy). *Suppose that κ is an extendible cardinal. Then exactly one of the two following holds.*

1. *If $\delta > \kappa$ is singular, then δ is singular in* HOD *and $\delta^+ = (\delta^+)^{\mathrm{HOD}}$.*
2. *If $\delta > \kappa$ is a regular cardinal, then it is measurable in* HOD*.*

The HOD Hypothesis is the first case of the two, and The HOD Conjecture states that the Hypothesis is in fact a theorem. Namely, under sufficiently strong large cardinal axioms, the universe of sets is provably close to HOD.

Theorem 10.18. *If there exists an extendible cardinal below an exacting cardinal, then the HOD Conjecture is false.*

Proof. Let $\kappa < \lambda$ be the extendible and exacting cardinals respectively. By Theorem 10.8, λ is regular in HOD, and so the first scenario of the HOD Dichotomy fails, and in particular the HOD Hypothesis fails, and so the HOD Conjecture is false. $\square$

Definition 10.19. The *Weak HOD Conjecture* states that if there is an extendible cardinal below a huge cardinal, then the HOD Hypothesis is provable.

Exercise 10.20. If there is an extendible cardinal below an exacting cardinal, then the Weak HOD Conjecture is false.

Fact 10.21. *Suppose that there is a Σ_3-correct Reinhardt cardinal and a supercompact above its critical sequence. Then* ZFC *is consistent with the existence of an extendible cardinal below an exacting cardinal. In particular, under this assumption, the HOD Conjecture is false.*

How does the HOD Conjecture affect the large cardinal structure of the universe?

Theorem 10.22. *If $\mathrm{cf}(\lambda) = \omega < \lambda$ is such that λ^+ is computed correctly in* HOD *and the club filter on λ^+ is λ^+-complete, then λ is not a Kunen cardinal.*

Proof. Suppose that λ was a Kunen cardinal and let $j\colon V_{\lambda+2} \to V_{\lambda+2}$ witness that. Note that by mapping each $R \subseteq V_{\lambda+1}$ which codes an extensional and well-founded relation to its transitive collapse, we can replace $V_{\lambda+2}$ by $M = \{a \mid \mathrm{tcl}(a) \leq^* V_{\lambda+1}\}$ and assume $j\colon M \to M$ instead.[31]

Let $S = \{\eta < \lambda^+ \mid \mathrm{cf}(\eta) = \omega\}$, then $S \in \mathrm{HOD}$, and it is stationary there, or otherwise λ^+ cannot be computed correctly in HOD. Partition S, in HOD, into a sequence of stationary sets, $\langle S_\alpha \mid \alpha < \lambda\rangle$. If this is impossible, then λ^+ must be measurable in HOD and thus $(\lambda^+)^{\mathrm{HOD}} < \lambda^+$.

Since $j(\lambda) = \lambda$, we have that $j(\langle S_\alpha \mid \alpha < \lambda\rangle) = \langle T_\beta \mid \beta < \lambda\rangle$. Since $j(S) = S$, the sequence of T_β is a stationary partition of S as well. Let $\kappa = \mathrm{crit}(j) < \lambda$ and let $C \subseteq \lambda^+$ be $\{\eta < \lambda^+ \mid j``\eta \subseteq \eta\}$. Since C is a club and T_κ is a stationary set, there is some $\eta_0 \in C \cap T_\kappa$, but if $\eta \in C \cap S$, $j(\eta) = \eta$. In particular, $j(\eta_0) = \eta_0$.

As the original sequence of S_α was a partition of S, there is some $\alpha_0 < \lambda$ with $\eta_0 \in S_{\alpha_0}$, and so $\eta_0 = j(\eta_0) \in j(S_{\alpha_0}) = T_{j(\alpha_0)}$. Therefore, $j(\alpha_0) = \kappa$, but $\kappa = \mathrm{crit}(j)$ so this is impossible. $\square$

Fact 10.23. *If λ is a singular limit of supercompact cardinals, then λ^+ is regular and the club filter on λ^+ is λ^+-complete.*

What the theorem tells us, along with the fact, is that in the first case of the HOD Dichotomy, if there is a proper class of supercompact cardinals, then there is no Reinhardt cardinals.

[31] If $V_{\lambda+1}$ can be well-ordered, then this is exactly $H((2^\lambda)^+)$.